Lecture 3

- So, we’ve reduced the 2-body problem to a one-body problem. Next, we reduce the dimensionality:
  - We have a central force, so \( \vec{F} \propto \vec{r} \)
    Thus we have no torque, since \( \vec{\tau} = \vec{r} \times \vec{F} = 0 \)
    Thus angular momentum is conserved in the 2-body system.
  - That angular momentum is always perpendicular to the orbital plane, since
    \[ \vec{L} \cdot \vec{r} = (\vec{r} \times \mu \vec{v}) \cdot \vec{r} \]
    \[ = (\vec{r} \times \vec{r}) \cdot \mu \vec{v} = 0 \]
  - Since the orbit is always in a single, constant plane we can just describe it using 2D polar coordinates, \( r \) and \( \phi \)
    Thus we have \( \vec{L} = \vec{r} \times \mu \vec{v} \) (by definition of \( L \))
    \[ = \mu r \dot{v} = \mu r^2 \dot{\phi} = \text{constant} \]
    \( \rightarrow \) Equal area law (Kepler’s 2nd) follows – true for any central force (not just \( 1/r^2 \))
  - Next, we go from 2D to 1D:
    - \( E = \frac{1}{2} \mu \vec{v} \cdot \vec{v} + V(r) \)
      \[ = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu r^2 \dot{\phi}^2 + V(r) \]
    - \( L = \mu r^2 \dot{\phi} \) (from above), and so \( \dot{\phi}^2 = \frac{L^2}{\mu^2 r^4} \)
    - and so \( E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2 \mu r^2} + V(r) \). We call those last two terms \( V_{\text{eff}} \).
    - The formal solution to solve for the orbital motion is:
      \[ dt = \frac{dr}{\sqrt{\frac{2}{\mu} [E - V_{\text{eff}}(r)]}} \]
    - For any given potential, one can integrate to get \( t(r) \) and then invert to find \( r(t) \).
      Usually one gets nasty-looking Elliptic integrals for a polynomial potential.
Get more insight from graphical analysis.

- Plot $V_{\text{eff}}$, and then the total system $E$ on the same graph. Given $L$ & $E$:
  - Must have $V_{\text{eff}} < E$ (otherwise $v^2 < 0$)
  - Motion shows a turning point whenever $V_{\text{eff}} = E$.

- For different energies plotted:
  - $E_1$: unbound orbit. Hyperbolic – interstellar comets!
  - $E_2$, $E_3$: bound, eccentric orbits (outer (apastron) and inner (periastron) points)
  - $E_4$: circular orbit (single radius).
  - $E < E_4$: not allowed!
Let’s look at this motion in the plane (for the bound case):

We see that the possible paths will fill in the regions between an inner and an outer radius \((r_1 \text{ and } r_2)\). But there’s no guarantee that the orbits actually repeat periodically.

We get periodic orbits, and closed ellipses, for two special cases:

\- \(V(r) \propto \frac{1}{r}\) (Keplerian motion)
\- \(V(r) \propto r^2\) (simple harmonic oscillator)
\- “Bertrand’s Theorem” says that these are the only two closed-orbit forms.

These closed-form cases are also special because they have an “extra” conserved quantity.

\- Consider gravity: \(V(r) = -\frac{G \mu M}{r}\) where \(M = m_1 + m_2\)

\- Define the “Laplace-Runge-Lenz” (LRL) vector,
\[ \vec{A} = \vec{p} \times \vec{L} - G M \mu^2 \vec{r} \]
\(\vec{A}\) is conserved! Describes shape & orientation of orbit

\[ \frac{d\vec{A}}{dt} = \frac{d\vec{p}}{dt} \times \vec{L} + \vec{p} \times \frac{d\vec{L}}{dt} - G M \mu^2 \frac{d\vec{r}}{dt} \]
(Second term goes to zero; \(\vec{L}\) conserved!)

\[ \frac{d\vec{p}}{dt} = -G \frac{\mu M}{r^2} \vec{r} \]

\[ \frac{d\vec{r}}{dt} = \frac{d\varphi}{dt} \vec{\hat{\varphi}} \]

\[ \vec{L} = \mu r^2 \vec{\hat{\varphi}} \vec{\hat{z}} \]

\[ \frac{d\vec{A}}{dt} = \left( -G \frac{\mu M}{r^2} \vec{r} \right) \times \left( \mu r^2 \vec{\hat{\varphi}} \vec{\hat{z}} \right) - G M \mu^2 \vec{\hat{\varphi}} \vec{\hat{\varphi}} \]
, which gives

\[ \frac{d\vec{A}}{dt} = +G \mu^2 \vec{\hat{\varphi}} \vec{\hat{\varphi}} - G M \mu^2 \vec{\hat{\varphi}} \vec{\hat{\varphi}} = 0 \]
\(\vec{A}\) is a conserved quantity!

But, what does the LRL vector mean?
\- It describes the elliptical equations of motion!
• A points in the orbital plane. Define it to point along the x-axis of our polar system:
\[ \vec{r} \cdot \vec{A} = r \cos \varphi = (\vec{r} \cdot (\vec{p} \times \vec{L})) - GM \mu^2 r \]
\[ \rightarrow \quad r A \cos \varphi = L^2 - GM \mu_2^2 r \]

We can solve this for \( r \):
\[ r(\varphi) = \frac{L^2 / GM \mu^2}{1 + (A/GM \mu^2) \cos \varphi} \]
and this is just the **equation of an ellipse** that we saw in Lecture 2, with
\[ e = \frac{A}{GM \mu^2} \quad \text{and} \quad L = \sqrt{GM \mu^2 a(1-e^2)} \]

• We defined \( A \) to point along the x-axis (\( \varphi = 0 \)). This is the same direction where \( r \) is minimized – so \( A \) (the LRL) points toward the closest approach in the orbit (“pericenter”).

### One remaining law: Kepler’s 3rd Law

- Consider the area of a curve in polar coordinates.
\[ d \text{Area} = \frac{1}{2} r^2 d \varphi \]  so
\[ \frac{d \text{Area}}{dt} = \frac{1}{2} r^2 \dot{\varphi} = \frac{1}{2} \frac{L}{\mu} = \text{constant} \]

If we integrate over a full period, we get the area of an ellipse:
\[ A_{\text{ellipse}} = \int_0^P d \text{Area} = \frac{1}{2} \int_0^P \frac{L}{\mu} dt = \frac{LP}{2\mu} . \]

And from geometry, \( A_{\text{ellipse}} = \pi ab \) (where \( b = a \sqrt{1-e^2} \) is the semiminor axis)

So set these equal:
\[ \frac{LP}{2\mu} = \pi a^2 \sqrt{1-e^2} . \]  Plugging the previous expression for \( L \) in:
\[ \frac{1}{2} \sqrt{GM \mu^2 a(1-e^2)} P = \pi a^2 \sqrt{1-e^2} , \]  which simplifies to
\[ \sqrt{\frac{GM}{a^3}} = \frac{2\pi}{P} = \Omega_{\text{Kepler}} \]

Rearranging to the more familiar form, we find:
\[ P^2 = \left( \frac{4\pi^2}{GM} \right) a^3 . \]  Or in Solar units,
\[ \left( \frac{P}{1 \text{yr}} \right)^2 = \left( \frac{M}{M_{\text{sun}}} \right)^{-1} \left( \frac{a}{1 \text{AU}} \right)^3 \]
- Other interesting bits and bobs:
  - A useful exercise for the reader is to show that \[ E = -\frac{G M \mu}{2a} \]
    (use \( rdot = 0 \) at pericenter, \( r = a (1 - e), \phi=0 \))
  - We have \( r(\phi) \) --- what about \( r(t) \) and \( \phi(t) \)?
    - Unfortunately there’s no general, closed-form solution – this is typically calculated iteratively using a numerical framework.
    - One can find parametric solutions (see Psets)
  - The position vector moves on an ellipse, but you can show that the velocity vector actually moves on a circle:

![Diagram](image)

- Really esoteric: all these conservation laws are tied to particular symmetries:
  - Energy conservation comes from time translation
  - Angular momentum conservation comes from SO(3) rotations
  - The RLR vector \( A \) is conserved because of rotations in 4D (!!). \( r \) & \( p \) map onto the 3D surface of a 4D Euclidean sphere. Cool – but not too useful.