12 Stability, Instability, and Convection

Now that we have the fundamental equations of stellar structure, we would like to examine some interesting situations in which they apply. One such interesting regime is the transition from stable stars to instability, either in a part of the star or throughout its interior. We will examine this by answering the following question: if we perturb the system (or a part of it), does it settle back into equilibrium?

12.1 *Thermal stability*

Suppose we briefly exceed thermal equilibrium; what happens? In equilibrium, the input luminosity from nuclear burning balances the energy radiated away:

$$\frac{dE_{\text{tot}}}{dt} = L_{\text{nuc}} - L_{\text{rad}} = 0 \tag{281}$$

If the star briefly overproduces energy, then (at least briefly) $L_{\text{nuc}} > L_{\text{rad}}$ and we overproduce a clump ΔE of energy. Over a star's main-sequence lifetime its core temperature steadily rises, so this slight imbalance is happening all the time. Whenever it does, the star must be responding on the photon diffusion timescale, $\tau_{\gamma,\text{diff}} \approx 10^4$ yr... but how?

From the virial theorem, we know that

$$\Delta E = -\Delta E_{\text{th}} = \frac{1}{2}\Delta E_{\text{grav}} \tag{282}$$

So if nuclear processes inject an extra ΔE into the star, we know we will *lose* an equivalent amount ΔE of thermal energy and simultaneously *gain* $2\Delta E$ of gravitational energy. Thus the star must have cooled, and – since its mass has not appreciably changed –its radius must have expanded.

The temperature change is the more relevant for thermal stability, because nuclear reaction rates depend very sensitively on temperature. For Sunlike stars, $\epsilon \propto T^{16}$ – so even a slight cooling will strongly diminish the nuclear energy production rate and will tend to bring the star back into thermal equilibrium. This makes sense, because stars are stable during their slow, steady evolution on the main sequence.

12.2 *Mechanical and Dynamical Stability*

Suppose that a fluid element of the star is briefly pushed away from hydrostatic equilibrium; what happens? We expect the star to respond on the dynamical timescale, $\tau_{\text{dyn}} \approx 30$ min... but how?

Let's consider a toy model of this scenario, in which we squeeze the star slightly and see what happens. (A full analysis would require us to compute a full eigenspectrum of near-equilibrium Navier-Stokes equations, and is definitely beyond the scope of our discussion here.) If we start with Eq. 228, we can integrate to find $P(M)$:

(283)
$$P(M) = \int_0^P dP$$

(284)
$$= \int_{M_{\rm tot}}^{M} \frac{dP}{dM} dM$$

(285)
$$= -\int_{M_{\rm tot}}^{M} \frac{GM}{4\pi r^4} dM$$

Initially, in equilibrium the gas pressure must be equal to the pressure required to maintain hydrostatic support – i.e., we must have $P_{hydro} = P_{gas}$.

If we squeeze the star over a sufficiently short period of time (shorter than the thermal diffusion timescale), heat transfer won't occur during the squeezing and so the contraction is adiabatic. If the contraction is also homologous, then we will have $r \longleftrightarrow r' = r(1-\epsilon)$ $\rho \longleftrightarrow \rho' = \rho(1+3\epsilon)$

How will the star's pressure respond? Since the contraction was sufficiently rapid, we have an adiabatic equation of state

(286) $P \propto \rho^{\gamma_{ad}}$,

where

(287) $\gamma_{ad} \equiv \frac{c_P}{c_V}$

where c_P and c_V are the heat capacities at constant pressure and volume, respectively. Statistical mechanics shows that we have $\gamma_{ad} = 5/3$ for an ideal monoatomic gas, and $4/3$ for photon radiation (or a fully relativistic, degenerate gas).

Thus our perturbed star will have a new internal gas pressure profile,

(288)
$$P'_{gas} = P_{gas}(1+3\epsilon)^{\gamma_{ad}}$$

(289)
$$\approx P_{gas}(1+3\gamma_{ad}\epsilon)$$

Will this new gas pressure be enough to maintain hydrostatic support of the star? To avoid collapse, we need $P_{gas} > P_{hydro}$ always. We know from Eq. 285 that the new *hydrostatic* pressure required to maintain equilibrium will be be

(290)
$$P'_{hydro} = -\int_{M_{\rm tot}}^{M} \frac{GM}{4\pi(r')^4} dM$$

(291)
$$= -(1+4\epsilon)\int_{M_{\rm tot}}^{M} \frac{GM}{4\pi r^4} dM$$

(292)
$$= (1+4\epsilon)P_{hydro}$$

Thus our small perturbation may push us out of equilibrium! The new pressures are in the ratio

(293) $$\frac{P'_{gas}}{P'_{hydro}} \approx \frac{1+3\gamma_{ad}\epsilon}{1+4\epsilon}$$

and so the star will only remain in equilibrium so long as

(294) $$\gamma_{ad} > \frac{4}{3}$$

This is pretty close; a more rigorous treatment of the same question yields

(295) $$\int_0^M \left(\gamma_{ad} - \frac{4}{3}\right) \frac{P}{\rho} dM > 0.$$

Regardless of the exact details, Eqs. 294 and 295 indicate that a star comes ever closer to collapse as it becomes more fully supported by relativistic particles (whether photon radiation, or a relativistic, degenerate gas).

The course reading from Sec. 3.6 of Prialnik shows another possible source of hydrostatic equilibrium, namely via partial ionization of the star. γ_{ad} can also drop below 4/3, thus also leading to collapse, via the reaction $\mathrm{H} \longleftrightarrow \mathrm{H}^+ + \mathrm{e}^-$.

In this reaction, both c_v and c_P change because added heat can go into ionization rather then into increasing the temperature. c_v changes more rapidly than c_P, so γ_{ad} gets smaller (as low as 1.2 or so). Qualitatively speaking, a stellar contraction reverses some of the ionization, reducing the number of particles and also reducing the pressure opposing the initial squeeze. As in the relativistic support case, when $\gamma_{ad} \leq 4/4$, the result is instability.

12.3 *Convection*

Not all structural instabilities lead to stellar collapse. One of the most common instabilities is almost ubiquitous in the vast majority of stars: convection. **Convection** is easily visualized by bringing a pot of water to a boil, and dropping in dark beans, rice grains, or other trace particles. We will now show how the same situation occurs inside of stars.

Convection is one of several dominant modes of energy transport inside of stars. Up until now, we have considered energy transport only by radiation, as described by Eq. ,

$$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB}T^3r^2}. \quad (296)$$

In a few cases energy can also be transported directly by conduction, which is important in the dense, degenerate white dwarfs and neutron stars. Whereas radiation transports heat via photon motions and conduction transports heat through microscopic particle motion, convection transports heat via bulk motions of large parcels of gas or fluid.

When a blob of stellar material is pushed upwards by some internal perturbation, how does it respond: will it sink back down, or continue to rise? Again, a consideration of different timescales is highly relevant here. An outward motion typically corresponds to a drop in both pressure and temperature. The pressure will equilibrate on $\tau_{dyn} \approx 30$ min, while heat will flow on the much slower $\tau_{\gamma,diff}10^4$ yr. So the motion is approximately adiabatic, and a rising blob will transport heat from the lower layers of the star into the outer layers.

The fluid parcel begins at r with some initial conditions $P(r)$, $\rho(r)$, and $T(r)$. After moving outward to $(r+dr)$ the parcel's temperature will remain unchanged even as the pressure rapidly equilibrates, so that the new pressure $P'(r+dr) = P(r+dr)$. Meanwhile (as in the previous sections) we will have an adiabatic equation of state (Eq. 286), which determines the parcel's new density ρ'.

The gas parcel will be stable to this radial perturbation so long as $\rho' > \rho(r+dr)$. Otherwise, if the parcel less dense than its surroundings, it will be like a child's helium balloon and continue to rise: instability! A full analysis shows that this stability requirement can be restated in terms of P and ρ as

$$\left(\frac{dP/dr}{d\rho/dr}\right) < \frac{dP}{d\rho}|_{\text{adiabatic}} \quad (297)$$

$$< \gamma_{ad}\frac{P}{\rho}. \quad (298)$$

Since dP/dr and $d\rho/dr$ are both negative quantities, this can be rearranged as

$$\frac{\rho}{\gamma_{ad}P}\frac{dP}{dr} > \frac{d\rho}{dr}. \quad (299)$$

If we also assume that the stellar material is approximately an ideal gas, then

$P = \rho kT/\mu m_p$ and so

$$\left|\frac{dT}{dr}\right| < \frac{T}{P}\left|\frac{dP}{dr}\right|\left(1 - \frac{1}{\gamma_{ad}}\right) \quad (300)$$

Eq. 300 is the **Schwarzchild stability criterion** again convection. The absolute magnitudes are not strictly necessary, but can help to mentally parse the criterion: as long as the thermal profile is shallower than the modified pressure profile, the star will remain stable to radial perturbations of material.

Modeling convection

Fully self-consistent models of stellar convection are an active area of research and require considerable computational resources to accurate capture the three-dimensional fluid dynamics. The simplest model of convection is to assume that the process is highly efficient – so much so that it drives the system to saturate the Schwarzchild criterion, and so

$$\frac{dT}{dr} = \frac{T}{P}\frac{dP}{dr}\left(1 - \frac{1}{\gamma_{ad}}\right) \quad (301)$$

The somewhat *ad hoc,* but long-tested, framework of **mixing length theory** (MLT) allows us to refine our understanding of convection. In MLT one assumes that gas parcels rise some standard length ℓ, deliver their heat there, and sink again. Accurately estimating ℓ can be as much art as science; at least for nearly Solar stars, ℓ can be calibrated against a host of other observations.

Another way to understand convection comes from examining the relevant equations of stellar structure. Since the star is unstable to convection when the thermal profile becomes too steep, let's consider the thermal transport equation:

$$\frac{dT}{dr} = -\frac{3}{64\pi}\frac{\rho\kappa}{\sigma_{SB}T^3}\frac{L}{r^2} \quad (302)$$

Convection may occur either when $|dT/dr|$ is especially large, or when the Schwarzchild criterion's factor of $(1 - 1/\gamma_{ad})$ is especially small:

1. **Large κ and/or low T**: sometimes met in the outer layers (of Sunlike stars);
2. **Large $F \equiv L/r^2$**: potentially satisfied near cores
3. bf Small γ_{ad}: near ionization layers and molecular disassociation layers.

Overall, we usually see convection across a range of stellar types. Descending along the main sequence, energy transport in the hottest (and most massive) stars is dominated by radiation. Stars of somewhat lower mass (but still with $M_* > M_\odot$) will retain radiative outer atmospheres but acquire interior convective regions. By the time one considers stars of roughly Solar mass, we see a convective exterior that surrounds an internal radiative core. Many years of Solar observations shows the outer surface of the Sun bubbling away, just like a boiling pot. Large, more evolved stars (e.g., red giants) also have convective outer layers; in these cases, the size of the convective cells $\ell \gtrsim R_\odot$!

As one considers still lower masses along the main sequence, the convection region deepens; below spectral types of M2V-M3V, the stars become **fully convection** — i.e., $\ell = R_*$. The Gaia DR2 color-magnitude diagram shows a narrow break in the main sequence which is interpreted as a direct observational signature of the onset of full convection for these smallest, coolest stars. These stars therefore have a fully adiabatic equation of state throughout their interior.

12.4 *Another look at convection vs. radiative transport*

Again, we already know that radiative transport is the default mechanism for getting energy from a star's center to its surface. However, it turns out that within a star, a second mechanism can take over from radiative transport and become dominant. To understand when this happens, we need to bring back two concepts we have previously discussed.

First, we have the temperature gradient. We will use the version that defines the temperature change as a function of radius:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L_r}{4\pi r^2} \tag{303}$$

Second, we have the definition of an adiabatic process: a process in which no heat is exchanged between a system and its environment.

Again, we begin by considering a blob of gas somewhere within a star. It has a temperature T_{blob} and is surrounded by gas at an ambient, local temperature T_*. At this point, they are in thermal equilibrium so that $T_{blob} = T_*$. What happens if this blob is given a quick nudge upward so that now it is warmer than the gas around it: $T_{blob} > T_*$? Just as warm air does, we expect that it will rise. In order for the blob to stop rising, it must become cooler than its environment.

There are two ways for our blob to cool. One way is for it to radiate (that is, exchange heat with its environment). The other way is for it to do work on its environment (essentially, to expand in order to reach pressure equilibrium with its surroundings). Which one is going to be more effective in a star? To answer this, we can just look at time scales. Heat exchange will occur on a roughly thermal (or Kelvin-Helmholz) time scale. For the sun, this time scale is on the order of ten million years. In contrast, work can be done on the blob's environment on a dynamical time scale (technically, the sound-crossing time scale, as this work is done by the expansion of the blob due to pressure). For the sun, this time scale is only about 30 minutes. The enormous difference in magnitude of these scales suggests that there will be almost no chance for the blob to exchange heat with its environment over the time scale in which it expands to reach pressure equilibrium with its environment: our blob will expand and cool nearly entirely adiabatically.

Once the blob has expanded enough to cool down to the ambient temperature, it will cease its upward motion and become stable again. The question is: how quickly will this happen? If an overly warm blob can quickly become cooler than the surrounding gas, then it will not travel far, and upward gas motions will be swiftly damped out. As the gas does not then move in bulk, energy in the star is transported just through the radiation field. However, if the blob cannot quickly become cooler than the ambient gas, it will rise and rise until it encounters a region where it finally satisfies this criterion. This sets up a convective zone in the star: an unstable situation that results in significant movement of gas in the star (think of a pot of water boiling). Warm gas travels upward through this region and eventually reaches the top of the convective zone (which could be the surface of the star, or a region inside the star where the physical conditions have changed significantly) where it is able to cool and return downward. Through the work that it does on its environment over this journey, it carries significant energy from the inner to the outer regions of the star. For this 'convective' zone of the star (which could be a small region or the entire star) convective transport is then the primary means by which energy is transported.

To determine whether convection will dominate, we compare the temperature gradient of the star (the ambient change in temperature as a function of radius) and the rate at which a parcel of gas will cool adiabatically (the so-called adiabatic temperature gradient). These are shown visually in Figure 25 for both convective stability and instability.

If the adiabatic temperature gradient is steeper than the temperature gradient in a star (as set by purely radiative energy transport) then the rate at which a blob of gas will rise and expand and cool will be more rapid than the rate at which the ambient gas in the star cools over the same distance. As a result, if a blob experiences a small displacement upward, it will very quickly become cooler than its surroundings, and sink back to its original position. No significant motion or convection will occur (this region is convectively stable) and the star will continue to transport energy radiatively.

However, if the adiabatic temperature gradient is shallower than the (radiative) temperature gradient in a star, then the rate at which a parcel of gas expands and cools as it rises will be slower than the rate at which the surrounding gas of the star cools over the same distance. Because of this, if a blob is displaced upward, it will remain hotter than its surroundings after it adiabatically expands to reach pressure equilibrium with its surroundings, and it will continue to rise. This sets up convection in the star: as long as the adiabatic temperature gradient is shallower than the radiative temperature gradient, the blob will rise. Only when the blob reaches an area of the star with difference physics (such that the temperature gradient becomes shallower than the adiabatic gradient) will it stop rising. The region in which

$$\left(\frac{dT}{dr}\right)_{ad} < \left(\frac{dT}{dr}\right)_{*} \tag{304}$$

defines the convective zone and the region in which convection dominates the energy transport.

Convection can then be favored in several ways. One way is through mak-

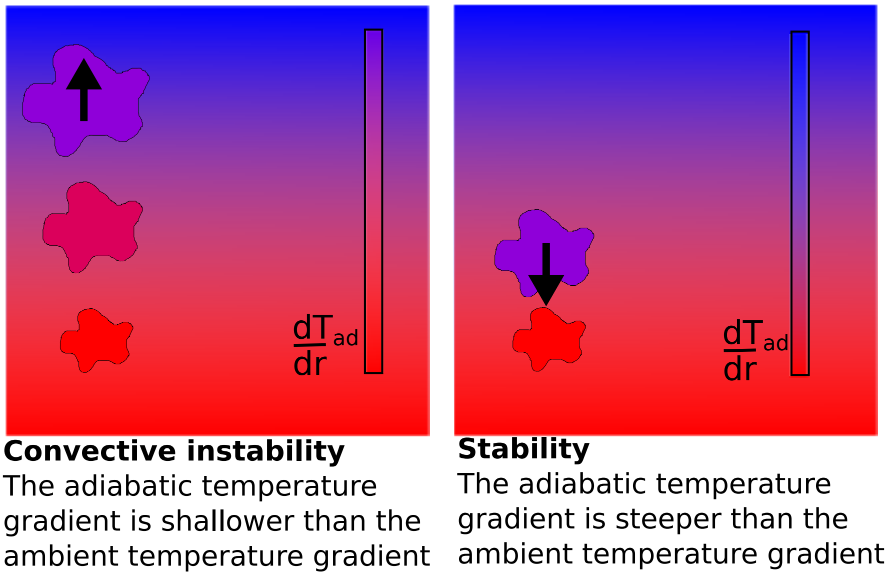

Figure 25: **Left**: An illustration of convective instability in a star. The ambient (background) temperature gradient, set by radiative transport of energy in the star, is steeper than the rate at which a parcel of gas can cool adiabatically. Because of this, a parcel of gas that becomes slightly warmer than the gas around it will rise uncontrollably, resulting in convection, which then is responsible for transporting energy in that region of the star. **Right**: An illustration of a stable situation. The ambient (background) temperature gradient, is shallower than the rate at which a parcel of gas can cool adiabatically. Because of this, a parcel of gas that becomes slightly warmer than the gas around it will very quickly become cooler than the gas around it, and will not rise significantly. In this situation, radiative energy transport dominates.

ing the adiabatic temperature gradient more shallow (this is set by the equation of state for the gas, and requires a deviation from the ideal gas law that lowers the adiabatic exponent). While this can and does occur, it is beyond the scope of this class, so we will not consider this in more detail. Alternatively then, we can ask what causes the temperature gradient of a star to steepen? Looking at Equation 303, we can see that the temperature gradient in a star is proportional to a number of variables, including the opacity κ and the energy flux L_r. Regions of high opacity are in fact a significant cause of convective zones in stars. As we saw in Section 18.1, many of the processes that cause opacity in stars favor conditions in which there are bound electrons. This will occur in cooler regions of a star, particularly in regions where the gas (Hydrogen or Helium) is only partially ionized. In fact, partially ionized gas also has a slightly lower adiabatic exponent than fully ionized gas, which further contributes to the development of convective instability. The sun's outer layers are convective for these reasons (its core is radiative, as this region is fully ionized). Cooler stars like red dwarfs are actually fully convective from their core to surface. As we will discuss further when we reach the topic of nuclear burning in stars, nuclear processes that release substantial amounts of energy can significantly increase L_r and thus also drive convection. This is the reason that stars more massive than the sun (which have a slightly different fusion reaction occurring) have convective cores.

Note that in defining convective instability we can also swap variables, and instead of temperature, consider the density and pressure of the blob (as both of these are connected to the temperature through our equation of state (the ideal gas equation, Equation 319, which is a good description of the conditions in the interior of stars). This approach is taken by Prialnik, and is the basis of the historical argument first made by Karl Schwarzschild in evaluating the convective stability of stars. Here, we assume that we have a blob that has pressure and density equal to the ambient values in the surrounding gas. When it is displaced upward, its pressure now exceeds the ambient pressure, and it expands adiabatically to reach pressure equilibrium with its surroundings. In expanding, not only has its pressure decreased, but its density as well. If the blob is now less dense than its surroundings, it will experiences a force that will displace it upward. However, if the blob remains more dense than its surroundings, it will instead experience a downward displacement force. This argument is in a sense more physical, as we are not appealing to 'warm air rising' but rather the underlying physical mechanism: the Archimedes buoyancy law. Using these variables, our condition for convective instability is now

$$\left(\frac{dP}{dr}\right)_{ad} < \left(\frac{dP}{dr}\right)_{*}. \tag{305}$$

All of this actually has an interesting application not just to stars, but to earth's atmosphere as well: the formation of thunderstorms! The formation of extremely tall (up to 12 miles!) clouds that lead to severe thunderstorms and tornadoes are driven by a convective instability in earth's lower atmosphere. The conditions that lead to this convective instability can be measured, and factor into forecasts of severe weather outbreaks. The criterion for convective instability is exactly the same as we just discussed for stars: the adiabatic temperature gradient, or the rate at which a parcel of gas displaced from ground level will cool as it rises, must be less than the temperature gradient (or profile) of the atmosphere:

$$\left(\frac{dT}{dr}\right)_{ad} < \left(\frac{dT}{dr}\right)_{atm} \tag{306}$$

The conditions that lead to thunderstorms have two things going on that make this more likely. First, weather systems that lead to thunderstorms are typically driven by the approach of a cold air mass (a cold front) that is pushing like a wedge into the upper atmosphere. This steepens the temperature gradient of the atmosphere: problem #1. The second thing that happens in advance of these weather systems is the buildup of a moist air mass in advance of the cold front, which drives high humidity. As you may have experienced firsthand (think of how quickly it gets cold at night in the desert, or alternatively, how warm a humid summer night can be, and how hard it is to stay cool on a humid day) air with a high moisture content is better at retaining heat, and thus cools more slowly. In essence, it has a shallower adiabatic temperature gradient: problem #2. Together, these two conditions are a recipe for strong convection. Humid air that is heated near the sunbaked ground will dramatically rise, unchecked, into the upper atmosphere, depositing energy and water vapor to make enormous, powerful cumulonimbus (thunderhead) clouds. The strength of the convection is measured by meteorologists with the CAPE (Convective Available Potential Energy) index. It measures this temperature differential, and uses it to determine how strong the upward buoyancy force will be. An extremely large CAPE for a given region could be a reason to issue a tornado watch.