11 Stellar Structure

Questions you should be able to answer after these lectures:

- What equations, variables, and physics describe the structure of a star?
- What are the two main types of pressure in a star, and when is each expected to dominate?
- What is an equation of state, and what is the equation of state that is valid for the sun?

11.1 *Formalism*

One of our goals in this class is to be able to describe not just the observable, exterior properties of a star, but to understand all the layers of these cosmic onions — from the observable properties of their outermost layers to the physics that occurs in their cores. This next part will then be a switch from some of what we have done before, where we have focused on the "surface" properties of a star (like size, total mass, and luminosity), and considered many of these to be fixed and unchanging. Our objective is to be able to describe the entire internal structure of a star in terms of its fundamental physical properties, and to model how this structure will change over time as it evolves.

Before we define the equations that do this, there are two points that may be useful to understand all of the notation being used here, and the way in which these equations are expressed.

First, when describing the evolution of a star with a set of equations, we will use mass as the fundamental variable rather than radius (as we have mostly been doing up until this point.) It is possible to change variables in this way because mass, like radius, increases monotonically as you go outward in a star from its center. We thus will set up our equations so that they follow individual, moving shells of mass in the star. There are several benefits to this. For one, it makes the problem of following the evolution of our star a more well-bounded problem. Over a star's lifetime, its radius can change by orders of magnitude from its starting value, and so a radial coordinate must always be defined with respect to the hugely time-varying outer extent of the star. In contrast, as our star ages, assuming its mass loss is insignificant, its mass coordinate will always lie between zero and its starting value M — a value which can generally be assumed to stay constant for most stars over most of stellar evolution. Further, by following shells of mass that do not cross over each other, we implicitly assume conservation of mass at a given time, and the mass enclosed by any of these moving shells will stay constant as the star evolves, even as the radius changes. This property also makes it easier to follow compositional changes in our star.

In general, the choice to follow individual fluid parcels rather than reference a fixed positional grid is known as adopting Lagrangian coordinates instead of Eulerian coordinates. For a **Lagrangian** formulation of a problem:

- This is a particle-based description, following individual particles in a fluid over time
- Conservation of mass and Newton's laws apply directly to each particle being followed
- However, following each individual particle can be computationally expensive
- This expense can be somewhat avoided for spherically-symmetric (and thus essentially '1D') problems

In contrast, for a **Eulerian** formulation of a problem:

- This is a field-based description, recording changes in properties at each point on a fixed positional grid in space over time
- The grid of coordinates is not distorted by the fluid motion
- Problems approached in this way are generally less computationally expensive, and are generally easier for 2D and 3D problems

There are thus tradeoffs for choosing each formulation. For stellar structure, Lagrangian coordinates are generally preferred, and we will rely heavily on equations expressed in terms of a stellar mass variable going forward.

Second, it might be useful to just recall the difference between the two types of derivatives that you may encounter in these equations. The first is a partial derivative, written as ∂f. The second is a total derivative, written as df. To illustrate the difference, let's assume that f is a function of a number of variables: $f(x,t)$. The partial derivative of f with respect to x is just $\frac{\partial f}{\partial x}$. Here, we have assumed in taking this derivative that x is held fixed with time and does not vary. However, most of the quantities that we will deal with in the equations of stellar structure *do* vary with time. The use of a partial derivative with respect to radius or mass indicates that we are considering the change in this space(like) coordinate for an instantaneous, fixed time value. In contrast, the total derivative does not hold any variables to be fixed, and considers how all of the dependent variables changes as a function of the variable considered. Note that when you see a quantity like $\dot{r}$ in an equation, this is actually the partial rather than total derivative with respect to time.

11.2 *Equations of Stellar Structure*

In this class, we will define four fundamental equations of stellar structure, and several additional relationships that, taken all together, will define the structure of a star and how it evolves with time. Depending on the textbook that you consult, you will find different versions of these equations using slightly different variables, or in a slightly different format.

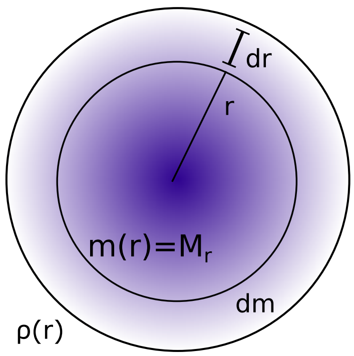

Figure 24: An illustration of a shell with mass dm and thickness dr. The mass enclosed inside of the shell is $m(r)$ (or M_r, depending on how you choose to write it). Assume that this object has a density structure $\rho(r)$

Mass continuity

The first two equations of stellar structure we have already seen before, as the conversion between the mass and radius coordinates

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \tag{227}$$

and as the equation of hydrostatic equilibrium (Eq. 192), now recast in terms of mass:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{228}$$

Eq. 227 and its variant forms are known variously as the **Mass Continuity Equation** or the **Equation of Conservation of Mass**. Either way, this is the first of our four fundamental equations of stellar structure, and relates our mass coordinate m to the radius coordinate r, as shown in Fig. 24.

Note that up until now we have been generally either been assuming a uniform constant density in all of the objects we have considered, or have been making approximations based on the average density $\langle\rho\rangle$. However, to better and more realistically describe stars we will want to use density distributions that are more realistic (e.g., reaching their highest value in the center of the star, and decreasing outward to zero at the edge of the star). This means we should start trying to think about ρ as a function rather than a constant (even when it is not explicitly written as $\rho(r)$ or $\rho(m)$ in the following equations).

Hydrostatic equilibrium

The second equation of stellar structure (Eq. 228, the equation of hydrostatic equilibrium) concerns the motion of a star, and we derived it in Sec. 10.2. As we noted earlier, stars can change their radii by orders of magnitude over the course of their evolution. As a result, we must consider how the interiors of stars move due the forces of pressure and gravity. We have already seen a specific case for this equation: the case in which gravity and pressure are balanced such that there is no net acceleration, and the star is in hydrostatic equilibrium (Equation 192).

We want to first consider a more general form of Eq. 228 that allows for the forces to be out of balance and thus there to be a net acceleration, and second to change variables from a dependence on radius to a dependence on mass. We can begin by rewriting our condition of force balance in Equation 192 as

$$0 = -\frac{Gm(r)}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}. \tag{229}$$

Each term in this equation has units of acceleration. Thus, this equation can be more generally written as

$$\ddot{r} = -\frac{Gm(r)}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}. \tag{230}$$

Using Equation **??** we can recast this expression in terms of a derivative with respect to m rather than r. This gives us the final form that we will use:

$$\ddot{r} = -\frac{Gm(r)}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}. \tag{231}$$

This is the most general form of our second equation of stellar structure. When $\ddot{r}$ is zero we are in equilibrium and so we obtain Eq. 228, the equation of **hydrostatic equilibrium**. This more general form, Eq. 231, is sometimes referred to as the **Equation of Motion** or the **Equation of Momentum Conservation**.

The Thermal Transport Equation

We also need to know how the temperature profile of a star changes with depth. If we do that, we can directly connect the inferred profile of temperature vs. optical depth (Eq. 164) to a physical coordinate within the star.

Assume there is a luminosity profile (determined by the energy equation, to be discussed next), such that the flux at radius r is

$$F(r) = \frac{L(r)}{4\pi 4^2} \tag{232}$$

In a plane-parallel atmosphere, we learned (Eq. 155) that the flux is related to the gradient of the radiation pressure. The assumptions we made then don't restrict the applicability of that relation only to the outer atmosphere, so we can apply it anywhere throughout the interior of our star. The only (minor) adjustment is that we replace dz with dr since we are now explicitly considering a spherical geometry, so we now have

$$F = -\frac{c}{\alpha}\frac{dP_{rad}}{dr} \tag{233}$$

Since we know that $P_{rad} = 4/3c\ \sigma_{SB} T^4$ (Eq. 257), we see that

$$\frac{dP_{rad}}{dr} = \frac{16\sigma_{SB}}{3c} T^3 \frac{dT}{dr}. \tag{234}$$

When combined with Eq. 232, we find the thermal profile equation,

$$\frac{dT}{dr} = -\frac{3\rho\kappa L(r)}{64\pi\sigma_{SB} T^3 r^2} \tag{235}$$

The Energy Equation

Eq. 235 shows that we need to know the luminosity profile in order to determine the thermal profile. In the outer photosphere we earlier required that flux is conserved (Sec. 9.2), but go far enough in and all stars (until the ends of their lives) are liberating extra energy via fusion.

Thus the next equation of stellar structure concerns the generation of energy within a star. As with the equation of motion, we will first begin with a simple case of equilibrium. In this case, we are concerned with the thermodynamics of the star: this is the equation for Thermal Equilibrium, or a constant flow of heat with time for a static star (a situation in which there is no work being done on any of our mass shells).

Consider the shell dm shown in Fig. 24. Inside of this shell we define a quantity ϵ_m that represents the net local gain gain of energy per time per unit mass (SI units of J s^{-1} kg^{-1}) due to local nuclear processes. Note that sometimes the volumetric power ϵ_r will also sometimes be used, but the power per unit mass ϵ_m is generally the more useful form. Regardless, we expect either ϵ to be very large deep in the stellar core and quickly go to zero in the outer layers where fusion is negligible – in those other regions, $\epsilon = 0$, L is constant, and we are back in the flux-conserving atmosphere of Sec. 9.2.

We then consider that the energy per time entering the shell is L_r (note that like M_r, this is now a local and internal rather than global or external property: it can be thought of as the luminosity of the star as measured at a radius r inside the star) and the energy per time that exits the shell is now $L_r + dL_r$ due to this local gain from nuclear burning in the shell. To conserve energy, we must then have (note that these are total rather than partial derivatives as there is no variation with time):

$$\frac{dL_r}{dm} = \epsilon_m. \tag{236}$$

This is the equation for **Thermal Equilibrium** in a star. While Thermal Equilibrium and Hydrostatic Equilibrium are separate conditions, it is generally unlikely that a star will be in Thermal Equilibrium without already being in Hydrostatic equilibrium, thus guaranteeing that there is no change in the energy flow in the star with time or with work being done. In general, Thermal Equilibrium and Eq. 236 require that any local energy losses in the shell (typically from energy propagating outward in the star) are exactly balanced by the rate of energy production in that shell due to nuclear burning. On a