Lecture 2

• Space is Big.
• Last time we talked about mass scales; today we’ll talk about size scales:
  ◦ Bohr radius $\rightarrow \frac{p^2/2m}{\hbar^2/2m} \sim \frac{\epsilon^2/a}{\epsilon^2/a} \rightarrow 5.3e-9$ cm
  ◦ $R_{\text{Earth}} = 6.3e8$ cm $= (20000 / \pi)$ km
  ◦ $R_{\text{Sun}} = 7e10$ cm
  ◦ $1$ AU $= 1.5e13$ cm $\sim 8$ light-seconds
  ◦ $1$ parsec $= 1$ pc $= 3e18$ cm $= 3.26$ light years (note that l.y. are ~never used in astrophysics)

• Parsec is the fundamental unit of distance; it is ~the typical distance between stars (though that’s just a coincidence). It is observationally defined:

  ◦ Over one year, the Earth’s displacement is $2$ AU and an object at distance $d$ changes apparent position by $2\theta$, where $\tan \theta = 1$ AU / $d$, or $d = 1$ AU / $\theta$, or $d / 1$ pc $= 1$ arcsec / $\theta$
  ◦ Nearest star: $1.3$ pc
    To our galactic center: $8$ kpc (kiloparsecs)
    To the nearest big galaxy: $620$ kpc !!!!

• Cosmic Distance Ladder:
  ◦ Distance is a key concept in astrophysics – e.g. the revolution currently underway thanks to ESA’s Gaia mission (measuring parallax for billions of objects with sub-milliarcsec precision)
  ◦ “Distance ladder” refers to the bootstrapping of distance measurements, from nearby stars to the furthest edges of the observable universe.
  ◦ Within solar system: light travel time. Radar, spacecraft communication, etc.
  ◦ Parallax: the first rung outside the Solar system. Measured by Gaia (2nd data release) for ~1 billion stars across ~half of the Galaxy. A revolution is underway!
Standard candles: if Luminosity $L$ is known, then observed flux $F$ gives distance:

$$F = \frac{L}{4\pi d^2} \quad \rightarrow \quad d = \left(\frac{L}{4\pi F}\right)^{1/2}$$

Most important types:
- Cepheid variables - giant pulsating stars, period varies with absolute magnitude
- Type Ia Supernovae – exploding stars (probably white dwarf)
- Neither of these are truly standard – only “standardizable” (which is almost as good)
- Other types (for other galaxies):
  - Tully-Fischer: $L \sim V^2$ (rotational velocity of spiral galaxies)
  - Faber-Jackson: $L \sim \sigma^2$ (velocity dispersion of elliptical galaxies)

Hubble’s Law – for very distant Galaxies.
- The universe is expanding at a nearly-constant rate (more on that in 8.902). Roughly, it expands evenly everywhere, so a distant galaxy’s apparent velocity is $v = H_0 d$. So,

$$d = \frac{v}{H_0}.$$ 
- Note that this doesn’t work for nearby galaxies like Andromeda (which is moving toward us due to gravity dominating over cosmic expansion).

**The two-body problem**  ← see Ch. 2 of Murray & Dermott
- The motion of two bodies about one another due to their mutual gravity.
  - Planets orbiting stars
  - Stars orbiting each other
  - Objects orbiting white dwarf; neutron star; black hole
- Certain quantities can be measured very precisely, enabling precise measurements of masses and sizes of bodies. E.g., binary pulsars (neutron stars): masses measured to within $10^{-3} M_{\odot}$ (~0.1%)
- Goal here: Go through the gravitational two-body problem with an eye on features that are observationally testable, and on features specific to the $1/r^2$ nature of gravity. Many “details” of the real world push us away from exact $1/r^2$ – e.g. physical sizes, non-spherical shapes, general relativity
- Key behavior we will use:
  - (1) Bodies move in elliptical trajectories (**Kepler’s 1st Law**)

\[ r(\varphi) = \frac{a(1-e^2)}{1+e\cos \varphi} \]

- **2nd Law:** Motion sweeps out equal areas in equal times: $dA/dt = \frac{1}{2} r^2 d\varphi/dt = \text{constant}$
- **3rd Law:** $a^3/P^2 \sim (M_{\text{tot}})$  ← our eventual goal. $P$=orbital period, $M_{\text{tot}} =$ total system mass
To fully describe two pointlike bodies in 3D, we need 6 position components and 6 velocity components. (If they are not pointlike we need even more – in this case we use, e.g., Euler angles to define bodies’ orientations.) Regardless, we need to reduce this to make things tractable!

FIRST, go from 2 bodies to 1 … we can do this for any central force (not just $1/r^2$). This means converting potential: $V(r_1, r_2) = V(|r_2 - r_1|)$ ← since potential depends only on relative position

$\vec{r} = \vec{r}_2 - \vec{r}_1$ (eq 1)

$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ (position of center of mass)

$\dot{\vec{R}} = 0$ IF no external forces operate on the system.

We can always set $Rdot = 0$ by choosing an inertial reference frame, and we can choose our origin so that $\vec{R} = 0$ too.

That means that $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$ ← combining with (eq 1) above shows that

$\vec{r}_2 = \frac{\vec{r}}{1 + m_2/m_1}$ and $\vec{r}_1 = -\frac{m_2}{m_1} \vec{r}_2$

… Which gives us the motions and velocities of both bodies in terms of the single variable $r$. So, we’ve reduced the 2-body problem to a one-body problem.