This separation is pleasant because it means whenever we consider one timescale, we can assume that the faster processes are in equilibrium while the slower processes are static.

Much excitement ensues when this hierarchy breaks down. For example, we see convection occur on τ_{dyn} which then fundamentally changes the thermal transport. Or in the cores of stars near the end of their life, τ_{nuc} becomes much shorter. If it gets shorter than τ_{dyn}, then the star has no time to settle into equilibrium – it may collapse.

10.8 *The Virial Theorem*

In considering complex systems as a whole, it becomes easier to describe important properties of a system in equilibrium in terms of its energy balance rather than its force balance. For systems in equilibrium– not just a star now, or even particles in a gas, but systems as complicated as planets in orbit, or clusters of stars and galaxies– there is a fundamental relationship between the internal, kinetic energy of the system and its gravitational binding energy.

This relationship can be derived in a fairly complicated way by taking several time derivatives of the moment of inertia of a system, and applying the equations of motion and Newton's laws. We will skip this derivation, the result of which can be expressed as:

$$\frac{d^2 I}{dt^2} = 2\langle K \rangle + \langle U \rangle, \tag{208}$$

where $\langle K \rangle$ is the time-averaged kinetic energy, and $\langle U \rangle$ is the time-averaged gravitational potential energy. For a system in equilibrium, $\frac{d^2 I}{dt^2}$ is zero, yielding the form more traditionally used in astronomy:

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle \tag{209}$$

The relationship Eq. 209 is known as the Virial Theorem. It is a consequence of the more general fact that whenever $U \propto r^n$, we will have

$$\langle K \rangle = \frac{1}{n}\langle U \rangle \tag{210}$$

And so for gravity with $U \propto r^{-1}$, we have the Virial Theorem, Eq. 209.

When can the Virial Theorem be applied to a system? In general, the system must be in equilibrium (as stated before, this is satisfied by the second time derivative of the moment of inertia being equal to zero). Note that this is not necessarily equivalent to the system being stationary, as we are considering the time-averaged quantities $\langle K \rangle$ and $\langle U \rangle$. This allows us to apply the Virial Theorem to a broad diversity of systems in motion, from atoms swirling within a star to stars orbiting in a globular cluster, for example. The system also generally must be isolated. In the simplified form we are using, we don't consider so-called 'surface terms' due to an additional external pressure from a medium in which our system is embedded. We also assume that there are not any other sources of internal support against gravity in the system apart from the its internal, kinetic energy (there is no magnetic field in the source, or rotation). Below, we introduce some of the many ways we can apply this tool.

Virial Theorem applied to a Star

For stars, the Virial Theorem relates the internal (i.e. thermal) energy to the gravitational potential energy. We can begin with the equation of hydrostatic equilibrium, Eq. 192. We multiply both sides by $4\pi r^3$ and integrate as follows

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -\int_0^R \left(\frac{GM(r)}{r}\right)\left(4\pi r^2 \rho(r)\right) dr \tag{211}$$

The left-hand side can be integrated by parts,

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = 4\pi r^3 P|_0^R - 3\int_0^R P 4\pi r^2 dr \tag{212}$$

and since $r(0) =$ and $P(R) = 0$, the first term equals zero. We can deal with the second term by assuming that the star is an ideal gas, replacing $P = nkT$, and using the thermal energy density

$$u = \frac{3}{2}nkT = \frac{3}{2}P \tag{213}$$

This means that the left-hand side of Eq. 211 becomes

$$-2\int_0^R u(4\pi r^2 dr) = -2E_{th} \tag{214}$$

Where E_{th} is the total thermal energy of the star.

As for the right-hand side of Eq. 211, we can simplify it considerably by recalling that

$$\Phi_g = -\frac{GM(r)}{r} \tag{215}$$

and

$$dM = 4\pi r^2 \rho(r) dr. \tag{216}$$

Thus the right-hand side of Eq. 211 becomes simply

$$\int_0^R \Phi_g(M')dM' = E_{\rm grav} \tag{217}$$

And so merely from the assumptions of hydrostatic equilibrium and an ideal gas, it turns out that

$$(218)\quad E_{\text{grav}} = -2E_{th}$$

or alternatively,

$$(219)\quad E_{tot} = -E_{th} = E_{\text{grav}}/2$$

The consequence is that the total energy of the bound system is negative, and that it has negative heat capacity – a star heats up as it loses energy! Eq. 219 shows that if the star radiates a bit of energy so that E_{tot} decreases, E_{th} increases while E_{grav} decreases by even more. So energy was lost from the star, causing its thermal energy to increase while it also becomes more strongly gravitationally bound. This behavior shows up in all gravitational systems with a thermal description — from stars to globular clusters to Hawking radiation near a black hole to the gravitational collapse of a gas cloud into a star.

Virial Theorem applied to Gravitational Collapse

We can begin by restating the Virial Theorem in terms of the average total energy of a system $\langle E \rangle$:

$$(220)\quad \langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle$$

A classic application of this relationship is then to ask, if the sun were powered only by energy from its gravitational contraction, how long could it live? To answer this, we need to build an expression for the gravitational potential energy of a uniform sphere: our model for the gravitational potential felt at each point inside of the sun. We can begin to put this into equation form by considering what the gravitational potential is for an infinitesimally thin shell of mass at the surface of a uniformly-dense sphere.

Using dM as defined previously, the differential change in gravitational potential energy that this shell adds to the sun is

$$(221)\quad dU = -\frac{GM(r)dM}{r}.$$

The simplest form for $M(r)$ is to assume a constant density. In this case, we can define

$$(222)\quad M(r) = \frac{4}{3}\pi r^3 \rho$$

To determine the total gravitational potential from shells at all radii, we must integrate Equation 221 over the entire size of the sphere from 0 to R, substituting our expressions for dM and $M(r)$ from Equations **??** and 222:

$$U = -\frac{G(4\pi\rho)^2}{3}\int_0^R r^4 dr. \tag{223}$$

Note that if this were not a uniform sphere, we would have to also consider ρ as a function of radius: $\rho(r)$ and include it in our integral as well. That would be a more realistic situation for a star like our sun, but we will keep it simple for now.

Performing this integral, and replacing the average density ρ with the quantity $\frac{3M}{4\pi R^3}$, we then find

$$U = -\frac{G(3M)^2}{R^6}\frac{R^5}{5} = -\frac{3}{5}\frac{GM^2}{R} \tag{224}$$

which is the gravitational potential (or binding energy) of a uniform sphere. All together, this is equivalent to the energy it would take to disassemble this sphere, piece by piece, and move each piece out to a distance of infinity (at which point it would have zero potential energy and zero kinetic energy).

To understand how this relates to the energy available for an object like the sun to radiate as a function of its gravitational collapse, we have to perform one more trick, and that is to realize that Equation 220 doesn't just tell us about the average energy of a system, but how that energy has evolved. That is to say,

$$\Delta E = \frac{1}{2}\Delta U \tag{225}$$

So, the change in energy of our sun as it collapsed from an initial cloud to its current size is half of the binding energy that we just calculated. How does our star just lose half of its energy as it collapses, and where does it go? The Virial Theorem says that as a cloud collapses it turns half of its potential energy into kinetic energy (Equation 209). The other half then goes into terms that are not accounted for in the Virial Theorem: radiation, internal excitation of atoms and molecules and ionization (see the Saha Equation, Equation 124).

Making the simplistic assumption that all of the energy released by the collapse goes into radiation, then we can calculate the energy available purely from gravitational collapse and contraction to power the luminosity of the sun. Assuming that the initial radius of the cloud from which our sun formed is not infinity, but is still large enough that the initial gravitational potential energy is effectively zero, the energy which is radiated from the collapse is half the current gravitational potential energy of the sun, or

$$E_{radiated} = -\frac{3}{10}\frac{GM_\odot^2}{R_\odot} \tag{226}$$

Eq. 226 therefore links the Virial Theorem back to the Kelvin-Helmholtz timescale of Sec. 10.5. For the sun, this is a total radiated energy of $\sim 10^{41}$ J. If we assume that the sun radiates this energy at a rate equal to its current luminosity ($\sim 10^{26}$ W) then we can calculate that the sun could be powered at its current luminosity just by this collapse energy for 10^{15} s, or 3×10^{7} years. While this is a long time, it does not compare to our current best estimates for the age of the earth and sun: ~4.5 billion years. As an interesting historical footnote, it was Lord Kelvin who first did this calculation to estimate the age of the sun (back before we knew that the sun must be powered by nuclear fusion). He used this calculation to argue that the Earth must only be a few million years old, he attacked Charles Darwin's estimate of hundreds of millions of years for the age of the earth, and he argued that the theory of evolution and natural selection must be bunk. In the end of course, history has shown who was actually correct on this point.