5 Gravitational Waves

A subset of binary objects can be studied in an entirely different way than astrometry, spectroscopy, and eclipses: this is through **gravitational waves,** undulations in the fabric of spacetime itself caused by rapidly-orbiting, massive objects. For our description of that, I follow Choudhuri's textbook, parts of chapters 12 and 13. Note that in much of what follows, we skip details about a number of different factors (e.g. "projection tensors") that introduce angular dependencies, and enforce certain rules that radiation must obey. For a detailed treatment of all this, consult a modern gravitational wave textbook (even Choudhuri doesn't cover everything that follows, below).

Recall that in relativity we describe spacetime through the four-vector

$$x^i = (x^0, x^1, x^2, x^3) \tag{11}$$
$$= (ct, x, y, z) \tag{12}$$

(note that those are indices, not exponents!). The *special* relativistic metric that describes the geometry of spacetime is

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{13}$$
$$= \eta_{ik} dx^i dx^k \tag{14}$$

But this is only appropriate for special (not general) relativity – and we definitely need GR to treat accelerating, inspiraling compact objects. For 8.901, we'll assume weak gravity and an only slightly modified form of gravity; "first-order general relativity." Then our new metric is

$$g_{ik} = \eta_{ik} + h_{ik} \tag{15}$$

where it's still true that $ds^2 = g_{ik} dx^i dx^k$, and h_{ik} is the GR perturbation. For ease of computation (see the textbook) we introduce a modified definition,

$$\bar{h}_{ik} = h_{ik} - \frac{1}{2}\eta_{ik} h \tag{16}$$

where here h is the trace (the sum of the elements on the main diagonal) of h_{ik}.

Now recall that Newtonian gravity gives rise to the gravitational Posson equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{17}$$

– this is the equivalent of Gauss' Law in electromagnetism. The GR equations above then lead to an equivalent expression in GR – the **inhomogeneous wave equation**,

$$\Box^2 \bar{h}_{ik} = -\frac{16\pi G}{c^4} T_{ik} \tag{18}$$

where $\Box^2 = -1\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ is the 4D differential operator and T_{ik} is the energy-momentum tensor, describing the distribution of energy and momentum in spacetime. This tensor is a key part of the **Einstein Equation** that describes how mass-energy leads to the curvature of spacetime, which unfortunately we don't have time to fully cover in 8.901.

One can solve Eq. 18 using the Green's function treatment found in almost all textbooks on electromagnetism. The solution is that

$$(19)\quad \bar{h}_{ik}(t,\vec{r}) = \frac{4G}{c^4}\int\limits_S \frac{T_{ik}(t - |\vec{r}-\vec{r}'|/c, \vec{r}')}{|\vec{r}-\vec{r}'|} d^3r'$$

(where $t_r = t - |\vec{r}-\vec{r}'|/c$ is the 'Retarded Time'; see Fig. 3 for the relevant geometry). This result implies that the effects of gravitation propagate outwards at speed c, just as do the effects of electromagnetism.

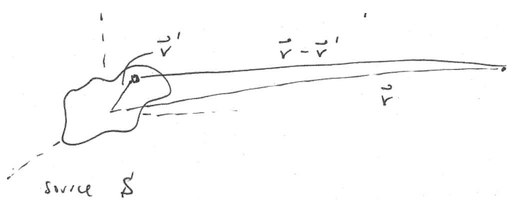

Figure 3: General geometry for Eq. 19.

We can simplify Eq. 19 in several ways. First, assuming than an observer is very far from the source implies $|\vec{r}| >> |\vec{r}'|$ for all points in the source S. Therefore,

$$(20)\quad \frac{1}{|\vec{r}-\vec{r}'|} \approx \frac{1}{r}$$

If the mass distribution (the source S) is also relatively small, then

$$(21)\quad t - |\vec{r}-\vec{r}'|/c \approx t - r/c$$

. Finally, general relativity tells us that the timelike components of h_{ik} do not radiate (see GR texts) – so we can neglect them in the analysis that follows.

Putting all this together, we have a simplified solution of Eq. 19, namely

$$(22)\quad \bar{h}_{ik} = \frac{4G}{c^4 r}\int\limits_S T_{ik}(\vec{r}', t_R) d^3r'$$

We can combine this with one more trick. The properties of the stress-energy tensor (see text, again) turn out to prove that

$$(23)\quad \int\limits_S T_{ik}(\vec{r}')d^3r' = \frac{1}{2}\frac{d^2}{dt^2}\int\limits_S T_{00}(\vec{r}') \cdot x_i' x_k' d^3r'$$

This is possibly the greatest help of all, since in the limit of a weak-gravity source $T_{00} = \rho c^2$, where ρ is the combined density of mass and energy.

If we then define the **quadrupole moment tensor** as

$$(24) \quad I_{ik} = \int_S \rho(\vec{r}')x'_i x'_j d^3r'$$

then we have as a result

$$(25) \quad h_{ik} = \frac{2G}{c^4 r}\frac{d^2}{dt^2}I_{ik}$$

which is the **quadrupole formula** for the gravitational wave amplitude.

What is this quadrupole moment tensor, I_{ik}? We can use it when we treat a binary's motion as approximately Newtonian, and then use I_{ik} to infer how gravitational wave emission causes the orbit to change. If we have a circular binary orbiting in the xy plane, with separation r and $m1$ at $(x > 0, y = 0)$. In the reduced description, we have a separation r, total mass M, reduced mass $\mu = m_1 m_2/M$, and orbital frequency $\Omega = \sqrt{GM/r^3}$. This means that the binary's position in space is

$$(26) \quad x_i = r(\cos\Omega t, \sin\Omega t, 0)$$

Treating the masses as point particles, we have $\rho = \rho_1 + \rho_2$ where

$$(27) \quad \rho_n = \delta(x - x_n)\delta(y - y_n)\delta(z)$$

so the moment tensor becomes simply $I_{ik} = \mu x_i x_k$, or

$$(28) \quad I_{ik} = \mu r^2 \begin{bmatrix} \cos^2\Omega t & \sin\Omega t\cos\Omega t & 0 \\ \sin\Omega t\cos\Omega t & \sin^2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As we will see below, this result implies that gravitational waves are emitted at twice the orbital frequency.

5.1 *Gravitational Radiation*

Why gravitational *waves*? Eq. 18 above implies that in empty space, we must have simply

$$(29) \quad \Box^2 \bar{h}_{lm} = \Box^2 h_{lm} = 0$$

This implies the existence of the aforementioned propagating gravitational waves, in an analogous fashion to the implication of Maxwell's Equations for traveling electromagnetic waves. In particular, if we define the wave to be traveling in the x^3 direction then a plane gravitational wave has the form

$$(30) \quad h_{lm} = A_{lm}e^{ik(ct-x^3)}$$

(where i and k now have their usual wave meanings, rather than referring to indices). It turns out that the A_{lm} tensor can be written as simply

$$(31)\quad A_{lm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The implication of just two variables in A_{lm} is that gravitational waves have just two polarizations, "+" and "×". This is why each LIGO and VIRGO detector needs just two arms – one per polarization mode.

Just like EM waves, GW also carry energy. The **Isaacson Tensor** forms part of the expression describing how much energy is being carried , namely:

$$(32)\quad \frac{dE}{dAdt} = \frac{1}{32\pi} c^3/G < \dot{h}_{ij}\dot{h}_{ij} >$$

This is meaningful only on distance scales of at least one wavelength, and when integrated over a large sphere (and accounting for better-unmentioned terms like the projection tensors), we have

$$(33)\quad \frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5} < \dddot{I}_{ij}\dddot{I}_{ij} >$$

which is the **quadrupole formula** for the energy carried by gravitational waves.

5.2 *Practical Effects*

In practice, this means that the energy flux carried by a gravitational wave of frequency f and amplitude h is

$$(34)\quad F_{gw} = 3 \text{ mW m}^{-2} \left(\frac{h}{10^{-22}}\right)^2 \left(\frac{f}{1 \text{ kHz}}\right)^2$$

In contrast, the Solar Constant is about 1.4×10^6 mW m^{-2}. But the full moon is $\sim 10^6\times$ fainter than the sun, and gravitational waves carry an energy comparable to that!

For a single gravitational wave event of duration τ, the observed "strain" (amplitude) h scales approximately as:

$$(35)\quad h = 10^{-21} \left(\frac{E_{GW}}{0.01 M_\odot c^2}\right)^{1/2} \left(\frac{r}{20 \text{ Mpc}}\right)^{-1} \left(\frac{f}{1 \text{ kHz}}\right)^{-1} \left(\frac{\tau}{1 \text{ ms}}\right)^{-1/2}$$

With today's LIGO strain sensitivity of $< 10^{-22}$, this means they should be sensitive to events out to at least the Virgo cluster (or further for stronger signals).

And as a final aside, note that the first detection of the presence of gravitational waves came not from LIGO but from observations of binary neutron stars. As the two massive objects rapidly orbit each other, gravitational waves steadily sap energy from the system, causing the orbits to steadily decay. When at least one of the neutron stars is a pulsar, this orbital decay can be measured to high precision.