10 Timescales in Stellar Interiors

Having dealt with the stellar photosphere and the radiation transport so relevant to our observations of this region, we're now ready to journey deeper into the inner layers of our stellar onion. Fundamentally, the aim we will develop in the coming chapters is to develop a connection between M, R, L, and T in stars (see Table 10 for some relevant scales).

More specifically, our goal will be to develop equilibrium models that describe stellar structure: $P(r)$, $\rho(r)$, and $T(r)$. We will have to model gravity, pressure balance, energy transport, and energy generation to get everything right. We will follow a fairly simple path, assuming spherical symmetric throughout and ignoring effects due to rotation, magnetic fields, etc.

Before laying out the equations, let's first think about some key timescales. By quantifying these timescales and assuming stars are in at least short-term equilibrium, we will be better-equipped to understand the relevant processes and to identify just what stellar equilibrium means.

10.1 *Photon collisions with matter*

This sets the timescale for radiation and matter to reach equilibrium. It depends on the **mean free path** of photons through the gas,

(180) $$\ell = \frac{1}{n\sigma}$$

So by dimensional analysis,

(181) $$\tau_\gamma \approx \frac{\ell}{c}$$

If we use numbers roughly appropriate for the average Sun (assuming full

Table 3: Relevant stellar quantitites.

Quantity	Value in Sun	Range in other stars
M	2×10^{33} g	$0.08 \lesssim (M/M_\odot) \lesssim 100$
R	7×10^{10} cm	$0.08 \lesssim (R/R_\odot) \lesssim 1000$
L	4×10^{33} erg s^{-1}	$10^{-3} \lesssim (L/L_\odot) \lesssim 10^6$
$T_{\rm eff}$	5777 K	3000 K $\lesssim (T_{\rm eff}/mathrmK) \lesssim$ 50,000 K
ρ_c	150 g cm^{-3}	$10 \lesssim (\rho_c/\text{g cm}^{-3}) \lesssim 1000$
T_c	1.5×10^7 K	$10^6 \lesssim (T_c/\text{K}) \lesssim 10^8$

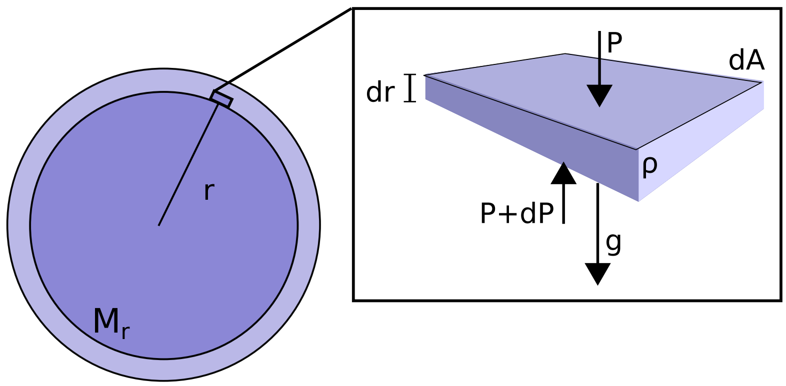

Figure 22: The state of hydrostatic equilibrium in an object like a star occurs when the inward force of gravity is balanced by an outward pressure gradient. This figure illustrates that balance for a packet of gas inside of a star

ionization, and thus Thomson scattering), we have

(182)
$$\ell = \frac{1}{n\sigma}$$
(183)
$$= \frac{m_p}{\rho\sigma_T}$$
(184)
$$= \frac{1.7 \times 10^{-24}\text{ g}}{(1.4\text{ g cm}^{-3})(2/3 \times 10^{-24}\text{ cm}^{-2})}$$
(185)
$$\approx 2\text{ cm}$$

So the matter-radiation equilibration timescale is roughly $\tau_\gamma \approx 10^{-10}$ s. Pretty fast!

10.2 *Gravity and the free-fall timescale*

For stars like the sun not to be either collapsing inward due to gravity or expanding outward due to their gas pressure, these two forces must be in balance. This condition is known as hydrostatic equilibrium. This balance is illustrated in Figure 22

As we will see, gravity sets the timescale for fluid to come into mechanical equilibrium. When we consider the balance between pressure and gravity on a small bit of the stellar atmosphere with volume $V = Adr$ (sketched in Fig. 22), we see that in equilibrium the vertical forces must cancel.

The small volume element has mass dm and so will feel a gravitational force equal to

(186) $F_g = \frac{GM_r\, dm}{r^2}$

where M_r is the mass of the star enclosed within a radius r,

(187) $M(r) \equiv 4\pi \int_{r'=0}^{r'=r} \rho(r') r'^2 dr'$

Assuming the volume element has a thickness dr and area dA, and the star has a uniform density ρ, then we can replace dm with $\rho dr dA$. This volume element will also feel a mean pressure which we can define as dP, where the pressure on the outward facing surface of this element is P and the pressure on the inward facing surface of this element is $P + dP$. The net pressure force is then $dPdA$, so

(188)
$$F_P(r) = F_g(r)$$
(189)
$$A\left(P(r) - P(r+dr)\right) = -\rho V g$$
(190)
$$= \rho A dr g$$
(191)

which yields the classic expression for **hydrostatic equilibrium**,

(192) $\frac{dP}{dr} = \rho(r) g(r)$

where

(193) $g \equiv -\frac{GM(r)}{r^2}$

and $M(r)$ is defined as above.

When applying Eq. 192 to stellar interiors, it's common to recast it as

(194) $\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$

In Eqs. 192 and 194 the left hand side is the pressure gradient across our volume element, and the right hand side is the gravitational force averaged over that same volume element. So it's not that pressure balances gravity in a star, but rather gravity is balanced by the gradient of increasing pressure from the center to the surface.

The gradient dP/dr describes the pressure profile of the stellar interior in equilibrium. What if the pressure changes suddenly – how long does it take

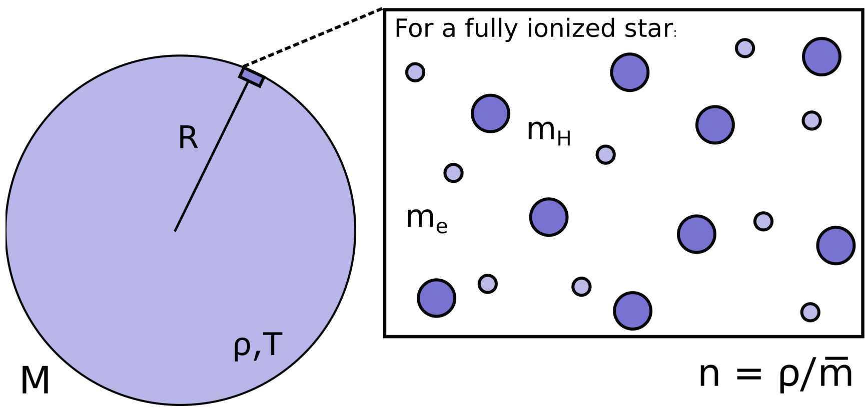

Figure 23: A simple model of a star having a radius R, mass M, constant density ρ, a constant temperature T, and a fully ionized interior. This simple model can be used to derive a typical free-fall time and a typical sound-crossing time for the sun.

us to re-establish equilibrium? Or equivalently: if nothing were holding up a star, how long would it take to collapse under its own gravity? Looking at Figure 29, we can model this as the time it would take for a parcel of gas on the surface of a star, at radius R, to travel to its center, due to the gravitational acceleration from a mass M.

Looking at Figure 29, we can model this as the time it would take for a parcel of gas on the surface of a star, at radius R, to travel to its center, due to the gravitational acceleration from a mass M. To order of magnitude, we can combine the following two equations

$$a = -\frac{GM}{r^2} \tag{195}$$

and

$$d = -\frac{1}{2}at^2. \tag{196}$$

Setting both r and d equal to the radius of our object R, and assuming a constant density $\rho = \frac{3M}{4\pi R^3}$, we find

$$\tau_{ff} \sim \frac{1}{\sqrt{G\rho}} \tag{197}$$

which is within a factor of two of the exact solution,

$$\tau_{ff} = \sqrt{\frac{3\pi}{32G\rho}} \tag{198}$$

Note that the free-fall timescale does not directly depend on the mass of an object or its radius (or in fact, the distance from the center of that object). It only depends on the density. Since $G \approx 2/3 \times 10^{-7}$ (cgs units), with $\langle \rho_\odot \rangle \approx 1$ g cm^{-3}, the average value is $\tau_{dyn} \sim 30$ min.

In real life, main-sequence stars like the sun are stable and long-lived structures that are not collapsing. Even if you have a cloud of gas that is collapsing under its own gravity to form a star, it does not collapse all the way to $R = 0$ thanks to its internal hydrostatic pressure gradient.

10.3 *The sound-crossing time*

We have an expression for the time scale upon which gravity will attempt to force changes on a system (such changes can either be collapse, if a system is far out of hydrostatic equilibrium and gravity is not significantly opposed by pressure, or contraction, if a system is more evenly balanced). What is the corresponding time scale upon which pressure will attempt to cause a system to expand?

The pressure time scale in a system can be characterized using the sound speed (as sound is equivalent to pressure waves in a medium). This isothermal sound speed is given by the relation

$$c_s = \sqrt{\frac{P}{\rho}} \tag{199}$$

Although gas clouds in the interstellar medium may be reasonably approximated as isothermal, the same is not true for stars. We will ignore that fact for now, but will return to this point later.

Referring back to Figure 29, we can define the sound-crossing time for an object as the time it takes for a sound wave to cross the object. Using a simple equation of motion $d = vt$ and approximating $2R$ just as R we can then define a sound-crossing time as

$$\tau_s \sim R\sqrt{\frac{\rho}{P}} \tag{200}$$

Using the ideal gas equation, we can substitute $\frac{\rho}{\bar{m}}kT$ for P and get an expression for the sound crossing time in terms of more fundamental parameters for an object:

$$\tau_s \sim R\sqrt{\frac{\bar{m}}{kT}} \tag{201}$$

Unlike the free-fall time we derived earlier, the sound-crossing time depends directly upon the size of the object, and its temperature. At the center of the Sun, $T_c \approx 1.5 \times 10^7$ K and $\bar{m} \sim m_p$ and so the sound-crossing timescale is roughly 30 min.

Note that by Eq. 198 we see that τ_s is also approximately equal to the free-fall timescale τ_{ff}. For an object not just to be in hydrostatic equilibrium but to remain this way, the pressure must be able to respond to changes in gravity, and vice versa. This response requires that a change in one force is met with a change in another force on a timescale that is sufficiently fast to restore the force balance. In practice, this means that for objects in hydrostatic equilibrium, the free-fall time is more or less equivalent to the sound-crossing time. In that way, a perturbation in pressure or density can be met with a corresponding response before the object moves significantly out of equilibrium.

10.4 *Radiation transport*

If photons streamed freely through a star, they'd zip without interruption from the core to the stellar surface in $R_\odot/c \approx 2$ s. But as we saw above in Eq. 185, the photons actually scatter every ∼1 cm. With each collision they "forget" their history, so the motion is a random walk with N steps. So for a single photon[1] to reach the surface from the core requires

$$(202)\quad \ell\sqrt{N} \sim R_\odot$$

which implies that the **photon diffusion timescale** is

$$(203)\quad \tau_{\gamma,\mathrm{diff}} \sim \frac{N\ell}{c} \sim \frac{R_\odot^2}{\ell}\frac{1}{c}$$

or roughly 10^4 yr.

10.5 *Thermal (Kelvin-Helmholtz) timescale*

The thermal timescale answers the question, How long will it take to radiate away an object's gravitational binding energy? This timescale also governs the contraction of stars and brown dwarfs (and gas giant planets) by specifying the time it takes for the object to radiate away a significant amount of its gravitational potential energy. This is determined by the **Kelvin-Helmholtz timescale**. This thermal time scale can generally be given as:

$$(204)\quad \tau_{KH} = \frac{E}{L},$$

where E is the gravitational potential energy released in the contraction to its final radius and L is the luminosity of the source. Approximating the Sun as a uniform sphere, we have

$$(205)\quad \tau_{KH} \sim \frac{GM_\odot^2}{R_\odot}\frac{1}{L_\odot}$$

[1]This is rather poetic – of course a given photon doesn't survive to reach the surface, but is absorbed and re-radiated as a new photon $\sim (R_\odot/\ell)^2$ times. Because of this, it may be better to think of the timescale of Eq. 203 as the **radiative energy transport timescale**.

which is roughly 3×10^7 yr.

Before nuclear processes were known, the Kelvin-Helmholtz timescale was invoked to argue that the Sun could be only a few 10^7 yr old – and therefore much of geology and evolutionary biology (read: Darwin) must be wrong. There turned out to be missing physics, but τ_{KH} turns out to still be important when describing the contraction of large gas clouds as they form new, young stars.

The time that a protostar spends contracting depends upon its mass, as its radius slowly contracts. A 0.1 $M_\odot$ star can take 100 million years on the Hayashi track to finish contracting and reach the main sequence. On the other hand, a 1 $M_\odot$ star can take only a few million years contracting on the Hayashi track before it develops a radiative core, and then spends up to a few tens of millions of years on the Henyey track before reaching the main sequence and nuclear burning equilibrium. The most massive stars, 10 $M_\odot$ and above, take less than 100,000 years to evolve to the main sequence.

10.6 *Nuclear timescale*

The time that a star spends on the main sequence – essentially the duration of the star's nuclear fuel under a constant burn rate – is termed the **the nuclear timescale**. It is a function of stellar mass and luminosity, essentially analogous to the thermal time scale of Equation 298. Here, the mass available (technically, the mass difference between the reactants and product of the nuclear reaction) serves as the energy available, according to $E = mc^2$.

If we fuse 4 protons to form one He^4 nucleus (an **alpha particle**), then the fractional energy change is

$$\frac{\Delta E}{E} = \frac{4m_p c^2 - m_{He} c^2}{4m_p c^2} \approx 0.007 \tag{206}$$

This is a handy rule of thumb: fusing H to He liberates roughly 0.7% of the available mass energy. As we will see, in more massive stars heavier elements can also fuse; further rules of thumb are that fusing He to C and then C to Fe (through multiple intermediate steps) each liberates another 0.1% of mass energy. But for a solar-mass star, the main-sequence nuclear timescale is

$$\tau_{nuc} \approx \frac{0.007 M_\odot c^2}{L_\odot} \approx 10^{11} \text{ yr} \tag{207}$$

which implies a main-sequence lifetime of roughly 100 billion years. The actual main-sequence lifetime for a $1M_\odot$ star is closer to 10 billion years; it turns out that significant stellar evolution typically occurs by the time ∼10% of a star's mass has been processed by fusion.

10.7 *A Hierarchy of Timescales*

So if we arrange our timescales, we find a strong separation of scales:

$$\begin{array}{ccccccccc} \tau_{nuc} & \gg & \tau_{KH} & \gg & \tau_{\gamma,diff} & \gg & \tau_{dyn} & \gg & \tau_\gamma \\ 10^{11}\text{ yr} & \gg & 3\times10^7\text{ yr} & \gg & 10^4\text{ yr} & \gg & 30\text{ min} & \gg & 10^{-10}\text{ s} \end{array}$$