

# MEASURING “DARK MATTER” IN ASSET PRICING MODELS

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OCTOBER 11, 2019

We introduce an information-based fragility measure for GMM models that are potentially misspecified and unstable. A large fragility measure signifies a model’s lack of internal refutability (weak power of specification tests) and external validity (poor out-of-sample fit). The fragility of a set of model-implied moment restrictions is tightly linked to the quantity of additional information the econometrician can obtain about the model parameters by imposing these restrictions. Our fragility measure can be computed at little cost even for complex dynamic structural models. We illustrate its applications via two models: a rare-disaster risk model and a long-run risk model. (JEL Codes: C52, G12, D81, E32)

KEYWORDS: Fragile beliefs, Unstable models, Misspecification, Time series, Out-of-sample fit, Disaster risk, Long-run risk, Semiparametric efficiency.

## 1. INTRODUCTION

In cosmology, dark matter is a form of matter that is not directly observable, yet its presence is required for Einstein’s theory of general relativity to be consistent with the observable motions of stars and galaxies. Certain economic models rely on an analogous form of “dark matter,” namely model components or parameters that are difficult to verify and measure directly in

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the data, despite having significant effects on the models' performance.<sup>1</sup>

Should we be worried about models with dark matter features? Two common defensive arguments for economic models relying on dark matter are: (i) these features are not inconsistent with the data, i.e., they cannot be rejected by the data; and (ii) the fact that they improve the model's ability to fit the data provides indirect evidence supporting their very existence, analogous to dark matter as posited in cosmology. In this paper, we propose a measure for economic dark matter and show that the measure helps quantify the degree of model fragility. Specifically, models with more dark matter tend to lack refutability and have higher overfitting tendency.

We define a measure of economic dark matter for a general class of GMM models ([Hansen, 1982](#)) that summarize a structural model with a set of unconditional moment restrictions. Our measure is based on the (relative) informativeness of the cross-equation restrictions imposed by the structural model. Essentially, the measure compares the asymptotic variances of two efficient GMM estimators for the model parameters  $\theta$ , one based on the full set of moment restrictions which we refer to as the full model, and the other based on a subset of the moment restrictions which we refer to as the baseline model. The dark matter measure is obtained by searching for the largest discrepancy between the two asymptotic variances in all linear directions.

We then show that the dark matter measure is linked to two key model properties: refutability and overfitting tendency. Specifically, we show that the power of the optimal specification tests vanishes as the dark matter measure for a model approaches infinity. We also show that those models with more dark matter tend to overfit the data; namely, the out-of-sample fit of the moment restrictions can deteriorate significantly relative to its in-sample fit once we take into account the possibility that the model is potentially misspecified and the true data-generating process (DGP) is subject to local instability (e.g., [Li and Müller, 2009](#)).

The dark matter measure has an intuitive sample-size interpretation, which can be thought of as the amount of additional data needed for the baseline estimator to match or exceed the precision that the full model estimator can achieve with the aid of the cross-equation restrictions. On the technical front, we extend the results on semiparametric local minimax efficiency bounds by [Hansen \(1985\)](#) and [Chamberlain \(1987\)](#) for unconditional moment restrictions to

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<sup>1</sup>For example, John Campbell referred to rare disasters as “dark matter for economists” in his 2008 Princeton Lectures in Finance ([Campbell, 2018](#)).

Markov processes with local instability. This step is necessary to formalize the information interpretation of our measure.

The fact that the informativeness of the cross-equation restrictions is linked to model fragility is intuitive. The cross-equation restrictions are informative under two conditions: (i) the additional moments from the full model are sensitive to small changes in the parameter values; and (ii) these moments can be measured precisely in the data relative to the baseline moments. Thus, computing the dark matter measure resembles the dual of conducting a form of sensitivity analysis. Practically, a model is considered fragile if its key implications are excessively sensitive to small perturbations of the data-generating process. Formally, specifying the relevant magnitude of “local perturbations” and defining “excessive sensitivity” are necessary. In multivariate settings, there is the additional challenge that the simultaneous responses of model parameters to the perturbations in the data-generating process must be considered in order to assess the full scope of model fragility.

Our dark matter measure effectively formalizes the sensitivity analysis by (i) benchmarking the local perturbation in the data-generating process against the identification derived from the baseline model, and (ii) defining the excessive sensitivity of efficient GMM estimators to the local perturbations in the data-generating process based on the sampling variability of the efficient GMM estimators. Naturally, we require the baseline model to be a correct benchmark similar to [Eichenbaum, Hansen, and Singleton \(1988\)](#), because we need the identification provided by the baseline model to define the reasonable perturbations in the data-generating process. In addition, our measure identifies the worst direction of perturbation for the multivariate setting by searching for the direction in the model parameter space in which the cross-equation restrictions are the most informative.

An intuitive justification for the concern regarding model sensitivity is the high effective degrees of freedom in those models with a high sensitivity, i.e., those models can be fitted to a wide range of empirical moments with minor changes to the parameter values. Models with high degrees of freedom are well-documented as being more prone to overfitting, hence statistical model selection procedures impose penalties for model complexity, as measured by AIC, BIC, LASSO, etc.

Formally, the concern regarding model sensitivity originates in potential misspecification and local instability of the data-generating process. Under the assumption that a model is

correctly specified, a high sensitivity of the moments to parameter perturbations is a beneficial feature. Imposing these model restrictions will facilitate estimating the model parameters significantly more precisely, which is the foundation for structural estimation (e.g., [Andrews, Gentzkow, and Shapiro, 2017](#)). However, if the model restrictions are potentially misspecified, the information obtained from imposing such restrictions may no longer be valid. [Kocherlakota \(2007\)](#) similarly emphasizes the “fallacy of fit.” The nontestable assumption of identification strength is related to the informativeness of the cross-equation restrictions. They indicate the implicit degrees of freedom postulated by the modeler, as the modeler effectively passes the statistical challenge of learning about model parameters from the data onto economic agents.

We analyze the consequences of misspecification and local instability by generalizing the local instability framework of [Li and Müller \(2009\)](#) to the semiparametric setting.<sup>2</sup> We show that models with large dark matter measures tend to generate an excessively high quality of in-sample fit. Due to this finding, these models are difficult to reject even when they are misspecified, hence the lack of refutability. Moreover, we show that under local instability, models with larger dark matter measures have a higher worst-case asymptotic expected degree of overfitting, as measured by the gap between the in-sample model fit and the out-of-sample model fit based on the Sargan-Hansen  $J$  statistic. Thus, we generalize the notion of sensitivity analysis from the perturbations of model parameters to the perturbations of the underlying data-generating processes.

The recursive (two-stage) GMM estimation (e.g., [Christiano and Eichenbaum, 1992](#); [Ogaki, 1993](#); [Newey and McFadden, 1994](#); [Hansen and Heckman, 1996](#); [Hansen, 2007b](#); [Lee, 2007](#); [Hansen, 2012](#)) has necessarily worse in-sample fit than does the efficient GMM estimation ([Hansen, 1982](#)); however, the former can in fact deliver better out-of-sample fit when the dark matter measure (i.e., the model fragility) is excessively high. Although the original impetus of the recursive (two-stage) GMM estimation was primarily computational, we advocate it as a robust estimation procedure against high model fragility. Thorough analyses on optimal robust estimation, and even on optimal model selection, however, are beyond the scope of this paper.

We evaluate the fragility of two models from the asset pricing literature. The first example is a rare-disaster model. In this model, parameters describing the likelihood and magnitude

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<sup>2</sup>We need a semiparametric framework for at least three reasons: (i) it provides a formal general econometric framework for local perturbations in the space of local data-generating processes; (ii) it is needed for justifying the information matrix interpretation of our dark matter measure based on semiparametric efficiency bounds; and (iii) it is a natural way to connect the GMM model to the structural economic model.

of economic disasters are difficult to estimate from the data unless information in asset prices is used. We derive the dark matter measure in this example analytically. We also illustrate how to incorporate uncertainty about the structural parameters (in this context, preference parameters) when computing model fragility. The second example is a long-run risk model with a nine-dimensional parameter space. We use this example to show that two calibrations of the model with similar in-sample fit can differ vastly in fragility properties. We conduct Monte Carlo simulation experiments for both examples and show that the calibrated models with large dark matter measures lack in-sample refutability and have poor out-of-sample fit, consistent with the theory.

### *Related Literature*

The idea that a model’s fragility is connected to its degrees of freedom (i.e., its complexity) dates back at least to Fisher (1922). Traditionally, effective degrees of freedom of a model are measured by the number of parameters, simply because the two coincide in Gaussian-linear models (e.g. Ye, 1998; Efron, 2004). Numerous statistical model selection procedures are based on this idea.<sup>3</sup>

However, the limitations of using the number of parameters to measure model’s degrees of freedom have been well documented. Hence, new methods have been developed to measure the sensitivity of model implications to parameter perturbations in the statistics literature.<sup>4</sup> A common feature of these proposals is that they rely on the same model being evaluated to determine the parameter perturbations; this is potentially problematic when evaluating economic models that are themselves fragile, partly due to a lack of internal refutability. In contrast, we propose using a baseline model to assign weights to potential alternative underlying data-generating processes or to determine the possible “reasonable” perturbations of data-generating processes.

To be more specific, our fragility measure is different from the extant measures in four aspects. First, we use a semiparametric framework to allow for general local perturbations of

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<sup>3</sup>Examples include the Akaike information criterion (AIC) (Akaike, 1973), the Bayesian information criterion (BIC) (Schwarz, 1978), the risk inflation criterion (RIC) (Foster and George, 1994), and the covariance inflation criterion (CIC) (Tibshirani and Knight, 1999).

<sup>4</sup>Extant statistics literature covers several alternative approaches to measuring the “implicit degrees of freedom” or “generalized degrees of freedom” (e.g., Ye, 1998; Shen and Ye, 2002; Efron, 2004; Spiegelhalter, Best, Carlin, and van der Linde, 2002; Ando, 2007; Gelman, Hwang, and Vehtari, 2013).

data-generating processes similar to [Hansen and Sargent \(2001\)](#), but not only the local perturbations of model parameters that fit the model. Second, the reasonable local perturbations of data-generating processes are generated by the baseline model that is less likely to be misspecified, and we use the baseline model as a benchmark to respect the primary purpose of the economic structural model in the fragility assessment. Third, we directly connect the model fragility measure to a model’s internal refutability, i.e., the optimal power of specification tests. Fourth, we also directly connect the model fragility measure to a model’s out-of-sample fit, emphasized by, for example, [Schorfheide and Wolpin \(2012\)](#) and [Athey and Imbens \(2017, 2019\)](#), for assessing economic models.<sup>5</sup>

Further, we have built our model fragility measure based on a multivariate sensitivity analysis. [Müller \(2012\)](#) studies multivariate sensitivity analysis in Bayesian inference and the worst-case direction. He focuses on the sensitivity of the posterior distribution with respect to the prior distribution by asking how much the posterior distribution changes in response to a perturbation of the prior distribution. Differently, we perturb the underlying data-generating process, and ask how much the moment restrictions and the baseline model parameters change. Similar to [Müller \(2012\)](#), we consider a multivariate sensitivity problem using asymptotic methods and eigenvalue decomposition to identify the worst-case direction.

Our work contributes to the literature on local instability in time series analysis. Evidence abounds on structural changes and nonstationarity in asset pricing (e.g., [Pesaran and Timmermann, 1995](#); [Bossaerts and Hillion, 1999](#); [Pastor and Stambaugh, 2001](#); [Lettau and Van Nieuwerburgh, 2008](#); [Lettau, Ludvigson, and Wachter, 2008](#); [Welch and Goyal, 2008](#); [Kojen and Van Nieuwerburgh, 2011](#); [Dangl and Halling, 2012](#)). Econometric theory has largely focused on testing whether or not the model is stable (see, e.g., [Nyblom, 1989](#); [Andrews, 1993](#); [Andrews and Ploberger, 1994](#); [Sowell, 1996](#); [Bai and Perron, 1998](#); [Hansen, 2000](#); [Andrews, 2003](#); [Elliott and Müller, 2006](#), for recent contributions). However, little research has explored the next step: what implications arise once instabilities are suspected? One exception is [Li and Müller \(2009\)](#) who show that the standard GMM inference ([Hansen, 1982](#)), despite ignoring the partial instability of a subset of model parameters, remains asymptotically valid for the

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<sup>5</sup>The terms “in-sample fit” and “out-of-sample fit”, as well as other similar terms, are used commonly in statistics, econometrics, and empirical asset pricing (e.g., [Hastie, Tibshirani, and Friedman, 2001](#); [Ferson, Nallareddy, and Xie, 2013](#); [Varian, 2014](#); [Athey and Imbens, 2015](#); [Mullainathan and Spiess, 2017](#); [Müller and Watson, 2016](#)). The terms “out-of-sample fit” and “external validity” have been used interchangeably in finance and economics literature (e.g., [Bossaerts and Hillion, 1999](#); [Stock and Watson, 2002](#); [Schorfheide and Wolpin, 2012](#)).

subset of stable parameters. We show that the GMM models tend to have poor out-of-sample fit if their dark matter measure, i.e., model fragility measure, is excessively large.

Our work also connects to the literature on structural estimation, including rational expectations econometrics, in which economic assumptions (the cross-equation restrictions) have been used extensively to increase efficiency in estimating the structural parameters. Classic examples include [Saracoglu and Sargent \(1978\)](#), [Hansen and Sargent \(1980\)](#), [Campbell and Shiller \(1988\)](#), among others, and textbook treatments by [Lucas and Sargent \(1981\)](#), [Hansen and Sargent \(1991\)](#). In a fragile model, cross-equation restrictions may imply excessively tight confidence regions for the parameters, with low coverage probability under reasonable parameter perturbations. An important potential source of fragility in this context is that the structural model relies heavily on the agents possessing accurate knowledge of hard-to-estimate parameters.

[Hansen \(2007a\)](#) offers an extensive discussion of the informational burden that rational expectations models place on the agents, which is one of the key motivations for research in Bayesian learning, model ambiguity, and robustness (e.g., [Gilboa and Schmeidler, 1989](#); [Hansen and Sargent, 2001](#); [Epstein and Schneider, 2003](#); [Klibanoff, Marinacci, and Mukerji, 2005](#)). This literature recognizes that the traditional assumption that agents possess precise knowledge of the relevant probability distributions is not justifiable in certain contexts, and explicitly incorporates robustness considerations into agents' decision problems. Our approach complements this line of research, in that our measure of fragility helps diagnose situations in which incorporating parameter uncertainty and agents' robustness considerations within an economic model could be particularly important.

Our analysis of the disaster-risk model relates to studies that have highlighted the challenges in testing such models. One implication of the low probability of disasters is the so-called “peso problem” (see [Lewis, 2008](#), for an overview): if observations of disasters in a particular sample under-represent their population distribution, standard inference procedures may lead to distorted conclusions. Thus, the peso problem is a particular case of the weak identification problem. Our analysis highlights that in applications subject to the peso problem, it is important to guard against model fragility. On this front, [Zin \(2002\)](#) shows that certain specifications of higher-order moments of the endowment growth distribution may help the model fit the asset pricing moments while being difficult to reject in the endowment data. Our analysis of model fragility encapsulates such considerations in a general quantitative measure.

## 2. AN INTUITIVE EXAMPLE

We now use a version of the Gordon growth model to illustrate how the dark matter measure connects to model fragility. Suppose dividends follow the process

$$(1) \quad \frac{Y_{t+1}}{Y_t} = 1 + \theta + \sigma_Y \epsilon_{Y,t+1}, \quad \epsilon_{Y,t} \text{ is i.i.d., with } E[\epsilon_{Y,t}] = 0, E[\epsilon_{Y,t}^2] = 1.$$

According to the Gordon growth model, the price of the claim on the dividends is the present value of expected future dividends. Assuming the risk-adjusted discount rate is  $r$ , then

$$(2) \quad P_t = \sum_{s=1}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}.$$

Therefore, the price-dividend ratio according to the Gordon growth model is

$$(3) \quad E\left[\frac{P_t}{Y_t}\right] = F(\theta), \text{ with } F(\theta) \equiv \frac{1+\theta}{r-\theta}.$$

The econometrician tries to evaluate a GMM version of this model by focusing on the first moments of dividend growth and price-dividend ratio. For simplicity, we assume that the econometrician knows all parameters except the average dividend growth rate  $\theta$ . The moment conditions are

$$(4) \quad \mathbb{E}[m(\mathbf{y}_t, \theta)] = 0, \text{ with } m(\mathbf{y}_t, \theta) \equiv \begin{bmatrix} Y_{t+1}/Y_t - 1 - \theta \\ P_t/Y_t - F(\theta) \end{bmatrix},$$

where  $\mathbf{y}_t \equiv (Y_t, P_t)^T$  for all  $t = 1, \dots, n$ . We denote the first element of  $m(\mathbf{y}_t, \theta)$  by  $m^{(1)}(\mathbf{y}_t, \theta)$  and refer to it as the baseline moment restriction. We focus on the GMM model (4) to evaluate its fragility. We assume that the Gordon growth model can be misspecified and that the true local data-generating processes can be described by

$$(5) \quad \begin{bmatrix} Y_{t+1}/Y_t \\ P_{t+1}/Y_{t+1} \end{bmatrix} = \frac{f_{n,t}}{\sqrt{n}} + \begin{bmatrix} 1 + \theta_0 \\ F(\theta_0) \end{bmatrix} + \begin{bmatrix} \sigma_Y \epsilon_{Y,t+1} \\ \sigma_P \epsilon_{P,t+1} \end{bmatrix},$$

where the shocks  $\epsilon_{Y,t}$  and  $\epsilon_{P,t}$  are mutually independent with mean zero and standard deviation



one; and, the term  $f_{n,t}$  captures the potential local bias and instability:

$$(6) \quad f_{n,t} = \lambda_1 + \lambda_2 b(t/n), \text{ with } t = 1, \dots, n, \text{ and } \lambda_i \equiv \begin{bmatrix} \lambda_i^{(1)} \\ \lambda_i^{(2)} \end{bmatrix} \text{ for } i \in \{1, 2\},$$

where  $b(\cdot)$  is an unknown deterministic function on  $[0, 1]$  whose path has a finite number of discontinuities and one-sided limits everywhere. We assume that  $\sup_{u \in [0, 1]} |b(u)| \leq 1$  and  $\int_0^1 b(u) du = 0$ .<sup>6</sup> Instability (namely, structural changes and nonstationarity) in the data-generating process of price-dividend ratios is prevalent, and incorporating the consideration of instability has first-order importance in studying the out-of-sample fit of asset pricing models.<sup>7</sup> Our modeling specification in (6) follows the econometric setup in the literature of local instability (e.g., [Andrews, 1993](#); [Sowell, 1996](#); [Li and Müller, 2009](#)). Intuitively,  $\lambda_1$  captures the stable local moment bias, while  $\lambda_2$  captures the extent to which the system is unstable.

For calibration, we set  $\theta_0 = 0$ ,  $r = 3\%$ ,  $\sigma_Y = 4\%$ , and  $\sigma_P = 5$ . Evaluated at the calibrated parameter value  $\theta_0 = 0$ , the Jacobian matrices of the baseline moment restrictions and the full moment restrictions are  $D_{11} = -1$  and  $D = [-1 \ F'(\theta_0)]^T$  with  $F'(\theta_0) = (1+r)/r^2$ , respectively. Under the calibrated distribution  $Q_0$  of  $\mathbf{y}_t$ , the spectral density matrices (evaluated at zero frequency) for the baseline and full models are  $\Omega_{11} = \sigma_Y^2$  and  $\Omega = \text{diag}\{\sigma_Y^2, \sigma_P^2\}$ , respectively.

### ***Dark matter measure.***

Panel A of [Figure 1](#) displays the asymptotic distribution of the baseline estimator  $\tilde{\theta}_n$  (the efficient GMM estimator for  $\theta$  based on the baseline moment of (4)). The graph illustrates the degree of uncertainty about the value of  $\theta$  according to the baseline model and the dividend data:  $\sqrt{n}(\tilde{\theta}_n - \theta_0) \xrightarrow{d} N(0, \mathbf{I}_B^{-1})$ , where  $\mathbf{I}_B \equiv D_{11}^T \Omega_{11}^{-1} D_{11} = 6.25 \cdot 10^2$ . More specifically, when  $n = 100$ , the 95% asymptotic confidence interval for the growth rate  $\theta$  is approximately  $[-0.8\%, 0.8\%]$ .

Panel B plots the model-implied average price-dividend ratio  $F(\theta)$  as a function of the dividend growth rate  $\theta$ . Because  $F(\theta)$  rises quickly with  $\theta$ , there is only a narrow range of  $\theta$  for which the model-implied average price-dividend ratios would be consistent with the sample

<sup>6</sup>Such normalization is without loss of generality, because we can always subsume the scaling and average into the constant vectors  $\lambda_1$  and  $\lambda_2$ .

<sup>7</sup>e.g., [Pesaran and Timmermann \(1995\)](#); [Pastor and Stambaugh \(2001\)](#); [Lettau, Ludvigson, and Wachter \(2008\)](#); [Lettau and Van Nieuwerburgh \(2008\)](#); [Welch and Goyal \(2008\)](#); [Kojen and Van Nieuwerburgh \(2011\)](#).

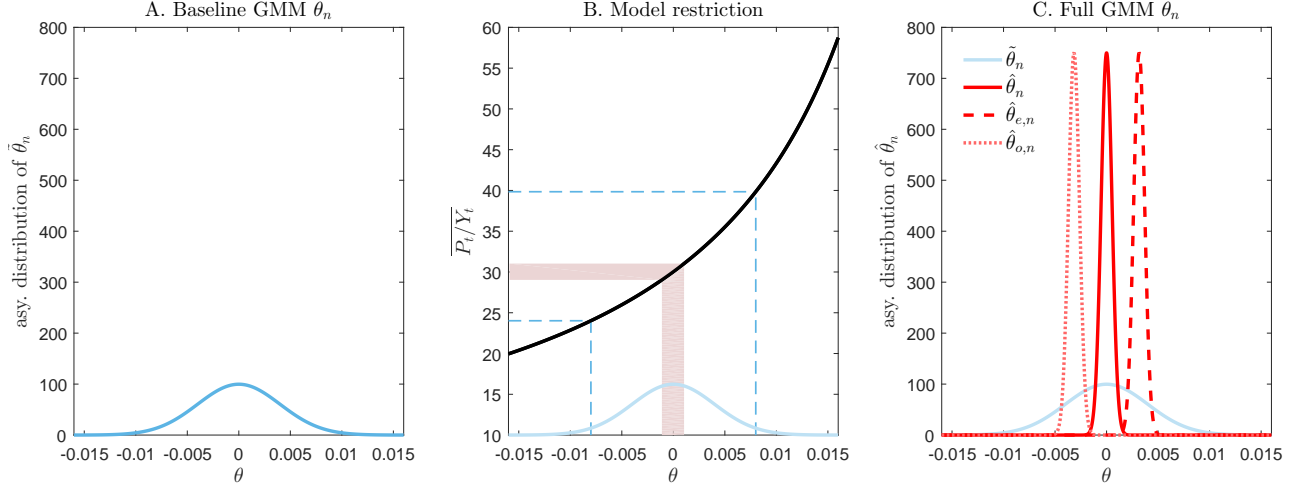


FIGURE 1.— An example of an informative asset pricing restriction on the parameters. Panel A plots the asymptotic distribution of  $\sqrt{n}(\tilde{\theta}_n - \theta_0)$  with  $n = 100$ . Panel B plots the average price-dividend ratio as a function of  $\theta$ . The light blue dashed vertical lines represent the confidence band for  $\tilde{\theta}_n$  with sample size  $n = 100$ , and the corresponding light blue dashed horizontal lines represent the model-implied average price-dividend ratio when perturbing  $\theta$  within the confidence band. The shaded area represents the confidence band of the average price-dividend ratio  $\overline{P_t/Y_t}$  according to the data with  $n = 100$ . Panel C plots the asymptotic distribution of  $\hat{\theta}_n$  based on the assumed asset pricing restriction displayed in Panel B and the whole sample  $\mathbf{y}^n$  with  $n = 100$ ; the normal density indicated by the red solid line is the asymptotic distribution of  $\sqrt{n}(\hat{\theta}_n - \theta_0)$  when the model is correctly specified ( $f_{n,t} \equiv 0$ ). By contrast, the normal density indicated by the red dashed line is the asymptotic distribution of  $\sqrt{\pi n}(\hat{\theta}_{e,n} - \theta_0)$ , and the normal density indicated by the red dotted line is the asymptotic distribution of  $\sqrt{\pi n}(\hat{\theta}_{o,n} - \theta_0)$ , when the model is locally misspecified and unstable ( $\lambda_1 = 0, \lambda_2 = 30$  and  $b(\cdot)$  in equation (11)). The estimators  $\hat{\theta}_{e,n}$  and  $\hat{\theta}_{o,n}$  are based on  $\mathbf{y}_e^n$  and  $\mathbf{y}_o^n$ , respectively, with sample size  $n = 200$  and  $\pi = 1/2$ .

mean. To see this, we use the shaded horizontal region near 30 on the  $y$ -axis to denote the 95% confidence interval for the average price-dividend ratio from the data (assuming there is no local misspecification, i.e.  $f_{n,t} \equiv 0$ ). The corresponding shaded vertical region on the  $x$ -axis shows the range of  $\theta$  consistent with the model, which is significantly more concentrated than the asymptotic distribution of the baseline estimator.

Consequently, the full estimator  $\hat{\theta}_n$ , which is the efficient GMM estimator based on the full set of moment restrictions in (4), is significantly more precise than the baseline estimator (only if  $f_{n,t} \equiv 0$ ):  $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \mathbf{I}_F^{-1})$ , where  $\mathbf{I}_F \equiv D^T \Omega^{-1} D = 5.24 \cdot 10^4$ . Specifically, when  $n = 100$ , the 95% asymptotic confidence interval for the growth rate  $\theta$  is approximately  $[-0.08\%, 0.08\%]$ . Indeed, a comparison between the asymptotic distributions of the baseline and full estimators in panel C of Figure 1 reveals how informative the asset pricing restriction in (4) is about  $\theta$ . This incremental informativeness of the asset pricing restrictions is the exact

focus of our dark matter measure. More precisely, we define the dark matter measure as

$$(7) \quad \varrho(\theta_0) \equiv \mathbf{I}_F / \mathbf{I}_B - 1 = [F'(\theta_0)\sigma_Y / \sigma_P]^2 = 83.82.$$

This simple example shows that the informativeness ( $\varrho(\theta_0)$ ) increases in the sensitivity of the asset pricing moment to the parameter value ( $F'(\theta_0)$ ) and decreases in the variability of the asset pricing moment ( $\sigma_P$ ).

### ***Dark matter and model fragility.***

From the above example, we can see the link between our dark matter measure and model fragility. We follow convention to define model fragility as the excessive sensitivity of the model's implications, for example, how well it fits the data, to small perturbations of the data-generating process. To formalize this procedure, we need to be precise on (i) what constitutes small perturbations in data-generating processes, and (ii) how to define excessive sensitivity of the model fit. Intuitively, if trusting the baseline moment restrictions but lacking confidence in the additional moment restrictions from the structural model, it makes sense to benchmark the magnitude of the perturbation to the uncertainty about  $\theta$  in the baseline moment restrictions.<sup>8</sup>

Following the rules of the sensitivity analysis set above, panel B of [Figure 1](#) indicates that a perturbation of  $\theta$  within the 95% confidence region of the baseline estimator  $\tilde{\theta}_n$  (marked by the two vertical dashed lines) can move the average model-implied price-dividend ratio far away from the confidence region implied by the empirical moments (the narrow shaded region near 30 on the  $y$ -axis), which means the perturbed model will likely be rejected by the data. Similar to the informativeness of the asset pricing restriction, model sensitivity is higher when the asset pricing moment has a larger gradient with respect to the model parameter  $\theta$ , and when the error of the asset pricing moment is smaller.

More generally, the dark matter measure can be viewed as a multivariate model sensitivity measure. The direction in which the two efficient asymptotic variances based on the baseline and full models differ the most is also the one in which a local perturbation of the parameter vector  $\theta$  results in the largest changes in the model fit.

It is also worth noting that the connection between the informativeness of the structural

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<sup>8</sup>We formally justify this benchmark in [Section 4.3](#) by considering local instability and misspecification in the data-generating process.

model restrictions and model fragility only makes sense in light of the possibility of model misspecification. If a model is correctly specified, the more sensitive the moments, then the more precisely will be the estimate of the parameters by imposing valid restrictions (e.g., [Andrews, Gentzkow, and Shapiro, 2017](#)); this can be seen from panel C of [Figure 1](#), where the asymptotic distribution of  $\hat{\theta}_n$  with  $f_{n,t} \equiv 0$  is tightly distributed around the true value  $\theta_0 = 0$ . However, if the asset pricing restrictions are potentially misspecified and unstable, then the extra information they provide may not be valid, thereby making inference problematic.

### *Lack of refutability.*

We are interested in testing the validity of the asset pricing moment restriction (i.e., whether  $\lambda_1^{(2)} = 0$ ) given the prior information that the baseline moment restriction is correctly specified (i.e.,  $\lambda_1^{(1)} = 0$ ). Considering an upper bound for the maximin local power, we can focus on a subset of alternatives satisfying  $b \equiv 0$  to obtain an upper bound characterized by the dark matter measure, albeit it is not necessarily the tightest bound. In this example, the  $C$  test ([Eichenbaum, Hansen, and Singleton, 1988](#)) is numerically equivalent to the  $J$  test ([Hansen, 1982](#)); and [Newey \(1985a\)](#) shows that it is asymptotically optimal for these particular alternatives. We consider the set of alternatives  $\mathcal{A}_\kappa(Q_0) \equiv \left\{ \lambda_1 \in \mathbb{R}^2 : \lambda_1^{(1)} = 0, |\lambda_1^{(2)}| \geq \kappa \right\}$ .

The  $C$  test statistic is

$$(8) \quad C_n = \left[ Z_n + \sqrt{\frac{\varrho(\theta_0)}{1 + \varrho(\theta_0)}} \left( \frac{\lambda_1^{(1)}}{\sigma_Y} \right) + \sqrt{\frac{1}{1 + \varrho(\theta_0)}} \left( \frac{\lambda_1^{(2)}}{\sigma_P} \right) \right]^2 + o_p(1),$$

where  $Z_n = \sqrt{\varrho(\theta_0)/(1 + \varrho(\theta_0))} (n^{-1/2} \sum_{t=1}^n \epsilon_{Y,t}) + \sqrt{1/(1 + \varrho(\theta_0))} (n^{-1/2} \sum_{t=1}^n \epsilon_{P,t})$  converges to a standard normal variable in distribution. According to [Newey \(1985a\)](#) and [Chen and Santos \(2018\)](#), the maximin asymptotic power of the GMM specification tests of size  $\alpha$  is bounded from above by

$$(9) \quad \inf_{\lambda_1 \in \mathcal{A}_\kappa(Q_0)} \lim_{n \rightarrow \infty} \mathbb{P} \{ C_n > c_{1-\alpha} \} \leq \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \left[ Z_n + \sigma_P^{-1} \kappa / \sqrt{1 + \varrho(\theta_0)} \right]^2 > c_{1-\alpha} \right\}$$

$$(10) \quad = \mathbb{P} \{ \chi_1^2(\mu) > c_{1-\alpha} \},$$

where  $c_{1-\alpha}$  is the  $1 - \alpha$  quantile of a chi-square distribution with degree of freedom one, and  $\chi_1^2(\mu)$  represents a noncentral chi-square distribution with degree of freedom one and the

noncentrality parameter  $\mu \equiv \frac{(\kappa/\sigma_P)^2}{1 + \varrho(\theta_0)}$ . The upper bound in (9) is the asymptotic probability when  $\lambda_1^{(1)} = 0$  and  $\lambda_1^{(2)} = \kappa$ . Thus, when the model has too much dark matter, the noncentrality parameter  $\mu$  is very close to zero and thus the upper bound  $\mathbb{P}\{\chi_1^2(\mu) > c_{1-\alpha}\}$  is very close to  $\alpha$ , which means that the test power is close to zero.

Intuitively, the baseline moment restrictions have little ability to refute the cross-equation restrictions implied by the model when the dark matter measure  $\varrho(\theta_0)$  is excessively large. It is clear from panel B of Figure 1 that, by tuning the parameter value of  $\theta$  in acceptable region imposed by the baseline model (i.e., within the 95% confidence region of the baseline estimator  $\tilde{\theta}_n$ , marked by the two vertical dashed lines), the econometrician can fit the average model-implied price-dividend ratio over an immensely wide range (i.e., the range between the two horizontal dashed lines), which means the model-implied cross-equation restriction can hardly be rejected by the data.

### *Poor out-of-sample fit.*

A common method for evaluating out-of-sample fit of models is to hold out data from the model estimation (e.g., Schorfheide and Wolpin, 2012; Müller and Watson, 2016). We split the entire time series  $\mathbf{y}^n \equiv \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$  into two non-overlapping subsamples  $\mathbf{y}_e^n \equiv \{\mathbf{y}_1, \dots, \mathbf{y}_{\lfloor \pi n \rfloor}\}$  and  $\mathbf{y}_o^n \equiv \{\mathbf{y}_{\lfloor \pi n \rfloor + 1}, \dots, \mathbf{y}_n\}$  with  $\pi \in (0, 1/2]$ . The first segment  $\mathbf{y}_e^n$  is used as the estimation sample, while the second segment  $\mathbf{y}_o^n$  is used as the holdout sample. In particular, for the true local data-generating process in (5) – (6), we assume that  $\lambda_1^{(1)} = \lambda_1^{(2)} = \lambda_2^{(1)} = 0$ ,  $\lambda_2^{(2)} = 30$ ,  $\pi = 1/2$ , and

$$(11) \quad b(t/n) = \begin{cases} 1, & \text{when } 1 \leq t \leq \lfloor \pi n \rfloor \\ -1, & \text{when } \lfloor \pi n \rfloor < t \leq n, \end{cases}$$

where the sequence  $b(t/n)$  captures the structural breaks and  $\pi$  is the break point.

We consider the overfitting measure of the efficient GMM estimator  $\hat{\theta}_{e,n}$  based on the estimation sample  $\mathbf{y}_e^n$ , defined as the extent to which the out-of-sample fitting error exceeds

the in-sample fitting error:<sup>9</sup>

$$(12) \quad \mathcal{O}(\hat{\theta}_{e,n}, \mathbf{y}^n) \equiv \frac{1}{2} \left[ \mathcal{L}(\hat{\theta}_{e,n}, \mathbf{y}_o^n) - \mathcal{L}(\hat{\theta}_{e,n}, \mathbf{y}_e^n) \right],$$

with  $\mathcal{L}(\theta, \mathbf{y}_s^n) \equiv J(\theta, \mathbf{y}_s^n) - J(\theta_0, \mathbf{y}_s^n)$  for  $s \in \{e, o\}$  and the parameter value  $\theta_0$  ensures that the baseline moment restriction  $E[m^{(1)}(\mathbf{y}_t, \theta_0)] = 0$  is perfectly satisfied. It can be shown that (see Theorem 2) the expected overfitting measure can be approximated by

$$(13) \quad E[\mathcal{O}(\hat{\theta}_{e,n}, \mathbf{y}^n)] \approx 1 + \left( \frac{\lambda_2^{(2)}}{\sigma_P} \right)^2 \frac{\varrho(\theta_0)}{2}, \quad \text{as } n \text{ approaches infinity.}$$

The out-of-sample fit should be poor if the efficient GMM estimators  $\hat{\theta}_{e,n}$  and  $\hat{\theta}_{o,n}$ , based on  $\mathbf{y}_e^n$  and  $\mathbf{y}_o^n$  respectively, are statistically separate from each other. This can be seen in the case of the asymptotic distributions of  $\hat{\theta}_{e,n}$  and  $\hat{\theta}_{o,n}$  (panel C of Figure 1), which are centered away from the correct value  $\theta_0 = 0$  for the baseline model and therefore distant from each other. The distance between the in-sample estimator  $\hat{\theta}_{e,n}$  and the out-of-sample estimator  $\hat{\theta}_{o,n}$  is significant according to the asymptotic distribution, which implies that the estimator  $\hat{\theta}_{e,n}$  has a poor out-of-sample fit.

### *General setting.*

In the remainder of the paper, we formally develop the set of results illustrated by the simple example above. We consider the setting of weakly dependent time series data, which are prevalent in financial and macroeconomic studies, and allow for local perturbations (e.g., Hansen and Sargent, 2001) and instability (e.g., Li and Müller, 2009) of data-generating processes in a semiparametric framework. We then define the dark matter measure for a general class of GMM models, and formally establish the connection between the dark matter measure, the model's refutability, and its out-of-sample fit.

## 3. DARK MATTER MEASURE

In this section, we set up the model and introduce the dark matter measure. We then discuss the connections between the dark matter measure, sensitivity analysis, and model testability.

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<sup>9</sup>A similar measure and concept of overfitting is considered by Mullainathan and Spiess (2017) and Hansen and Dumitrescu (2018).

### 3.1. Model Setup

Let  $\mathcal{Y} = \mathbb{R}^{d_y}$ , the  $d_y$ -dimensional Euclidean space with Borel  $\sigma$ -field  $\mathcal{F}$ . Let  $\mathcal{P}$  denote the collection of all probability measures on the measurable space  $(\mathcal{Y} \times \mathcal{Y}, \mathcal{F} \otimes \mathcal{F})$  with the product sample space  $\mathcal{Y} \times \mathcal{Y}$  and the product  $\sigma$ -field  $\mathcal{F} \otimes \mathcal{F}$ .

#### *Markov processes.*

We consider a subspace of  $\mathcal{P}$ , denoted by  $\mathcal{H}$ , in which each probability measure is the bivariate marginal distribution  $Q$  for a time-homogeneous Harris ergodic and stationary Markov process  $\{\mathbf{y}_t : t = 0, 1, \dots\}$  satisfying the Doeblin condition.<sup>10</sup> Following [Bickel and Kwon \(2001\)](#), we parameterize time-homogeneous Markov processes by the bivariate marginal distributions  $Q$  of  $(\mathbf{y}_{t-1}, \mathbf{y}_t)$  for any  $t \geq 1$ . We denote the  $(n+1)$ -variate joint distribution of  $\mathbf{y}^n \equiv \{\mathbf{y}_0, \dots, \mathbf{y}_n\}$  corresponding to  $Q$  by  $\mathbb{P}_n$ . A Markov process is Harris ergodic if it is aperiodic, irreducible, and positive Harris recurrent (e.g. [Jones, 2004](#); [Meyn and Tweedie, 2009](#)). Harris ergodicity guarantees the existence of a unique invariant probability measure (e.g., [Meyn and Tweedie, 2009](#)). Given Harris ergodicity, stationarity only requires that the initial distribution of  $\mathbf{y}_0$  is the unique invariant probability measure. The Doeblin condition implies that the  $\phi$ -mixing coefficients  $\phi(n)$  decay to zero exponentially fast (e.g. [Bradley, 2005](#), Section 3.2 and Theorem 3.4), which is useful for establishing the uniform law of large numbers (ULLN) ([White and Domowitz, 1984](#)) and the central limit theorem (CLT) (see, e.g., [Jones, 2004](#), Theorem 9).<sup>11</sup>

#### *Structural model and moment restrictions.*

Consider a stable structural model denoted by  $\mathcal{Q}$ , which aims to capture certain statistical features of the observed data  $\mathbf{y}^n$ . The parameters of such a “stable” model, denoted by  $\theta$ , are constant over time (e.g., [Li and Müller, 2009](#)).<sup>12</sup> We assume that the restrictions imposed by the model on the data can be summarized by a set of moment restrictions, and that the model’s performance in a given data sample can be measured by the degree to which these

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<sup>10</sup>The set of Markov processes satisfying the Doeblin condition includes a broad class of time series commonly used in finance and macroeconomics; see, for example, [Stokey and Lucas \(1989\)](#) and [Ljungqvist and Sargent \(2004\)](#).

<sup>11</sup>First-order Markov models are widely adopted for approximating financial and economic time series. Many prominent structural asset pricing models feature state dynamics as first-order Markovian processes (e.g., [Campbell and Cochrane, 1999](#); [Bansal and Yaron, 2004](#); [Gabaix, 2012](#); [Wachter, 2013](#)).

<sup>12</sup>Technically, the model parameters may vary with the sample size  $n$ , though they do not depend on the time index  $t \in \{1, \dots, n\}$ .

moment restrictions are violated (i.e., the fit of moment restrictions). As we will explain in Section 3.4 below, our notion of model fragility is also based on the moment restrictions, specifically their sensitivity to local perturbations of the underlying data-generating process.<sup>13</sup> We follow the literature (e.g., [Li and Müller, 2009](#); [Chen and Santos, 2018](#)), and refer to these models as GMM models. As reflected in the original applications of GMM in asset pricing ([Hansen and Singleton, 1982, 1983](#)) and recently emphasized by [Hansen \(2014\)](#), structural asset pricing models are typically partially specified. Further, GMM has proven particularly valuable for analyzing structural models via focusing on key cross-equation restrictions such as Euler equations, without being overly influenced by the details and potential singularities of the remainder of the structural model. Accordingly, in this paper we adopt the semiparametric framework of GMM models.

Specifically, we assume the moment function corresponding to the full structural model is  $m(\cdot, \theta) \in \mathbb{R}^{d_m}$ , defined on a compact parameter set  $\Theta \in \mathbb{R}^{d_\theta}$  with nonempty interior, and denote the full GMM model by  $\mathcal{Q}$ ,

$$(14) \quad \mathcal{Q} = \{Q \in \mathcal{H} : E^Q[m(\cdot, \theta)] = 0, \text{ for some } \theta \in \Theta\},$$

which is a collection of probability measures under which the moment restrictions hold for some parameter vector  $\theta$ . The system of moment restrictions is over-identified; that is,  $d_\theta < d_m$ .

We assume that the moment function  $m(\cdot, \theta)$  has a recursive structure:

$$(15) \quad m(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta) = \begin{bmatrix} m^{(1)}(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta^{(1)}) \\ m^{(2)}(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta^{(1)}, \theta^{(2)}) \end{bmatrix}, \quad \text{with} \quad \theta = \begin{bmatrix} \theta^{(1)} \\ \theta^{(2)} \end{bmatrix}.$$

Here  $\theta^{(1)}$  is a  $d_{\theta,1}$ -dimensional sub-vector of  $\theta$ , with  $d_{\theta,1} \leq d_\theta$ , and  $m^{(1)}(\cdot, \theta^{(1)})$  has dimension  $d_{m,1}$  such that  $d_{\theta,1} \leq d_{m,1}$ . The baseline moments can be represented by a selection matrix  $\Gamma_{m,1}$ :

$$(16) \quad m^{(1)}(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta^{(1)}) = \Gamma_{m,1} m(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta), \quad \text{with} \quad \Gamma_{m,1} \equiv [I, 0_{d_{m,1} \times (d_m - d_{m,1})}].$$

The assumption of the recursive structure for the moment function enables us to examine the

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<sup>13</sup>[Kocherlakota \(2016\)](#) adopts a similar notion in studying the sensitivity of real macro models to the specification of the Phillips curve.



fragility of a subset of moment restrictions, namely those in  $m^{(2)}(\cdot, \theta)$ . Such recursive structures are common in asset pricing. For example, the moment restrictions  $m^{(1)}(\cdot, \theta^{(1)})$  could be derived from a statistical model of the real quantities (such as consumption), while the additional moment restrictions  $m^{(2)}(\cdot, \theta)$  may apply to the joint dynamics of the real quantities and asset prices.

Since the first coordinate block of the moment function  $m(\cdot, \theta)$  only depends on  $\theta^{(1)}$ , the Jacobian matrix  $D(\theta)$  is block lower triangular:

$$(17) \quad D(\theta) = \begin{bmatrix} D_{11}(\theta) & 0 \\ D_{12}(\theta) & D_{22}(\theta) \end{bmatrix}, \text{ where } D_{ij}(\theta) \equiv \mathbb{E}^Q [\nabla_{\theta^{(j)}} m^{(i)}(\cdot, \theta)] \text{ and } i, j = 1, 2.$$

Corresponding to the first coordinate block  $m^{(1)}(\cdot, \theta^{(1)})$  of the moment function  $m(\cdot, \theta)$  in (15), we define the baseline GMM model  $\mathcal{Q}^{(1)}$ ,

$$(18) \quad \mathcal{Q}^{(1)} = \{Q \in \mathcal{H} : \mathbb{E}^Q [m^{(1)}(\cdot, \theta^{(1)})] = 0 \text{ for some } \theta^{(1)} \in \Theta^{(1)}\}.$$

The baseline model is a collection of probability measures under which the first block of moment restrictions, hereafter referred to as the baseline moments, hold for some parameter vector  $\theta^{(1)}$ . This definition is analogous to the definition of the full model, and clearly  $\mathcal{Q} \subset \mathcal{Q}^{(1)}$ . The subvector  $\theta^{(2)}$  can only be identified by the moment restrictions not contained in the baseline model. We refer to  $\theta^{(2)}$  as the nuisance parameters.

Although economic models often feature conditional moment restrictions, for estimation and testing, it is common to focus on a finite number of implied unconditional moment restrictions by using nonlinear instrumental variables (e.g., Hansen and Singleton, 1982, 1983; Hansen, 1985; Nagel and Singleton, 2011). For simplicity, we take these unconditional moments as the starting point in our analysis.

Following the definition of the full structural model (14), we define a mapping from the probability measure of the bivariate marginal distribution  $Q \in \mathcal{Q}$  to model parameters  $\theta$ ,  $\theta = \vartheta(Q)$ , such that

$$(19) \quad \mathbb{E}^Q [m(\cdot, \vartheta(Q))] = 0.$$

***Calibrated models.***

Consider a calibrated model parameter value  $\theta_0 \in \text{int}(\Theta)$ , the interior of  $\Theta$ , such that the moment restrictions evaluated at  $\theta_0$  hold under some  $Q_0 \in \mathcal{Q}$ :

$$(20) \quad E^{Q_0} [m(\cdot, \theta_0)] = 0.$$

Note that  $Q_0$  remains unknown to the econometrician, even though  $\theta_0$  may be known. The calibrated full and baseline GMM models are sets of probability measures satisfying

$$(21) \quad \mathcal{Q}(\theta_0) \equiv \{Q \in \mathcal{H} : E^Q [m(\cdot, \theta_0)] = 0\}, \text{ and}$$

$$(22) \quad \mathcal{Q}^{(1)}(\theta_0^{(1)}) \equiv \{Q \in \mathcal{H} : E^Q [m^{(1)}(\cdot, \theta_0^{(1)})] = 0\}, \text{ respectively.}$$

By definition,  $\mathcal{Q}(\theta_0) \subset \mathcal{Q}$ . Moreover,  $\mathcal{Q}(\theta_0)$  is non-empty (due to the assumption for  $\theta_0$ ). We pick one distribution from  $\mathcal{Q}(\theta_0)$  and denote it by  $Q_0$ , which is a distribution under which the moment restrictions of the full model hold under the calibrated parameters  $\theta_0$ .

*3.2. The Efficient GMM Estimation*

Under the distribution  $Q_0$ , the spectral density matrices (at zero frequency) for the baseline and full models are

$$(23) \quad \Omega_{11} = \sum_{t=-\infty}^{\infty} E^{Q_0} [m^{(1)}(\mathbf{y}_0, \mathbf{y}_1, \theta_0) m^{(1)}(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta_0)^T], \text{ and}$$

$$(24) \quad \Omega = \sum_{t=-\infty}^{\infty} E^{Q_0} [m(\mathbf{y}_0, \mathbf{y}_1, \theta_0) m(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta_0)^T], \text{ respectively,}$$

where  $\Omega_{11}$  is the upper-left block of  $\Omega$ . We assume that both  $\Omega$  and the Jacobian matrix  $D$  are known. In general, computing the expectations requires knowledge of the distribution  $Q_0$ . When  $Q_0$  is unknown in practice, expectations can be replaced by their consistent estimators. For example, several consistent estimators of the covariance matrices are provided by [Newey and West \(1987\)](#), [Andrews \(1991\)](#), and [Andrews and Monahan \(1992\)](#). These estimation methods usually require a two-step plug-in procedure introduced by [Hansen \(1982\)](#) when  $\theta_0$  is unknown. We further assume that  $\Omega = I$ , which is innocuous because we can always rotate the system without altering the structure ([Hansen, 2007b](#)). More details are provided in [Appendix G.1](#).

For any given  $\theta$ , we define

$$m_t(\theta) \equiv m(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta) \quad \text{and} \quad m_t^{(1)}(\theta^{(1)}) \equiv m^{(1)}(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta^{(1)}),$$

and denote the empirical moment functions for the full and baseline GMM models by

$$\hat{m}_n(\theta) \equiv \frac{1}{n} \sum_{t=1}^n m_t(\theta) \quad \text{and} \quad \hat{m}_n^{(1)}(\theta^{(1)}) \equiv \frac{1}{n} \sum_{t=1}^n m_t^{(1)}(\theta^{(1)}), \quad \text{respectively.}$$

Then, the efficient GMM estimator  $\hat{\theta}_n$  of the full model and that of the baseline model  $\tilde{\theta}_n^{(1)}$  minimize

$$(25) \quad J(\theta, \mathbf{y}^n) \equiv n |\hat{m}_n(\theta)|^2 \quad \text{and} \quad J^{(1)}(\theta^{(1)}, \mathbf{y}^n) \equiv n |\hat{m}_n^{(1)}(\theta^{(1)})|^2, \quad \text{respectively.}$$

### 3.3. Information Matrices Based on Unconditional Moment Restrictions

We now introduce information matrices for the GMM models. In statistics and econometrics, information regarding model parameters is often quantified by the efficiency bound on parameter estimators. One example is the Fisher information matrix for a given parametric family of likelihood functions, which is justified by the Cramér-Rao efficiency bound under the minimax criterion. The same idea can be extended to semiparametric models (e.g., [Bickel, Klaassen, Ritov, and Wellner, 1993](#)). [Hansen \(1985\)](#) and [Chamberlain \(1987\)](#) study semiparametric local minimax efficiency bounds for both conditional and unconditional moment restrictions. [Hansen \(1985\)](#) derives the efficiency bounds from the perspective of characterizing the optimal instrument in the estimation of generalized instrumental variables. [Chamberlain \(1987\)](#) focuses on moment restrictions parameterized in terms of a finite-dimensional vector, as do we, but posed in an i.i.d. context.

In this paper, we extend the semiparametric efficiency bound result for unconditional moment restrictions of [Chamberlain \(1987\)](#) from i.i.d. data-generating processes to Markov processes with local instability. In [Appendix B](#), we show that the optimal GMM covariance matrix derived by [Hansen \(1982\)](#) achieves the semiparametric minimax efficiency bound for unconditional moment restrictions with Markov data-generating processes that are locally unstable.

We denote

$$(26) \quad D \equiv D(\theta_0) \text{ and } D_{ij} \equiv D_{ij}(\theta_0) \text{ for } i, j = 1, 2.$$

Then, the information matrices for  $\theta^{(1)}$  in the baseline model and for  $\theta$  in the full model, evaluated at  $\theta_0^{(1)}$  and  $\theta_0$ , respectively, are

$$(27) \quad \mathbf{I}_B = D_{11}^T D_{11} \text{ and } \mathbf{I}_Q = D^T D = \begin{bmatrix} D_{11}^T D_{11} + D_{21}^T D_{21} & D_{21}^T D_{22} \\ D_{22}^T D_{21} & D_{22}^T D_{22} \end{bmatrix}.$$

We define the marginal information matrix for  $\theta^{(1)}$  in the full model, evaluated at  $\theta_0^{(1)}$ , as

$$(28) \quad \mathbf{I}_F = [\Gamma_{\theta,1} \mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T]^{-1}, \text{ where } \Gamma_{\theta,1} \equiv [I, 0_{d_{\theta,1} \times (d_{\theta} - d_{\theta,1})}],$$

which accounts for the uncertainty concerning the nuisance parameters  $\theta^{(2)}$  when gauging the information about  $\theta^{(1)}$  provided by the moment restrictions. Based on the inversion rule of partitioned matrices, the marginal information matrix  $\mathbf{I}_F$  can be rewritten as

$$(29) \quad \mathbf{I}_F = D_{11}^T D_{11} + D_{21}^T \Lambda_2 D_{21}, \text{ with } \Lambda_2 \equiv I - D_{22} (D_{22}^T D_{22})^{-1} D_{22}^T.$$

Although our objective is to measure model fragility, in Section 3.4 we introduce a measure of the incremental informativeness of the moment restrictions about model parameters. We then argue that this informativeness measure is intuitively connected to the notion of model sensitivity described earlier in Section 2, and we shall formally establish the link between our informativeness measure and model fragility in Sections 4 and 5.

### 3.4. The Dark Matter Measure

In this section, we ask how informative the moment restrictions are regarding the model parameters  $\theta^{(1)}$ . Since  $\theta^{(1)}$  appears in both the baseline moment restrictions and the additional moment restrictions in the full model, the cross-equation restrictions provide additional information about  $\theta^{(1)}$  above and beyond the baseline model. The informativeness of the moment restrictions naturally depends on the sensitivity of the moments to changes in model parameters. If a small change in the parameter values can dramatically change the value of the moment

function (i.e., high sensitivity), then imposing the moment restrictions empirically will tend to greatly restrict the parameter estimates (i.e., the moment restrictions are informative).

Before introducing our information measure, we discuss the relevant regularity conditions, including smoothness, rank, and identification.

**ASSUMPTION 1 (GMM Regularity Conditions)** *We assume that the moment function  $m(\cdot, \theta)$ , defined on a compact set  $\Theta$ , satisfies the following regularity conditions:*

- (i) *The moment restrictions are over-identified:  $d_\theta < d_m$ ;*
- (ii)  *$E^{Q_0}[m_t^{(1)}(\theta^{(1)})] = 0$  and  $E^{Q_0}[m_t(\theta)] = 0$  only when  $\theta^{(1)} = \theta_0^{(1)}$  and  $\theta = \theta_0$ ;*
- (iii)  *$m_t(\theta)$  is continuously differentiable in  $\theta$ , and  $D$  has full column rank.*

**REMARK 1** *The compactness of  $\Theta$  and the assumption  $\theta_0 \in \text{int}(\Theta)$  are the standard regularity conditions to ensure the uniform law of large numbers (ULLN) and the first-order-condition characterization of GMM estimators, respectively. Condition (i) is the standard over-identification condition in GMM (Hansen, 1982). Condition (ii) is also a standard identification assumption to ensure that the sequence of GMM estimators has a unique limit (Hansen, 1982). Condition (iii) is the rank condition for moment restrictions, and is the sufficient condition for local identification enabling us to consistently estimate  $\theta_0$ .*

We now introduce our dark matter measure.

**DEFINITION 1** *Let the incremental information matrix of the full model relative to the baseline models be*

$$(30) \quad \Pi \equiv \mathbf{I}_F^{1/2} \mathbf{I}_B^{-1} \mathbf{I}_F^{1/2} - I.$$

*The dark matter measure is defined as the largest eigenvalue of  $\Pi$ , denoted by<sup>14</sup>*

$$(31) \quad \varrho(\theta_0) \equiv \max_{|\mathbf{v}|=1} \mathbf{v}^T \Pi \mathbf{v}.$$

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<sup>14</sup>We focus on the one-dimensional worst-case fragility. There are straightforward extensions to the cases in which  $\mathbf{v}$  is a matrix.

To better understand the dark matter measure, we rewrite it as

$$(32) \quad \varrho(\theta_0) = \max_{|\mathbf{v}|=1} \frac{\mathbf{v}^T \mathbf{I}_B^{-1} \mathbf{v}}{\mathbf{v}^T \mathbf{I}_F^{-1} \mathbf{v}} - 1.$$

As Equation (32) shows, our measure effectively compares the asymptotic covariance matrices of the two estimators of  $\theta^{(1)}$ : the matrix based on the baseline model, and the matrix based on the full model. It is the largest ratio, over all possible directions  $\mathbf{v} \in \mathbb{R}^{d_{\theta,1}}$ , of the two asymptotic variances of the efficient GMM estimator for a linear combination of model parameters  $\mathbf{v}^T \theta^{(1)}$  under the baseline and full models.

Equation (32) shows that the dark matter measure has a natural “effective-sample-size” interpretation. This equation gives the minimum sample size required for the estimator of the baseline model to match the asymptotic precision (the inverse of the variance) of the estimator of the full structural model in all directions of the parameter space. Because asymptotic variance scales inversely with the sample size, the effective sample size is  $n [\varrho(\theta_0) + 1]$ .

Our dark matter measure isolates the information provided by the structural model above and beyond the baseline model. For the same structural model, alternative choices of the baseline model affect the magnitude of the dark matter measure. To this point, we have been silent on the question of how the baseline model should be chosen in relation to the full structural model. In general, there is no hard rule for this choice, beyond the technical requirement that the associated baseline parameters  $\theta^{(1)}$  be identified by the baseline model. Desirable choices of the baseline model depend on which aspects of the model the fragility analysis aims to capture.

#### 4. LOCAL MISSPECIFICATION AND INSTABILITY

A formal analysis of model fragility requires a framework for misspecification and instability. We adopt a statistical method similar to that of [Hansen and Sargent \(2001\)](#): the econometrician treats  $\mathbb{P}_0$  as an approximation of the true data-generating process by taking into account a class of alternative data-generating processes that are statistically difficult to distinguish from  $Q_0$  (i.e. a neighborhood of  $Q_0$  in the space of probability measures) and assuming that the true process lies in such a collection of local alternatives. To model instability, we generalize the local instability framework of [Li and Müller \(2009\)](#) to the semiparametric setting, which provides a general econometric playground within which we analyze GMM model properties.

We first specify the true local data generating process in Subsection 4.1. We then intro-

duce the concept of model misspecification in Subsection 4.2, and extend our framework by incorporating the concept of local instability in Subsection 4.3.

#### 4.1. Local Data-Generating Processes

Our analysis is local in nature. We focus on a calibrated model with model parameter  $\theta_0$  as defined in (20), with the corresponding bivariate marginal distribution  $Q_0$ . To characterize the locally perturbed models, we define the collection of local perturbations of  $Q_0$ , denoted by  $\mathcal{N}(Q_0)$ , as follows. Note that  $\mathcal{N}(Q_0)$  is a subset of  $L^2(Q_0)$ , the space of square-integrable random variables on the probability space  $(\mathcal{Y} \times \mathcal{Y}, \mathcal{F} \otimes \mathcal{F}, Q_0)$ .

**DEFINITION 2** *The collection  $\mathcal{N}(Q_0)$  consists of the one-dimensional parametric family of bivariate distributions  $Q_{s,f}$  indexed by  $s \in (-\epsilon, \epsilon)$  for some  $\epsilon > 0$  and  $f \in L^2(Q_0)$ , such that the path  $Q_{s,f} \in \mathcal{H}$  passes through the probability measure  $Q_0 \in \mathcal{H}$  at  $s = 0$ , and  $Q_{s,f}$  satisfies the smoothness condition (Hellinger-differentiability condition):<sup>15</sup>*

$$(33) \quad \frac{dQ_{s,f}}{dQ_0} = 1 + sf + s\Delta(s),$$

where  $\Delta(s)$  converges to 0 in  $L^2(Q_0)$  as  $s \rightarrow 0$ . Here, we refer to the scalar measurable function  $f \in L^2(Q_0)$  as the “score” of the parametric model  $s \mapsto Q_{s,f}$ .

**PROPOSITION 1** (Necessary Properties of Scores) *If  $f \in L^2(Q_0)$  satisfies (33), then it follows that (i)  $E^{Q_0}[f] = E^{Q_0}[\Delta(s)] = 0$  for all  $s$ , and (ii)  $E^{Q_0}[f(\mathbf{y}, \mathbf{y}')|\mathbf{y}] = E^{Q_0}[f(\mathbf{y}', \mathbf{y})|\mathbf{y}]$ .*

Now, appealing to the concept in Definition 2, we specify the true local data-generating process. We denote the joint distribution of  $\mathbf{y}^n$  corresponding to the bivariate marginal distribution  $Q_0$  by  $\mathbb{P}_{0,n}$ . Deviating from  $\mathbb{P}_{0,n}$ , the true local data-generating process for  $\mathbf{y}^n$  has the joint distribution  $\mathbb{P}_n^*$  with a sequence of bivariate marginal distributions for each consecutive pair

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<sup>15</sup>The smoothness condition (33) is equivalent to the Hellinger-differentiability, shown in Appendix G.2. It is a common regularity condition adopted for (semi)parametric inference (e.g., van der Vaart, 1988).

$(\mathbf{y}_{t-1}, \mathbf{y}_t)$ ,  $Q_n^* \equiv Q_{1/\sqrt{n}, f_{n,t}^*}$ , which is characterized by

$$(34) \quad \frac{dQ_{1/\sqrt{n}, f_{n,t}^*}}{dQ_0} = 1 + \frac{f_{n,t}^*}{\sqrt{n}} + \Delta_n, \text{ where}$$

$$(35) \quad f_{n,t}^* \equiv g^*(\mathbf{y}_{t-1}, \mathbf{y}_t)^T \begin{bmatrix} 1 \\ b^*(t/n) \end{bmatrix} \text{ and } \sqrt{n}\Delta_n \rightarrow 0 \text{ in } L^2(Q_0).$$

The vector  $g^*$  has two elements:  $g^* = [g_1^*, g_2^*]^T$  with  $g_1^*, g_2^* \in L^2(Q_0)$ . In other words,  $f_{n,t}^* = g_1^*(\mathbf{y}_{t-1}, \mathbf{y}_t) + g_2^*(\mathbf{y}_{t-1}, \mathbf{y}_t)b^*(t/n)$  where  $g_1^*$  represents time-invariant perturbation, while  $g_2^*$  multiplied by  $b(t/n)$  represents time-varying perturbation (i.e., local instability). The unknown function  $b^*(\cdot)$  is a deterministic function on  $[0, 1]$  that generates local instability.<sup>16</sup> When  $n$  is large,  $1 + f_{n,t}^*/\sqrt{n}$  is approximately the Radon-Nikodym density of  $Q_{1/\sqrt{n}, f_{n,t}^*}$  with respect to  $Q_0$ .

Prior to imposing additional regularity conditions on the true score  $f_{n,t}^*$ , we define a set of square-integrable variables corresponding to  $Q_0$ .

DEFINITION 3 (Set of Scores) *For  $Q_0 \in \mathcal{Q}$ , define*

$$(36) \quad L_0^2(Q_0) \equiv \{ \varsigma \in L^2(Q_0) : E^{Q_0} [\varsigma(\mathbf{y}, \mathbf{y}')] = 0 \text{ and } E^{Q_0} [\varsigma(\mathbf{y}, \mathbf{y}')|\mathbf{y}] = E^{Q_0} [\varsigma(\mathbf{y}', \mathbf{y})|\mathbf{y}] \}.$$

Given the notation  $L_0^2(Q_0)$ , the necessary conditions for scores derived in Proposition 1 can be restated as  $f \in L_0^2(Q_0)$ . We then make the following assumption about the true  $f_{n,t}^*$ .

ASSUMPTION 2 (Local Data-Generating Process) *The true local data-generating process in (34) satisfies the following conditions:*

(i)  $g^* \in \mathcal{G}(Q_0)$ , which is defined as

$$\mathcal{G}(Q_0) \equiv \{ g = [g_1, g_2]^T : E^{Q_0} [g_2(\mathbf{y}, \mathbf{y}')|\mathbf{y}] = 0 \text{ and } g_1, g_2 \in L_0^2(Q_0) \};$$

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<sup>16</sup>Similar to, for example, Andrews (1993), Sowell (1996), and Li and Müller (2009), we assume instability to be non-stochastic in contrast to, for example, Stock and Watson (1996, 1998), Primiceri (2005), and Cogley and Sargent (2005). The assumption is for technical simplicity. We can extend from non-stochastic to stochastic instability following the arguments in Li and Müller (2009).



(ii)  $b^* \in \mathcal{B}$ , which is defined as

$$\mathcal{B} \equiv \left\{ b : \begin{array}{l} |b(u)| \leq 1 \text{ for all } u \in [0, 1] \text{ and } \int_0^1 b(u) du = 0, \text{ whose path has a} \\ \text{finite number of discontinuities and one-sided limits everywhere.} \end{array} \right\}.$$

REMARK 2 *The first part of Assumption 2 (i) implies that  $E^{Q_0} [f_{n,t}^*(\mathbf{y}, \mathbf{y}') | \mathbf{y}] = E^{Q_0} [g_1^*(\mathbf{y}, \mathbf{y}') | \mathbf{y}]$  is invariant over time, which further ensures that the univariate marginal distribution of the true joint distribution  $\mathbb{P}_n$  is invariant over time (Proposition 3). The second part of Assumption 2 (i) that  $g_1^*, g_2^* \in L_0^2(Q_0)$  is not restrictive since it is guaranteed by Proposition 1.*

Next, we impose additional assumptions about the heteroskedasticity of the locally unstable data-generating process under consideration, thereby extending the statistical setting of Andrews (1993), Sowell (1996) and Li and Müller (2009) to the semiparametric setting.

ASSUMPTION 3 *(Tail Properties of Local Instability) As  $n \rightarrow \infty$ , it holds that under  $Q_0$*

(i)  $n^{-1} \max_{1 \leq t \leq n} |g(\mathbf{y}_{t-1}, \mathbf{y}_t)|^2 = o_p(1)$ ;

(ii)  $E^{Q_0} [|g(\mathbf{y}_{t-1}, \mathbf{y}_t)|^{2+\nu}] < \infty$ , for some  $\nu > 0$ .

REMARK 3 *Condition (i) is needed for establishing the results on the law of large numbers (LLN) of Lemma 4 of Li and Müller (2009), which we use throughout in our proofs. Condition (ii) implies  $n^{-1} \sum_{t=1}^n E_{t-1}^{Q_0} [|g(\mathbf{y}_{t-1}, \mathbf{y}_t)|^{2+\nu}] = O_p(1)$  and  $n^{-1} \sum_{t=1}^n |g(\mathbf{y}_{t-1}, \mathbf{y}_t)|^{2+\nu} = O_p(1)$ . Condition (ii) is needed for establishing the local asymptotic normality (LAN) for time-inhomogeneous Markov processes (see Proposition 5 in Appendix A) and thus ensuring that the locally unstable data-generating process is contiguous to the stable data-generating process (see Corollary 2 in Appendix A). Condition (ii) is also a commonly adopted assumption (e.g., Li and Müller, 2009, Lemma 1). A direct implication of Assumption 3 is the LLN and CLT of partial summations of score functions.*

Finally, we extend the global identification condition in Assumption 1 (ii) from the reference distribution  $Q_0$  to its perturbations.

ASSUMPTION 4 *(Global Identification Condition) There exists  $\epsilon > 0$  such that  $\vartheta(Q_{s,f})$  is unique if it exists, for all  $Q_{s,f} \in \mathcal{N}(Q_0)$  with the Hellinger distance  $\mathbf{H}^2(Q_{s,f}, Q_0) < \epsilon$ .*

REMARK 4 We define the collection of perturbed distributions  $\mathcal{N}(\mathbf{Q}_0)$  in Definition 2, and the mapping  $\vartheta(\cdot)$  in (19). This assumption ensures that the sequence of GMM estimators has a unique limit when the true distribution is a perturbation of  $\mathbf{Q}_0$ .

#### 4.2. Misspecification of GMM Models

##### **Regularity conditions on moments.**

ASSUMPTION 5 (Tail Properties of Moments) We assume that the moment function  $m(\cdot, \theta)$ , defined on a compact set  $\Theta$ , satisfies the following conditions:

- (i)  $\mathbb{E}^{\mathbf{Q}_0} [|m_t(\theta_0)|^{2+\nu}] < \infty$  for some  $\nu > 0$ , and  $\mathbb{E}^{\mathbf{Q}_0} [\sup_{\theta \in \Theta} \|\nabla_{\theta} m_t(\theta)\|_S^2] < \infty$ ,
- (ii)  $n^{-1/2} \max_{1 \leq t \leq n} |m_t(\theta_0)| = o_p(1)$ ,
- (iii)  $\sum_{t=1}^{\infty} \sqrt{\mathbb{E}^{\mathbf{Q}_0} [|\gamma_t|^2]} < \infty$ , with  $\gamma_t \equiv \mathbb{E}^{\mathbf{Q}_0} [m_t(\theta_0)|\mathcal{F}_1] - \mathbb{E}^{\mathbf{Q}_0} [m_t(\theta_0)|\mathcal{F}_0]$ ,

where  $\|\cdot\|_S$  is the spectral norm of matrices, and the information set  $\mathcal{F}_t$  is the sigma-field generated by  $\{\mathbf{y}_{t-j}\}_{j=0}^{\infty}$ .

REMARK 5 Conditions (i) and (ii) are needed to establish the functional central limit theorem (invariance principle) of [McLeish \(1975b\)](#) and [Phillips and Durlauf \(1986\)](#). Condition (i) imposes restrictions on the amount of heteroskedasticity allowed in the observed moment series and their gradients, which also ensures the uniform square integrability of the moment function. This condition is commonly adopted in the literature (e.g., [Newey, 1985a](#); [Andrews, 1993](#); [Sowell, 1996](#); [Li and Müller, 2009](#), for similar regularity conditions). Condition (iii) states that the incremental information about the current moments between two consecutive information sets eventually becomes negligible as the information sets recede in history from the current observation. This condition ensures the martingale difference approximation for the temporal-dependent moment function as in [Hansen \(1985\)](#), which plays a key role in analyzing the semiparametric efficiency bound based on unconditional moment restrictions (see Proposition 7 in Appendix A and Theorem 4 in Appendix B).

##### **Tangent space and misspecification characterization.**

For a given  $f \in L_0^2(\mathbf{Q}_0)$ , we further require that the path of locally perturbed distributions satisfies  $\mathbf{Q}_{s,f} \in \mathcal{Q}$  and that  $\vartheta(\mathbf{Q}_{s,f})$ , as a function of  $s$ , is differentiable with respect to  $s$  at

$s = 0$ . The collection of such scores  $f$  is defined as follows:

$$\mathcal{T}(\mathbf{Q}_0) \equiv \left\{ f \in L_0^2(\mathbf{Q}_0) : \begin{array}{l} \exists \text{ a path } \mathbf{Q}_{s,f} \text{ such that } \mathbf{Q}_{s,f} \in \mathcal{Q} \cap \mathcal{N}(\mathbf{Q}_0) \text{ for all } s \in (-\epsilon, \epsilon) \\ \text{for some } \epsilon > 0 \text{ and } \vartheta(\mathbf{Q}_{s,f}) \text{ is differentiable at } s = 0 \end{array} \right\}.$$

We refer to the set  $\mathcal{T}(\mathbf{Q}_0)$  above as the *tangent set* of  $\mathcal{Q}$  at  $\mathbf{Q}_0$ . We further characterize the tangent set  $\mathcal{T}(\mathbf{Q}_0)$  as follows:

$$(37) \quad \mathcal{T}(\mathbf{Q}_0) = \{ f \in L_0^2(\mathbf{Q}_0) : \lambda(f) \in \text{lin}(D) \}, \text{ with } \lambda(f) \equiv E^{\mathbf{Q}_0} [m(\cdot, \theta_0)f],$$

where  $\lambda(f)$  is a linear operator on  $L_0^2(\mathbf{Q}_0)$  and linear space  $\text{lin}(D)$  is spanned by the columns of the Jacobian matrix  $D$  defined in (26). This characterization is standard in the literature (e.g., [Severini and Tripathi, 2013](#); [Chen and Santos, 2018](#)) and can be proved using an implicit function theorem. Equation (37) implies that  $\mathcal{T}(\mathbf{Q}_0)$  is a linear space. Whenever  $f_{n,t}^* \in L_0^2(\mathbf{Q}_0) \setminus \mathcal{T}(\mathbf{Q}_0)$ , the GMM model  $\mathcal{Q}$  is *locally misspecified* with respect to the true local data-generating process  $(\mathbf{y}_{t-1}, \mathbf{y}_t)$ ,  $\mathbf{Q}_n^* \equiv \mathbf{Q}_{1/\sqrt{n}, f_{n,t}^*}$ , defined in (34).

One direct implication of (37) is that, if  $d_\theta < d_m$ , then  $\mathcal{T}(\mathbf{Q}_0) \neq L_0^2(\mathbf{Q}_0)$ , and thus the distribution  $\mathbf{Q}_0$  is *locally overidentified* by  $\mathcal{Q}$  ([Chen and Santos, 2018](#)); further, if  $d_m = d_\theta$ , then  $\mathcal{T}(\mathbf{Q}_0) = L_0^2(\mathbf{Q}_0)$ , and thus  $\mathbf{Q}_0$  is *locally just identified* by  $\mathcal{Q}$ .

Similar to (37), the tangent set of the baseline GMM model  $\mathcal{Q}^{(1)}$  at  $\mathbf{Q}_0$  is characterized by

$$(38) \quad \mathcal{T}^{(1)}(\mathbf{Q}_0) \equiv \{ f \in L_0^2 : \lambda^{(1)}(f) \in \text{lin}(D_{11}) \}, \text{ with } \lambda^{(1)}(f) \equiv E^{\mathbf{Q}_0} [m^{(1)}(\cdot, \theta_0^{(1)})f],$$

where operator  $\lambda^{(1)}(f)$  is a linear operator on  $L_0^2(\mathbf{Q}_0)$ , linear space  $\text{lin}(D_{11})$  is spanned by the column vectors of  $D_{11}$ , and  $m^{(1)}(\cdot, \theta_0^{(1)})$  contains the top  $d_{m,1}$  elements of  $m(\cdot, \theta_0)$ .

#### 4.3. Local Instability of Data-Generating Processes

To formalize the analysis on out-of-sample fit (i.e. external validity), we need to consider data-generating processes that allow for structural breaks in a non-stationary manner. More precisely, we consider the set  $\mathcal{M}(\mathbf{Q}_0)$  consisting of all probability measures, each of which is a joint distribution for  $\mathbf{y}^n$  following a Markov process with local instability around a Markov process characterized by  $\mathbf{Q}_0$ . Now, we formalize the definition of  $\mathcal{M}(\mathbf{Q}_0)$  as follows.

DEFINITION 4 *The collection  $\mathcal{M}(\mathbf{Q}_0)$  contains all join distributions  $\mathbb{P}_{1/\sqrt{n},g,b}$ , for local Markov data-generating processes, characterized by a sequence of bivariate marginal distributions  $\mathbf{Q}_{s,f_{n,t}} \in \mathcal{N}(\mathbf{Q}_0)$  with  $t = 1, 2, \dots, n$  and index  $s \in (-\epsilon, \epsilon)$  for some  $\epsilon > 0$  such that*

$$(39) \quad f_{n,t} = g(\mathbf{y}_{t-1}, \mathbf{y}_t)^T \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix} \text{ with } g \in \mathcal{G}(\mathbf{Q}_0) \text{ and } b \in \mathcal{B}.$$

*The unique corresponding model parameter value is also time-varying:*

$$(40) \quad \theta_{n,t} \equiv \vartheta(\mathbf{Q}_{1/\sqrt{n},f_{n,t}}), \text{ for any } f_{n,t} \in \mathcal{T}(\mathbf{Q}_0) \text{ with } 1 \leq t \leq n \text{ and sufficiently large } n.$$

Definition 4 says that all the local data-generating processes in  $\mathcal{M}(\mathbf{Q}_0)$  are characterized by the pair  $(g, b) \in \mathcal{G}(\mathbf{Q}_0) \times \mathcal{B}$  and sample size  $n$ . The data-generating process is a time-homogeneous Markov process if  $b(u) \equiv 0$  or  $g_2(\mathbf{y}, \mathbf{y}') \equiv 0$ . Assumption 4 ensures the uniqueness of (40).

### **Local moment biases.**

Under the local data-generating process  $\mathbb{P}_{1/\sqrt{n},g,b}$  characterized by a sequence of bivariate marginal distributions  $\mathbf{Q}_{1/\sqrt{n},f_{n,t}}$  for  $t = 1, \dots, n$ , the moment restrictions evaluated at  $\theta_0$  are locally biased. We summarize the result in Proposition 2 with the proof in Appendix D.

PROPOSITION 2 (Local Biases of Moment Restrictions) *Suppose Assumptions 1 – 5 hold. Under the bivariate marginal distribution  $\mathbf{Q}_{1/\sqrt{n},f_{n,t}} \in \mathcal{M}(\mathbf{Q}_0)$  for the consecutive pair  $(\mathbf{y}_{t-1}, \mathbf{y}_t)$  where  $f_{n,t} = g_1(\mathbf{y}_{t-1}, \mathbf{y}_t) + g_2(\mathbf{y}_{t-1}, \mathbf{y}_t)b(t/n)$  and  $(g, b) \in \mathcal{G}(\mathbf{Q}_0) \times \mathcal{B}$ , the moment restrictions evaluated at  $\theta_0$  are locally biased:*

$$(41) \quad \mathbb{E}^{\mathbf{Q}_{1/\sqrt{n},f_{n,t}}} [m_t(\theta_0)] = \frac{\lambda(g_1) + \lambda(g_2)b(t/n)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where the linear operator  $\lambda(\cdot)$  is defined in (37).

## 5. WHY IS MODEL FRAGILITY A CONCERN?

We specify the notion of model fragility using high sensitivity of moment restrictions to local perturbations in the data-generating process. Using the semiparametric framework in Section 4, we now show that a fragile model lacks internal refutability and external validity. More

specifically, we show that our dark matter measure is inversely linked to the power of the  $C$  test (i.e. internal refutability) and the out-of-sample fit (i.e. external validity) in Sections 5.1 and 5.2, respectively.

The baseline GMM model plays a special role in the test power and out-of-sample fit analyses as a “benchmark”, characterizing the correct baseline parameter values  $\theta_{n,t}^{(1)}$  and discipline the asset pricing cross-equation restrictions.

**ASSUMPTION 6 (Correct Baseline GMM Model)** *We assume that the true local data-generating process with a joint distribution  $\mathbb{P}_{1/\sqrt{n}, g^*, b^*}$  satisfies*

$$(42) \quad \lambda^{(1)}(g_1^*) = 0 \text{ and } \lambda^{(1)}(g_2^*) \in \text{lin}(D_{11}),$$

where the linear operator  $\lambda^{(1)}(\cdot)$  is defined in (38).<sup>17</sup>

**REMARK 6** *Assumption 6 ensures that the baseline GMM model is correctly specified since  $\lambda^{(1)}(f_{n,t}) = \lambda^{(1)}(g_2)b(t/n) \in \text{lin}(D)$  for every  $t \in \{1, \dots, n\}$ . If we define*

$$(43) \quad \mathcal{G}_B(Q_0) \equiv \{g \in \mathcal{G}(Q_0) : \lambda^{(1)}(g_1) = 0 \text{ and } \lambda^{(1)}(g_2) \in \text{lin}(D_{11})\},$$

*Assumption 6 can be simply rewritten as  $g^* \in \mathcal{G}_B(Q_0)$ .*

The following corollary shows that the correct baseline parameters are invariant under Assumption 6.

**COROLLARY 1 (Correct Baseline Parameters)** *Suppose Assumptions 1 – 6 hold. Then, the correct baseline parameters  $\theta_{n,t}^{(1)} \equiv \vartheta^{(1)}(Q_{1/\sqrt{n}, f_{n,t}})$  exists for  $f_{n,t} = g_1(\mathbf{y}_{t-1}, \mathbf{y}_t) + g_2(\mathbf{y}_{t-1}, \mathbf{y}_t)b(t/n)$  with  $1 \leq t \leq n$ , and they can be approximated by*

$$(44) \quad \theta_{n,t}^{(1)} - \theta_0^{(1)} = -\frac{(D_{11}^T D_{11})^{-1} D_{11}^T \lambda^{(1)}(f_{n,t})}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

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<sup>17</sup>We can replace (42) by a seemingly weaker assumption  $\lambda^{(1)}(g_1^*), \lambda^{(1)}(g_2^*) \in \text{lin}(D_{11})$ . But, this does not add generality, because we can always replace  $\theta_0$  by a sequence of new reference points (reparametrization) to ensure that (42) is satisfied.

5.1. *Test Power and Dark Matter*

We now establish the connection between the dark matter measure and the local asymptotic maximin power. We focus on a subset of alternatives satisfying  $b \equiv 0$  to obtain an upper bound on the maximin local power, which is characterized by the dark matter measure. A specification test for a GMM model  $\mathcal{Q}$  against its baseline GMM model  $\mathcal{Q}^{(1)}$  is a test of the null hypothesis that there exists some parameter for which all moment restrictions hold under the true data-generating process against the alternative that there exists some parameter for which only the baseline moment restrictions hold. That is,

$$(45) \quad H_0 : \mathcal{Q}_n^* \in \mathcal{Q} \quad vs. \quad H_A : \mathcal{Q}_n^* \in \mathcal{Q}^{(1)} \setminus \mathcal{Q}.$$

Let  $\check{\varphi}_n$  be an arbitrary GMM test statistic that maps  $\mathbf{y}^n$  to  $[0, 1]$  (e.g. Hansen, 1982; Newey, 1985a). We restrict our attention to GMM specification tests  $\check{\varphi}_n$  that have local asymptotic level  $\alpha$  and possess an asymptotic local power function.<sup>18</sup> More precisely, we consider the local data-generating process  $\mathbb{P}_{1/\sqrt{n}, g, 0}$  for  $\mathbf{y}^n$  with a bivariate marginal distribution  $\mathcal{Q}_{1/\sqrt{n}, g_1}$  that converges to  $\mathcal{Q}_0 \in \mathcal{Q}(\theta_0)$  as  $n \rightarrow \infty$ .

The test  $\check{\varphi}_n$  has a local asymptotic level  $\alpha$  if

$$(46) \quad \limsup_{n \rightarrow \infty} \int \check{\varphi}_n d\mathbb{P}_{1/\sqrt{n}, g, 0} \leq \alpha, \quad \forall g \in \mathcal{G}(\mathcal{Q}_0) \text{ such that } g_1 \in \mathcal{T}(\mathcal{Q}_0),$$

and the test  $\check{\varphi}_n$  has a local asymptotic power function  $q(g, \check{\varphi})$  if

$$(47) \quad q(g, \check{\varphi}) \equiv \lim_{n \rightarrow \infty} \int \check{\varphi}_n d\mathbb{P}_{1/\sqrt{n}, g, 0}, \quad \forall g \in \mathcal{G}(\mathcal{Q}_0) \text{ such that } g_1 \in \mathcal{T}^{(1)}(\mathcal{Q}_0),$$

where  $\check{\varphi} \equiv \{\check{\varphi}_n\}_{n \geq 1}$  is the sequence of test statistics.

Finally, a test  $\check{\varphi}_n$  for (45) with a local asymptotic power function  $q(\cdot, \check{\varphi}) : \mathcal{G}(\mathcal{Q}_0) \rightarrow [0, 1]$  is said to be *locally unbiased* if  $q(g, \check{\varphi}) \leq \alpha$  for all  $g$  such that  $g_1 \in \mathcal{T}(\mathcal{Q}_0)$ , and  $q(g, \check{\varphi}) \geq \alpha$  for all  $g$  such that  $g_1 \in L_0^2(\mathcal{Q}_0) \setminus \mathcal{T}(\mathcal{Q}_0)$ . We denote the set of locally unbiased GMM specification

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<sup>18</sup>As the sample size  $n$  approaches infinity, the distance between the null hypothesis and the data-generating process necessarily diminishes according to  $n^{-1/2}$ . If this distance were held fixed, then the power of all consistent tests would tend to unity as  $n$  increases to infinity. Local power analysis, the evaluation of the behavior of the power function of a hypothesis test in a neighborhood of the null hypothesis invented by Neyman (1937), has become an important and commonly utilized technique in econometrics (e.g., Newey, 1985b; Davidson and MacKinnon, 1987; Saikkonen, 1989; McManus, 1991).

tests with level  $\alpha$  as  $\Phi_\alpha(Q_0)$ .

The guaranteed local asymptotic power of tests, over all feasible local data-generating processes, can be characterized by the power of maximin tests (e.g., [Lehmann and Romano, 1996](#), Chapter 8). Studies have demonstrated that the  $C$  test or incremental  $J$  test (e.g., [Eichenbaum, Hansen, and Singleton, 1988](#)) has the asymptotic optimality property in the maximin sense (e.g., [Newey, 1985a](#); [Chen and Santos, 2018](#)).<sup>19</sup> Based on this observation, we establish Theorem 1 below, which formally connects the maximin optimal power of tests to the dark matter measure. We present the proof in Appendix C.

We consider the set of alternatives:

$$(48) \quad \mathcal{A}_\kappa(Q_0) \equiv \{g \in \mathcal{G}_B(Q_0) : |\lambda^{(2)}(g_1)| \geq \kappa \text{ and } \lambda^{(2)}(g_1) \perp \text{lin}(D_{22})\}$$

where  $\lambda^{(2)}(g_1) \equiv E^{Q_0} [m^{(2)}(\cdot, \theta_0)g_1]$  is the bottom  $d_m - d_{m,1}$  elements of  $\lambda(g_1)$  defined in (37).

**THEOREM 1** *Suppose Assumptions 1 – 6 hold. The local asymptotic power of maximin tests is bounded above by*

$$(49) \quad \sup_{\check{\varphi} \in \Phi_\alpha(Q_0)} \inf_{g \in \mathcal{A}_\kappa(Q_0)} q(g, \check{\varphi}) \leq M_{\frac{d_{m,2}-d_{\theta,2}}{2}} \left( \sqrt{\frac{\kappa^2}{1 + \varrho(\theta_0)}}, \sqrt{c_{1-\alpha}} \right),$$

where  $c_{1-\alpha}$  is the  $1 - \alpha$  quantile of a chi-square distribution with degrees of freedom  $d_{m,2} - d_{\theta,2}$ , and  $M_\gamma(x_1, x_2)$  is the generalized Marcum Q-function. By definition of  $c_{1-\alpha}$ , it holds that

$$(50) \quad M_{\frac{d_{m,2}-d_{\theta,2}}{2}}(0, \sqrt{c_{1-\alpha}}) = \alpha.$$

Therefore, the local asymptotic power of maximin tests vanishes as the dark matter measure rises:

$$(51) \quad \sup_{\check{\varphi} \in \Phi_\alpha(Q_0)} \inf_{g \in \mathcal{A}_\kappa(Q_0)} q(g, \check{\varphi}) \rightarrow \alpha, \text{ as } \varrho(\theta_0) \rightarrow \infty.$$

The generalized Marcum Q-function  $M_\gamma(x_1, x_2)$  is strictly increasing in  $\gamma$  and  $x_1$ , and it is strictly decreasing in  $x_2$  (e.g., [Sun, Baricz, and Zhou, 2010](#), Theorem 1). Intuitive interpretation

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<sup>19</sup>Alternative asymptotically equivalent approaches can be found in the literature (e.g., [Newey, 1985a](#); [Chen and Santos, 2018](#)).

of (49) is that there exists a difficult specific alternative characterized by the score  $g \in \mathcal{A}_\kappa(Q_0)$  such that the power of the optimal locally unbiased GMM specification test with level  $\alpha$  cannot exceed  $M_{\frac{d_{m,2}-d_{\theta,2}}{2}} \left( \sqrt{\frac{\kappa^2}{1+\varrho(\theta_0)}}, \sqrt{c_{1-\alpha}} \right)$ , which is almost  $\alpha$  when  $\varrho(\theta_0)$  is extremely large. The coefficient  $\kappa$  captures the extent to which the alternatives under consideration are separate from the null, and thus, the upper bound for the test power (i.e., the right-hand side of (49)) naturally increases with  $\kappa$ .

### 5.2. Overfitting Tendency and Dark Matter

A common method adopted by economists and statisticians for assessing the external validity of models is to hold out data from the model estimation (e.g., [Schorfheide and Wolpin, 2012](#)). The assessment of external validity serves two important purposes: first, it mitigates the concern of in-sample overfitting (e.g., [Foster, Smith, and Whaley, 1997](#); [Kocherlakota, 2007](#); [Lettau and Van Nieuwerburgh, 2008](#); [Welch and Goyal, 2008](#); [Koijen and Van Nieuwerburgh, 2011](#); [Ferson, Nallareddy, and Xie, 2013](#); [Athey and Imbens, 2017, 2019](#)); and second, it serves as a primary criterion when the goal is long-run prediction (e.g., [Valkanov, 2003](#); [Müller and Watson, 2016](#)). The literature has emphasized that out-of-sample fit evaluation captures model specification uncertainty, model instability, calibration uncertainty, and estimation uncertainty, in addition to the usual uncertainty of future events ([Stock and Watson, 2008](#)).

The holdout approach to selecting a model among competing structural models amounts to splitting the entire time series  $\mathbf{y}^n \equiv \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$  into two non-overlapping subsamples  $\mathbf{y}_e^n \equiv \{\mathbf{y}_1, \dots, \mathbf{y}_{\lfloor \pi n \rfloor}\}$  and  $\mathbf{y}_o^n \equiv \{\mathbf{y}_{\lfloor \pi n \rfloor + 1}, \dots, \mathbf{y}_n\}$  with  $\pi \in (0, 1/2]$ .<sup>20</sup> Here,  $\lfloor x \rfloor$  is the largest integer less than or equal to the real number  $x$ . The first segment  $\mathbf{y}_e^n$  is used as the estimation sample, while the second segment  $\mathbf{y}_o^n$  is used as the holdout sample (e.g., [Schorfheide and Wolpin, 2012](#)).<sup>21</sup> This approach has been commonly adopted in the literature on forecasting and model selection. Further, the holdout approach is also a natural way to investigate the long-run forecast problems in financial and macroeconomic time series, because the salient definition of a long-run forecast is that the prediction horizon is long relative to the sample length of the estimation sample ([Müller and Watson, 2016](#), Section 5.2).

<sup>20</sup>We specify an upper bound for  $\pi$  to prevent the out-of-sample fit problem from becoming trivial. Without loss of generality, we choose the upper bound for  $\pi$  to be  $1/2$ .

<sup>21</sup>The non-overlapping equal-length estimation and holdout subsamples are standard exercises in cross-validation for out-of-sample fit evaluation; in the statistics and machine learning literature,  $\mathbf{y}_e^n$  is also referred to as training sample, and  $\mathbf{y}_o^n$  as testing sample (e.g., [Hastie, Tibshirani, and Friedman, 2001](#), Chapter 7).



To consider the out-of-sample fit of estimated time-series models, we focus on model instability, because the constant misspecification over time would not affect the out-of-sample fit of estimated models based on in-sample data. Thus, in this subsection we focus on the case with  $\lambda(g_1) \in \text{lin}(D)$ .

The idea is to quantify the overfitting tendency as a model property by focusing on the  $J$  statistic as the loss function. We define  $\theta_{n,t}^{(1)} \equiv \vartheta^{(1)}(Q_{1/\sqrt{n}, f_{n,t}})$  for  $t = 1, \dots, n$ , and

$$(52) \quad \theta_{e,n}^{(1)} \equiv \frac{1}{\lfloor \pi n \rfloor} \sum_{t=1}^{\lfloor \pi n \rfloor} \theta_{n,t}^{(1)} \quad \text{and} \quad \theta_{o,n}^{(1)} \equiv \frac{1}{\lfloor (1-\pi)n \rfloor} \sum_{t=\lfloor \pi n \rfloor+1}^n \theta_{n,t}^{(1)}.$$

More precisely, we consider the goodness-of-fit of the full set of moments under any given baseline parameters  $\theta^{(1)}$ :

$$(53) \quad \mathcal{L}(\theta^{(1)}; \mathbf{y}_s^n) \equiv J(\theta^{(1)}, \psi_s(\theta^{(1)}), \mathbf{y}_s^n) - J(\theta_{s,n}^{(1)}, \psi_s(\theta_{s,n}^{(1)}), \mathbf{y}_s^n), \quad \text{with } s \in \{e, o\}$$

where  $\theta_{s,n}^{(1)}$  is the average of those correct baseline parameter values that perfectly fit baseline moment restrictions (see (44)) and  $\psi_s(\theta^{(1)})$  is chosen to minimize the  $J$  statistic while taking  $\theta^{(1)}$  as given:<sup>22</sup>

$$(54) \quad \psi_s(\theta^{(1)}) \equiv \underset{\theta^{(2)}}{\text{argmin}} J(\theta^{(1)}, \theta^{(2)}, \mathbf{y}_s^n) \quad \text{for any fixed } \theta^{(1)} \text{ with } s \in \{e, o\}.$$

In the definition of  $\mathcal{L}(\theta^{(1)}; \mathbf{y}_s^n)$ , we benchmark the goodness-of-fit measure against the  $J$  statistic evaluated at the average of correct baseline parameter values  $\theta_{s,n}^{(1)}$  for  $s \in \{e, o\}$  to control the mechanical influence of instability on the  $J$  statistic. The lower the goodness-of-fit measure  $\mathcal{L}(\theta^{(1)}; \mathbf{y}_s^n)$ , the better the baseline parameter value  $\theta^{(1)}$  fits the moments in the sample  $\mathbf{y}_s^n$  with  $s \in \{e, o\}$ . Importantly, by minimizing over all possible values of the nuisance parameters  $\theta^{(2)}$ , the goodness-of-fit measure  $\mathcal{L}(\theta^{(1)}; \mathbf{y}_s^n)$  captures the best possible fit of the parameter value  $\theta^{(1)}$  only.

We consider a GMM estimator of the baseline parameters  $\theta^{(1)}$ , denoted by  $\check{\theta}_{e,n}^{(1)}$ , based on the estimation sample  $\mathbf{y}_e^n$  and all moment restrictions. We then assess the out-of-sample fit of  $\check{\theta}_{e,n}^{(1)}$  on the holdout sample by looking at the magnitude of the expected out-of-sample fitting

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<sup>22</sup>Mathematically, the formulation (53) and (54) follow the generic recursive GMM estimation procedure in Hansen (2007b) and Hansen (2012).

error  $\int \mathcal{L}(\check{\theta}_{e,n}^{(1)}, \mathbf{y}_o^n) d\mathbb{P}_{1/\sqrt{n},g,b}$ . The overfitting measure of the estimator  $\check{\theta}_{e,n}^{(1)}$  can be defined as the extent to which the out-of-sample fitting error is larger than the in-sample fitting error:

$$(55) \quad \mathcal{O}(\check{\theta}_{e,n}^{(1)}, \mathbf{y}^n) \equiv \frac{1}{2} [\mathcal{L}(\check{\theta}_{e,n}^{(1)}, \mathbf{y}_o^n) - \mathcal{L}(\check{\theta}_{e,n}^{(1)}, \mathbf{y}_e^n)].$$

The asymptotic expected overfitting measure of the sequence of estimators  $\check{\theta}_{e,n}^{(1)}$  is <sup>23</sup>

$$(56) \quad \omega(g, b, \check{\theta}_e^{(1)}) \equiv \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \int \mathcal{O}(\check{\theta}_{e,n}^{(1)}, \mathbf{y}^n) \mathbf{1}_{\{|\mathcal{O}(\check{\theta}_{e,n}^{(1)}, \mathbf{y}^n)| \leq l\}} d\mathbb{P}_{1/\sqrt{n},g,b},$$

where  $\check{\theta}_e^{(1)} \equiv \{\check{\theta}_{e,n}^{(1)}\}_{n \geq 1}$  is a sequence of GMM estimators.  $\omega(g, b, \check{\theta}_e^{(1)})$  quantifies the extent to which the structural model over-fits the data when the true local data-generating process is  $\mathbb{P}_{1/\sqrt{n},g,b}$ . Similar in spirit to information criteria in model selection such as AIC and BIC, models whose expected overfitting measures are sizable compared with those of other models that fit the sample data equally well in sample should be penalized.

### *Two types of estimators*

Here we focus on two particular estimation procedures – the efficient GMM estimation procedure and the recursive GMM estimation procedure.<sup>24</sup> The former is designed to use the identification strength provided by the additional asset pricing moment restrictions  $E^{Q_0} [m_t^{(2)}(\theta)] = 0$  as much as possible, while the latter does not use any identification assumptions imposed by the additional asset pricing moment restrictions  $E^{Q_0} [m_t^{(2)}(\theta)] = 0$ . The identification strength is a nontestable assumption postulated by the structural model. The literature on recursive GMM estimation is substantial and dates back decades (e.g., [Christiano and Eichenbaum, 1992](#); [Ogaki, 1993](#); [Newey and McFadden, 1994](#); [Hansen and Heckman, 1996](#); [Hansen, 2007b](#); [Lee, 2007](#); [Hansen, 2012](#)). While the original impetus of the recursive GMM estimation was primarily computational, we advocate it as a more robust procedure against potential instability and misspecification since the procedure barely relies on the nontestable assumption of identification strength of the additional moment restrictions  $m_t^{(2)}(\theta)$ ; the robustness of the recursive GMM estimation procedure is especially valuable when the dark matter measure is

<sup>23</sup>The method of first calculating the truncated statistic, then letting the ceiling  $l$  increase to infinity, is commonly adopted in the literature for technical simplification (e.g. [Bickel, 1981](#); [Le Cam and Yang, 2000](#); [Kitamura, Otsu, and Evdokimov, 2013](#)).

<sup>24</sup>The recursive GMM estimation procedure is also referred to as the sequential (two-step) GMM estimation procedure in the literature.

excessively large.

Characterized by selection matrices, the efficient GMM estimator and the recursive GMM estimator based on the estimation sample  $\mathbf{y}_e^n$ , denoted by  $\hat{\theta}_{e,n}$  and  $\tilde{\theta}_{e,n}$  respectively, have the selection matrices  $A = D$  and  $A = \text{diag}\{D_{11}, A_{22}\}$  with the (constrained) efficient selection matrix  $A_{22} \equiv [D_{21}(D_{11}^T D_{11})^{-1} D_{21}^T + I]^{-1} D_{22}$  (Hansen, 2007b).

### ***Overfitting of the efficient GMM estimator $\hat{\theta}_{e,n}$ .***

Recall that  $\lambda(g_2)$  captures the magnitude of instability. We consider the set of possible magnitudes of instability  $\mathcal{U}_\kappa(Q_0) \equiv \{g \in \mathcal{G}_B(Q_0) : |\lambda(g_2)| \leq \kappa\}$ . A larger  $\kappa$  allows for a higher degree of instability in the fragility analysis.

**THEOREM 2** *Suppose Assumptions 1 – 6 hold. The overfitting of the efficient GMM estimator  $\hat{\theta}_{e,n}$  based on the estimation sample  $\mathbf{y}_e^n$  is defined as the worst-case asymptotic expected overfitting measure. It is characterized by the dark matter measure:*

$$(57) \quad \sup_{g \in \mathcal{U}_\kappa(Q_0), b \in \mathcal{B}} \omega(g, b, \hat{\theta}_e^{(1)}) = d_{\theta,1} + \varrho(\theta_0) \pi \kappa^2.$$

*Therefore, the asymptotic expected overfitting of the efficient GMM estimator sequence  $\hat{\theta}_e \equiv \{\hat{\theta}_{e,n}\}_{n \geq 1}$  can be arbitrarily large, being linearly related to the dark matter measure.*

The overfitting of the efficient GMM estimator has two sources. The first term  $d_{\theta,1}$  captures the traditional overfitting due to the sampling uncertainty in the estimation sample (see Theorem 3), while the second term  $\varrho(\theta_0) \pi \kappa^2$  captures the overfitting due to potential misspecification and instability. The latter component is the focus of our paper and directly depends on the dark matter measure. The second component,  $\varrho(\theta_0) \pi \kappa^2$ , vanishes if there is no local instability (i.e.,  $\kappa = 0$ ).

### ***Overfitting of the recursive GMM estimator $\tilde{\theta}_{e,n}$ .***

**THEOREM 3** *Suppose Assumptions 1 – 6 hold. The overfitting of the recursive GMM estimator  $\tilde{\theta}_{e,n}$  based on the estimation sample  $\mathbf{y}_e^n$  is defined as the worst-case asymptotic expected*

overfitting measure, which only depends on the number of baseline parameters:

$$(58) \quad \sup_{g \in \mathcal{U}_\kappa(Q_0), b \in \mathcal{B}} \omega(g, b, \tilde{\theta}_e^{(1)}) = d_{\theta,1},$$

where  $\mathcal{U}_\kappa(Q_0) \equiv \{g \in \mathcal{G}_B(Q_0) : |\lambda(g_2)| \leq \kappa\}$ . Therefore, the overfitting of the recursive GMM estimator sequence  $\tilde{\theta}_e \equiv \{\tilde{\theta}_{e,n}\}_{n \geq 1}$  is determined by model parameter dimensionality, not affected by the dark matter of the model.

The results above echo the traditional information criteria such as AIC and BIC, where the number of parameters captures the overfitting tendency due to sampling uncertainty. The recursive GMM estimator is not affected by the nontestable assumptions of identification strength imposed by  $E^{Q_0} [m_t^{(2)}(\theta)] = 0$ , thus its overfitting is not affected by the misspecified (local) instability. Importantly, Theorem 3 suggests that the recursive GMM estimator provides a robust estimator for models with large dark matter measures (i.e. large  $\varrho(\theta_0)$ ) and thus subject to severe (local) instability concerns (i.e. large  $\mathcal{U}_\kappa(Q_0)$ ).

### ***Instability of the efficient GMM estimator.***

Intuitively, the formal results about out-of-sample fit above can be appreciated through the sensitivity of efficient GMM estimators to local instability (see panel C of Figure 1). We consider a local perturbation of the model from  $Q_0$  in the direction of  $g \in \mathcal{G}_B(Q_0)$  with instability  $b \in \mathcal{B}$ . According to Proposition 6 (in Appendix A),

$$(59) \quad \begin{bmatrix} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) \end{bmatrix} \xrightarrow{d} \begin{bmatrix} m_e \\ m_o \end{bmatrix}, \text{ with } E \begin{bmatrix} m_e \\ m_o \end{bmatrix} = \begin{bmatrix} \nu_e(g, b, \pi) \\ \nu_o(g, b, \pi) \end{bmatrix},$$

where  $(m_e, m_o)$  are independent normals with the identity covariance matrix and means:

$$(60) \quad \nu_e(g, b, \pi) \equiv \frac{\lambda(g^T)}{\sqrt{\pi}} \left[ \int_0^\pi b(u) du \right] \text{ and } \nu_o(g, b, \pi) \equiv \frac{\lambda(g^T)}{\sqrt{1-\pi}} \left[ \int_\pi^1 b(u) du \right].$$

Further, Proposition 9 (in Appendix A) shows that the in- and out-of-sample estimators satisfy

$$(61) \quad \begin{bmatrix} \sqrt{\pi n}(\hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)}) \\ \sqrt{(1-\pi)n}(\hat{\theta}_{o,n}^{(1)} - \theta_{o,n}^{(1)}) \end{bmatrix} \xrightarrow{d} \begin{bmatrix} \hat{\theta}_e^{(1)} \\ \hat{\theta}_o^{(1)} \end{bmatrix}, \text{ with } E \begin{bmatrix} \hat{\theta}_e^{(1)} \\ \hat{\theta}_o^{(1)} \end{bmatrix} = -(L_F - L_B) \begin{bmatrix} \nu_e(g, b, \pi) \\ \nu_o(g, b, \pi) \end{bmatrix},$$

where  $L_B \equiv \mathbf{I}_B^{-1} D_{11}^T \Gamma_{m,1}$ ,  $L_F \equiv \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T$ , and  $(\hat{\theta}_e^{(1)}, \hat{\theta}_o^{(1)})$  are independent normals with covariance matrix  $\mathbf{I}_F^{-1}$ . Therefore, the amount of estimator instability (normalized by covariance matrix  $\mathbf{I}_F$ ) as a function of moment instability is

$$(62) \quad \mathbf{I}_F^{1/2} E \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \hat{\theta}_{o,n}^{(1)} \end{bmatrix} = \beta E [m_o - m_e],$$

where  $\beta = -\mathbf{I}_F^{1/2}(L_F - L_B)$  is the so-called sensitivity matrix. The largest sensitivity can be captured by the spectral norm of the sensitivity matrix  $\beta$ ; that is  $\|\beta\|_s = \sqrt{\varrho(\theta_0)}$ . This result resembles that of [Andrews, Gentzkow, and Shapiro \(2017\)](#). Their goal is to determine which moments are most helpful in identifying parameters via sensitivity analysis. In our case, high sensitivity can lead to severe instability of the estimator out of sample versus in sample.

### 5.3. The Dark Matter Measure and Rational Expectations Assumption

Under the rational expectations framework, agents are assumed to know the true data-generating process. The rational expectations (RE) framework fits naturally into our setting, with the baseline model being the statistical model of the data-generating process, and the full model imposing additional restrictions between asset prices and real quantities, which conform with the true data-generating process.

In this case, a high level of the dark matter measure signifies that we can obtain a significant amount of information about the parameter values of the data-generating process from the cross-equation restrictions. Based on the rational expectations assumption, such a situation might be interpreted as evidence that the agents within the economic model have superior information relative to the econometrician. Our analysis offers a complementary perspective, and suggests caution on the grounds of model fragility.

When model parameters are weakly identified in the baseline model relative to the full model, they are essentially fitted to the moment restrictions from the full model. Additional

moment restrictions imposed by the full model may or may not be correctly specified, and, as we show, it is difficult to test their validity statistically within the scope of fragile models – models with high levels of the dark matter measure. Under fragile models, the rational expectations assumption is not effective at disciplining agents’ beliefs.<sup>25</sup> Such models are likely to over-fit the data in sample, with poor out-of-sample performance, especially when the data-generating process is unstable.

## 6. EXAMPLES AND SIMULATION STUDIES

In this section, we use the dark matter measure to analyze a rare disaster model, a widely used asset pricing model. We analyze the long-run risk model, another widely used and more quantitative asset pricing model, in Appendix H. We focus on the fragility of the GMM models capturing the main implications of calibrated structural asset pricing models.

Rare economic disasters are a natural source of “dark matter” in asset pricing models. It is difficult to evaluate the likelihood and magnitude of rare disasters statistically. Yet, agents’ aversion to large disasters can have large ex-ante effects on asset prices. In this subsection, we use our measure to analyze a disaster risk model similar to Barro (2006).

The model specifies the joint dynamics of the log growth rate of aggregate consumption (endowment)  $g_t$  and the excess log return on the market portfolio  $r_t$ . There is an observable state variable  $z_t$ , which follows an i.i.d. Bernoulli distribution and is equal to one with probability  $p$ . When  $z_t = 1$ , the economy is in a disaster regime, while the normal regime corresponds to  $z_t = 0$ . In the normal regime, the log consumption growth  $g_t = u_t$ , which is i.i.d. normal,  $u_t \sim N(\mu, \sigma^2)$ . In a disaster state,  $g_t = -v_t$ , where  $v_t$  follows a truncated exponential distribution with density

$$(63) \quad \nu_t \stackrel{\text{i.i.d.}}{\sim} \mathbf{1}\{v_t > \underline{v}\} \xi e^{-\xi(v_t - \underline{v})}.$$

Here the lower bound for disaster size is  $\underline{v}$  and the average disaster size is  $\underline{v} + 1/\xi$ .

The joint distribution of log consumption growth  $g_t$  and excess log return  $r_t$  changes with the underlying state  $z_t$ . When the economy is in the normal regime ( $z_t = 0$ ),  $g_t$  and  $r_t$  are

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<sup>25</sup>Hansen (2007a) discusses at length the concerns surrounding the informational burden that rational expectations models place on agents, which is one of the key motivations for research in Bayesian learning, model ambiguity, and robustness (e.g., Gilboa and Schmeidler, 1989; Hansen and Sargent, 2001; Epstein and Schneider, 2003; Klibanoff, Marinacci, and Mukerji, 2005).

jointly normal, and

$$(64) \quad r_t = \eta + \rho \frac{\tau}{\sigma} (g_t - \mu) + \sqrt{1 - \rho^2} \tau \varepsilon_{0,t},$$

where  $\varepsilon_{0,t}$  is i.i.d. standard normal. The parameter  $\tau$  is the return volatility in the normal regime, while  $\rho$  is the correlation between return and consumption growth in this regime. When the economy is in a disaster state ( $z_t = 1$ ), the joint distribution of  $g_t$  and  $r_t$  is

$$(65) \quad r_t = \ell g_t + \varsigma \varepsilon_{1,t}$$

where  $\varepsilon_{1,t}$  is i.i.d. standard normal.

Next, we assume that the representative agent has a constant relative risk aversion utility function  $u_t(c_t) = \delta_D^t c_t^{1-\gamma_D} / (1 - \gamma_D)$ , where  $\gamma_D > 0$  is the coefficient of relative risk aversion and  $\delta_D < 1$  is the time preference parameter. The log equity premium,  $\bar{r} \equiv \mathbb{E}[r_t]$ , is available in closed form (see Appendix F for details) as follows:

$$(66) \quad \bar{r}(p, \xi) = (1 - p)\eta - p\ell(\underline{v} + 1/\xi), \quad \text{where}$$

$$(67) \quad \eta = \eta(p, \xi) \approx \gamma_D \rho \sigma \tau - \frac{\tau^2}{2} + e^{\gamma_D \mu - \frac{\gamma_D^2 \sigma^2}{2}} \Delta(\xi) \frac{p}{1 - p}, \quad \text{and}$$

$$(68) \quad \Delta(\xi) = \xi \left( \frac{e^{\gamma_D \underline{v}}}{\xi - \gamma_D} - \frac{e^{\frac{\xi^2}{2} + (\gamma_D - \ell)\underline{v}}}{\xi + \ell - \gamma_D} \right).$$

The term  $\eta$  in (66) is the log equity premium in the normal regime. The first two terms of  $\eta$  in (67) describe the market risk premium due to Gaussian consumption shocks; the third term is the disaster risk premium, which explodes as  $\xi$  approaches  $\gamma_D$  from above. In other words, there is an upper bound on the average disaster size for the equity premium to remain finite, which also limits how heavy the tail of the disaster size distribution can be.

The fact that the equity premium explodes as  $\xi$  approaches  $\gamma_D$  is an important feature of our version of the disaster risk model. No matter how rare the disasters are (i.e., a very small  $p$ ), an arbitrarily large equity premium can be generated as long as the average disaster size is sufficiently large (or equivalently,  $\xi$  is sufficiently small). Extremely rare but large disasters can be consistent with the data in the sense that they are difficult to rule out based on the observable data (and standard statistical tests). Below we illustrate how our dark matter

measure can detect and quantify the fragility of these models.

To apply our framework to the disaster risk model, we first formulate the economic model above as a GMM model  $\mathcal{Q}$  with the (transformed) moments:

$$(69) \quad m_t(\theta) = \Omega(\theta)^{-1/2} \begin{bmatrix} z_t - p \\ g_t - (1 - z_t)\mu + z_t(\underline{v} + 1/\xi) \\ r_t - (1 - z_t) [\eta + \rho \frac{\tau}{\sigma}(g_t - \mu)] - z_t \ell g_t \end{bmatrix}.$$

The first two moments in  $m_t(\theta)$  are the baseline moments, the third is the asset pricing moment, and  $\Omega(\theta)$  is the asymptotic covariance matrix of the untransformed moments. To simplify the example, we focus on the parameters  $\theta = (p, \xi)'$  when constructing the dark matter measure, while treating the parameters  $(\gamma_D, \mu, \sigma, \underline{v}, \tau, \rho, \ell, \varsigma)$  as auxiliary parameters fixed at known values, making them a part of the functional-form specification. In other words, the nuisance parameter vector  $\theta^{(2)}$  is empty in this example.

Based on the approximation (67), the dark matter measure is (see Appendix F for details):

$$(70) \quad \varrho(\theta) \approx 1 + \frac{p\Delta(\xi)^2 + p(1-p)\xi^2\dot{\Delta}(\xi)^2}{(1-\rho^2)\tau^2(1-p)^2} e^{2\gamma_D\mu - \gamma_D^2\sigma^2},$$

where  $\dot{\Delta}(\xi)$  is the first derivative of  $\Delta(\xi)$ , and

$$(71) \quad \dot{\Delta}(\xi) = -\frac{e^{\gamma_D\underline{v}}\gamma_D}{(\xi - \gamma_D)^2} + \frac{e^{(\gamma_D - \ell)\underline{v}}(\gamma_D - \ell)}{(\xi - \gamma_D + \ell)^2} e^{\varsigma^2/2}.$$

All else equal, when  $\xi$  approaches  $\gamma_D$ , both  $\Delta(\xi)$  and  $\dot{\Delta}(\xi)$  approach infinity, which suggests that disaster risk models featuring large but rare disasters (i.e., small  $\xi$  and small  $p$ ) will be more fragile according to our measure.

### ***Quantitative analysis.***

To take the model to the data, we use annual real per-capita consumption growth (non-durables and services) from the National Income and Product Accounts (NIPA) and returns on the CRSP value-weighted market portfolio from 1929 to 2011. We fix the following auxiliary parameters at the values of the corresponding moments of the empirical distribution of consumption growth and excess stock returns:  $\mu = 1.87\%$ ,  $\sigma = 1.95\%$ ,  $\tau = 19.14\%$ ,  $\varsigma = 34.89\%$



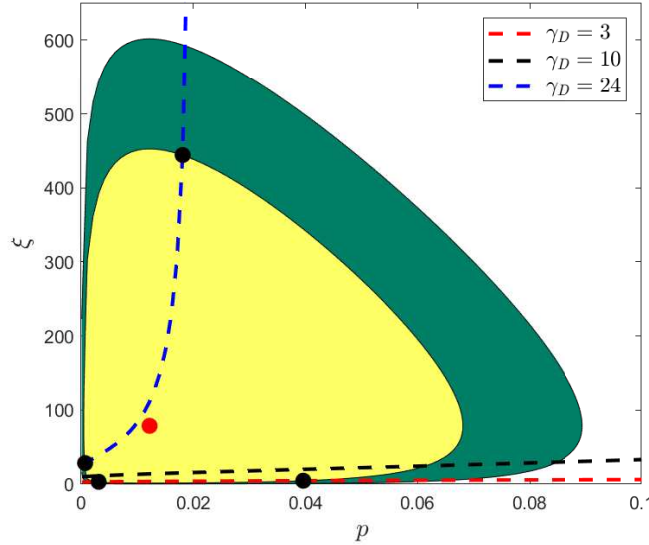


FIGURE 2.— The 95% and 99% confidence regions of  $(p, \xi)$  for the baseline model and the equity premium isoquants implied by the asset pricing moment restriction (66) for  $\gamma_D = 3, 10, 24$ .  $p$  is disaster probability, and  $\xi$  characterizes the inverse of average disaster size. The efficient GMM estimates are  $(\hat{p}, \hat{\xi}) = (0.012, 78.79)$ , indicated by the red dot around the center of the confidence region. Only  $p$  and  $\xi$  are treated as unknown to the econometrician, and all other parameters are treated as auxiliary parameters with fixed known values; therefore, the dark matter measure is defined only based on  $\theta = (p, \xi)$ .

and  $\rho = 0.59$ . The lower bound for disaster size is set to  $\underline{v} = 7\%$ , and the leverage factor in the disaster regime is  $\ell = 3$ .

In Figure 2, we plot the 95% and 99% confidence regions for  $(p, \xi)$  based on the baseline model. As expected, the confidence regions are large, which confirms that the baseline model provides limited information about  $p$  and  $\xi$ . We also plot the equity premium isoquants: for a given level of risk aversion  $\gamma_D$ , each dashed line in Figure 2 shows the different calibrations of the disaster risk model that match the unconditional equity premium of 5.09%. In particular, even for low risk aversion (e.g.,  $\gamma_D = 3$ ), there exist models that not only match the observed equity premium, but are also consistent with the macro data in the sense that the model parameters  $(p, \xi)$  remain inside the 95% confidence region.<sup>26</sup>

While it is difficult to distinguish among a wide range of calibrations based on the fit with the macro data, these calibrated models can differ vastly based on our dark matter measure. For illustration, we focus on the following four calibrations, which are the four points where

<sup>26</sup>Julliard and Ghosh (2012) estimate the consumption Euler equation using the empirical likelihood method and show that the model requires a high level of relative risk aversion to match the equity premium. Their empirical likelihood criterion rules out any large disasters that have not occurred in the historical sample, hence requiring the model to generate high equity premium using moderate disasters.

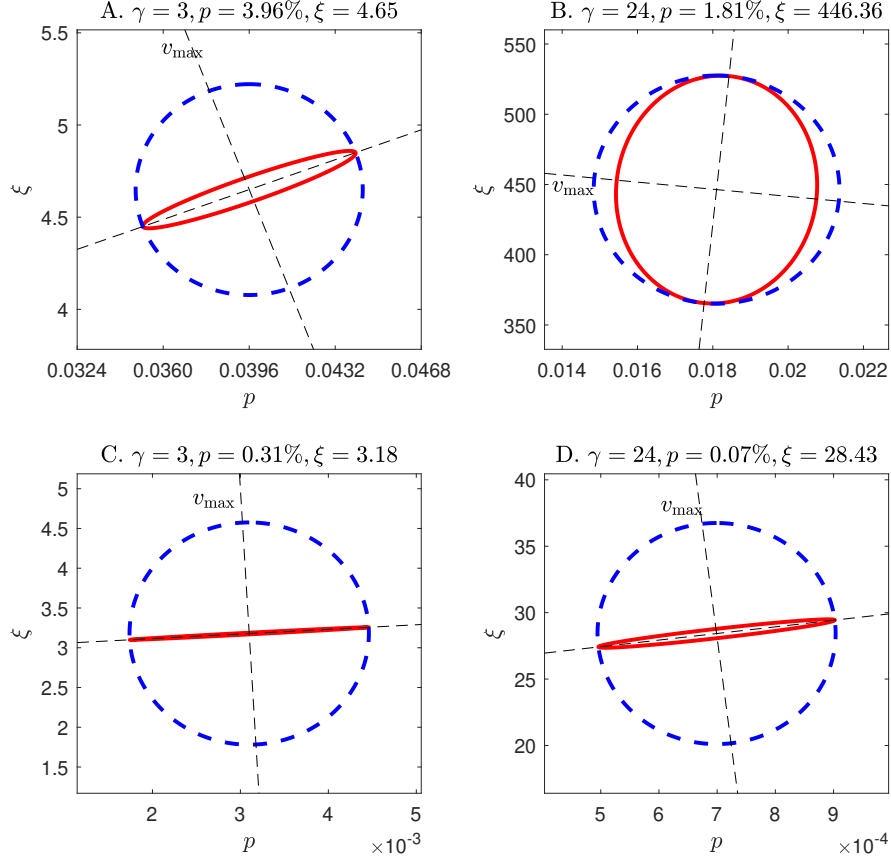


FIGURE 3.— Visualization of the dark matter measure through the 95% confidence regions for the asymptotic distribution of the efficient GMM estimators for four “acceptable” calibrations. In panels A through D, the dark matter measures are  $\varrho(\theta) = 74.03, 1.49, 1.78 \cdot 10^4$ , and  $5.60 \cdot 10^2$ , respectively, which are obtained in the direction marked by the vector  $v_{\max}$ . Parameter  $p$  is disaster probability, and  $\xi$  characterizes the inverse of average disaster size. Only  $p$  and  $\xi$  are treated as unknown to the econometrician, and all other parameters are treated as auxiliary parameters with fixed known values; therefore, the dark matter measure is defined only based on  $\theta = (p, \xi)$ .

the equity premium isoquants for  $\gamma_D = 3$  and 24 intersect the boundary of the 95% confidence region in Figure 2.<sup>27</sup>

With just two parameters in  $\theta = (p, \xi)$ , we can visualize the dark matter measure by plotting the asymptotic confidence regions for  $(p, \xi)$  in the baseline model and the full model, as determined by the respective information matrices  $\mathbf{I}_B$  and  $\mathbf{I}_F$ . In each panel of Figure 3, the largest dashed-line circle marks the 95% asymptotic confidence region for  $(p, \xi)$  under the baseline model. The smaller solid-line ellipse indicates the 95% asymptotic confidence region for  $(p, \xi)$  under the full model. Intuitively, the direction in Figure 3, along which the asset

<sup>27</sup>For  $\gamma_D = 3$ , the two points are  $(p = 3.96\%, \xi = 4.65)$  and  $(p = 0.31\%, \xi = 3.179)$ . For  $\gamma_D = 24$ , the two points are  $(p = 1.81\%, \xi = 446.36)$  and  $(p = 0.07\%, \xi = 28.43)$ .

pricing restriction does not provide additional information about the parameters  $\theta = (p, \xi)$ , is parallel to the tangent direction of the dashed lines (i.e., the equity premium isoquants) in Figure 2, evaluated at the black dots. This highlights the straightforward fact that the structural restriction does not really increase informativeness in the direction along which the equity premium does not change.

In panel A of Figure 3, with  $\gamma_D = 3, p = 3.96\%, \xi = 4.65, \varrho(\theta) = 74.07$ . This means that under the baseline model, we need to increase the amount of consumption data by a factor of 74.07 to match or exceed the precision in estimation of any linear combination of  $p$  and  $\xi$  afforded by the equity premium constraint. Panels C and D of Figure 3 correspond to the calibrations with “extra rare and large disasters,” and for  $\gamma_D = 3$  and 24, the dark matter measure  $\varrho(\theta)$  rises to  $1.78 \times 10^4$  and  $5.60 \times 10^2$ , respectively. If, in panel B of Figure 3, we raise  $\gamma_D$  to 24 while changing the annual disaster probability to 1.81% and lowering the average disaster size to 7.002% ( $\xi = 446.36$ ), the dark matter measure  $\varrho(\theta)$  declines to 1.49. The reason for the reduced fragility in this calibration is the combination of a higher disaster probability and a lower average disaster size.

### *Monte Carlo experiments.*

We use simulations to illustrate the connection between the dark matter measure and the model fragility (i.e., the internal refutability and external validity) of disaster risk models in finite samples. More precisely, we assume that the true local data-generating process has a time-varying relation between the expected log excess return and other dynamic parameters:

$$(72) \quad \bar{r}_n = \bar{r}(p_0, \xi_0) + \frac{\iota_t \delta_r}{\sqrt{n}}, \text{ with } \iota_t = \begin{cases} 1, & \text{when } 1 \leq t \leq \lfloor \pi n \rfloor \\ -1, & \text{when } \lfloor \pi n \rfloor < t \leq n, \end{cases}$$

where the time series  $\iota_t$  captures the structural breaks and  $\pi \in (0, 1/2]$  is the break point. Such a simple process  $\iota_t$  characterizes one structural break in the middle of the time-series sample. The corresponding moment biases, evaluated at  $\theta_0$ , are

$$(73) \quad E^{Q_0} [m_t(\theta_0)] = \begin{bmatrix} 0 \\ 0 \\ \frac{\lambda_t^{(2)}}{\sqrt{n}} \end{bmatrix} \text{ with } \lambda_t^{(2)} \equiv \frac{\iota_t \delta_r}{\sqrt{(1-p_0)(1-\rho^2)\tau^2 + p_0\varsigma^2}}.$$

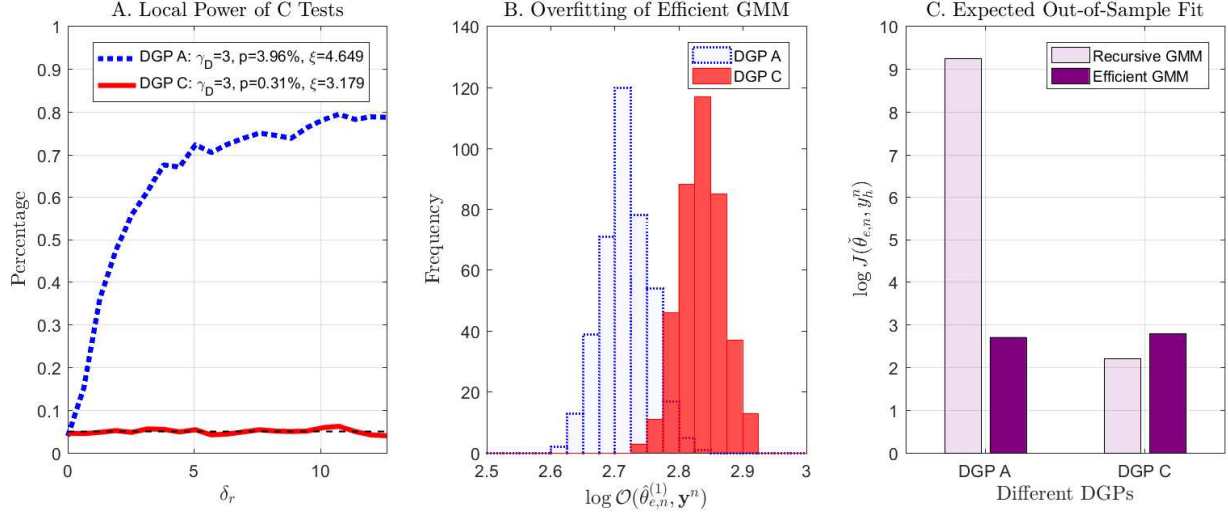


FIGURE 4.— Monte Carlo experiments for disaster risk models. In panel A, we simulate 1000 independent yearly time series with length  $n = 100$  (i.e. 100 years). In panels B and C, we set  $\delta_r = 0.4$ , and simulate 400 independent yearly time series with length  $n = 100$  (i.e. 100 years) and break point  $\pi = 1/2$ . Only  $p$  and  $\xi$  are treated as unknown to the econometrician, and all other parameters are treated as auxiliary parameters with fixed known values; therefore, the dark matter measure is defined only based on  $\theta = (p, \xi)$ .

Therefore, data-generating processes A and C in Figure 3 have identical moment's local biases  $\lambda_t^{(2)}$  after substituting the calibrated parameter values into (73), which guarantees that the comparisons across models in panels A and B of Figure 4 are valid.

Figure 4 shows three different simulation experiments. Panel A displays the local power functions of  $C$  tests. The solid and dotted curves reflect the test powers when the data-generating processes are characterized by calibrations A and C in Figure 3, respectively. In this experiment, we vary the local misspecification  $\delta_r$  in the risk premium moment restriction. The data-generating process under calibration C features an excessively large amount of dark matter according to panel C of Figure 3, and not surprisingly, it has little internal refutability (i.e. little test power) consistent with Theorem 1.

Panel B of Figure 4 displays the histograms of logged overfitting measures  $\log \mathcal{O}(\hat{\theta}_{e,n}^{(1)}, \mathbf{y}^n)$  of efficient GMM estimators for two data-generating processes under calibrations A and C in Figure 3. The estimator  $\hat{\theta}_{e,n}$  is based on the estimation sample  $\mathbf{y}_e^n = \{\mathbf{y}_1, \dots, \mathbf{y}_{[n/2]}\}$ . In this experiment, we specify a structural break in the risk premium in the middle of the time-series sample with  $\delta_r = 0.4$ . Panel B shows that the efficient GMM estimator is likely to be overfitting the data in the calibrated structural model with a high dark matter measure, which is consistent with Theorem 2.

Panel C of Figure 4 compares the expected out-of-sample fit of the recursive GMM estimator  $\tilde{\theta}_{e,n}$  with that of the efficient GMM estimator  $\hat{\theta}_{e,n}$ , based on the estimation sample  $\mathbf{y}_e^n$ . We describe the two types of estimators in Section 5.2. Consistent with the conventional intuition, under the data-generating process A, the efficient GMM estimator yields a better expected out-of-sample fit than the recursive GMM estimator. This is because the additional identification information is reliable and meaningful when the amount of dark matter is not excessively large. In contrast, the recursive GMM estimator delivers a better expected out-of-sample fit under the data-generating process C, which exhibits a much higher dark matter measure. This finding indicates that the concern about misspecification and instability may offset – and even reverse – the efficiency gain from the additional moment restrictions. The result for the data-generating process C suggests that the econometrician should prioritize robustness over efficiency when estimating models that rely heavily on dark matter (i.e., models with an excessively large  $\varrho(\theta_0)$ ).

### *What to do with fragile models?*

Fragile models lack of refutability and are prone to over-fitting the data in sample with poor out-of-sample fit. Our analysis highlights the parameter combinations in which the dark matter is embedded, suggesting which aspects of the model need to be enhanced to make the model useful in practice. One approach is to bring in more data to identify the problematic parameter combinations better under the baseline model, (as done in Barro and Ursúa, 2012; Nakamura, Steinsson, Barro, and Ursúa, 2013, who use international data to better estimate the distribution of consumption disasters). Another approach taken in a number of papers is to modify the preference specification or the belief formation mechanism so that the model parameters are better identified by the baseline moments and do not rely excessively on the restrictions implied by the asset pricing moments (e.g., Hansen and Sargent, 2010; Bidder and Dew-Becker, 2016; Collin-Dufresne, Johannes, and Lochstoer, 2016; Nagel and Xu, 2019). Yet another approach is to extend the model to connect the problematic parameter combinations of the baseline model to additional data – for example, Gârleanu, Panageas, and Yu (2012) and Kung and Schmid (2015) explicitly model production and innovation to endogenize the consumption dynamics, with a particular focus on low-frequency fluctuations. This ties the properties of R&D investment to those of the consumption process.

## 7. CONCLUSION

In this paper, we propose a new tractable measure of model fragility based on quantifying the informativeness of the cross-equation restrictions that a structural model imposes on the model parameters. We argue that our measure quantifies a useful model property related to the model's tendency to over-fit the data in sample.

Our fragility measure should be used as a model selection criterion. When faced with a set of candidate models consistent with available data, selecting the less fragile model can be an appealing criterion from the point of view of out of sample performance. We leave the formal development of model selection based on our measure of model fragility to future research.

Our model fragility measure is easy to implement, and the worst-case direction is particularly instructive. This direction provides guidance on which features of the model are most vulnerable to overfitting. Additional data or model elements would be needed to alleviate this tendency. Further, the worst-case direction is useful to consider when constructing robust estimators based on considerations of out-of-sample fit, e.g., following the idea of recursive GMM estimators.

Our methodology has a broad range of potential applications. In addition to the examples involving asset pricing, our measure can be used to assess the robustness of structural models in other areas of economics, such as industrial organization and corporate finance.

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## Appendix

We first list all additional theoretical results in Appendices A and B. Then, we show the proofs of the theorems in Appendix C and those of the propositions and corollaries in Appendix D. The derivation of the disaster risk model can be found in Appendix F. Other miscellaneous derivations and proofs are collected in Appendix G. In the appendix, we denote  $\sum_{t=1}^{\lfloor \pi n \rfloor}$  by  $\sum_{t \leq \pi n}$  and  $\sum_{t=\lfloor \pi n \rfloor+1}^n$  by  $\sum_{t > \pi n}$  for notational simplicity.

### APPENDIX A: AUXILIARY RESULTS

#### A.1. Auxiliary Results on Data-Generating Processes

Here we introduce auxiliary propositions that characterize the useful properties of the data-generating processes under regularity conditions. Proposition 3 derives the corresponding scores (or local perturbations) of the univariate marginal distribution  $\mu_{s,f}$  and the Markov transition kernel  $K_{s,f}$  when we perturb the bivariate distribution from  $Q_0$  to  $Q_{s,f}$  (i.e., the score of  $Q_{s,f}$  is  $f$ ). Proposition 4 considers local data-generating processes characterized by scores  $f_{n,t}$  and shows that the scores  $f_{n,t}$  satisfy the law of large numbers and central limit theorem. Proposition 4, together with Hellinger-differentiability, is needed to ensure the local asymptotic normality of the local data-generating processes, as established in Proposition 5. The LAN property is needed to establish the contiguity property of the locally unstable data-generating process  $\mathbb{P}_{1/\sqrt{n},g,b}$  as a local perturbation with respect to the reference process  $\mathbb{P}_0$  for asymptotic equivalence arguments.

**PROPOSITION 3** (Implied Scores of Marginal and Transition Distributions) *Suppose  $Q_{s,f} \in \mathcal{N}(Q_0)$  for some  $Q_0 \in \mathcal{H}$ . Let  $\mu$  and  $K$  be the univariate marginal distribution and the Markov transition kernel of  $Q_0$ , respectively. Then, the marginal distribution  $\mu_{s,f}$  and Markov transition kernel  $K_{s,f}$  of  $Q_{s,f}$  satisfy the Hellinger differentiability conditions:*

$$(74) \quad \frac{d\mu_{s,f}}{d\mu_0} = 1 + s\bar{f} + s\Delta_\mu(s) \quad \text{and} \quad \frac{dK_{s,f}(\cdot|y)}{dK_0(\cdot|y)} = 1 + s\tilde{f}(y, \cdot) + s\Delta_K(y, s) \quad \forall y \in \mathcal{Y},$$

where  $\Delta_\mu(s)$  and  $\Delta_K(y, s)$  converge to 0 in  $L^2(Q_0)$  for all  $y \in \mathcal{Y}$  as  $s \rightarrow 0$ , and the marginal score and the conditional score are

$$(75) \quad \bar{f}(\mathbf{y}) \equiv E^{Q_0} [f(\mathbf{y}, \mathbf{y}') | \mathbf{y}] = E^{Q_0} [f(\mathbf{y}', \mathbf{y}) | \mathbf{y}] \quad \text{and} \quad \tilde{f}(\mathbf{y}, \mathbf{y}') \equiv f(\mathbf{y}, \mathbf{y}') - \bar{f}(\mathbf{y}).$$

PROPOSITION 4 *Suppose Assumptions 2 – 3 hold. Let  $\tilde{f}_{n,t} \equiv f_{n,t} - \mathbb{E}_{t-1}^{\mathbb{Q}_0} [f_{n,t}]$  and  $\tilde{g}(\mathbf{y}_{t-1}, \mathbf{y}_t) \equiv g(\mathbf{y}_{t-1}, \mathbf{y}_t) - \mathbb{E}_{t-1}^{\mathbb{Q}_0} [g(\mathbf{y}_{t-1}, \mathbf{y}_t)]$ . Then it holds that under  $\mathbb{Q}_0$ ,*

$$(76) \quad n^{-1} \sum_{t \leq \pi n} \tilde{f}_{n,t}^2 \xrightarrow{p} \Upsilon(\pi) \quad \text{and} \quad n^{-1} \sum_{t \leq \pi n} \mathbb{E}_{t-1}^{\mathbb{Q}} [\tilde{f}_{n,t}^2] \xrightarrow{p} \Upsilon(\pi), \quad \text{where}$$

$$(77) \quad \Upsilon(\pi) \equiv \mathbb{E}^{\mathbb{Q}} [\tilde{g}^T B_{\pi} \tilde{g}] \quad \text{with} \quad B_{\pi} \equiv \begin{bmatrix} \pi & \int_0^{\pi} b(u) du \\ \int_0^{\pi} b(u) du & \int_0^{\pi} b(u)^2 du \end{bmatrix}.$$

Further, the asymptotic normality result follows

$$(78) \quad n^{-1/2} \sum_{t \leq \pi n} \tilde{f}_{n,t} \xrightarrow{d} N(0, \Upsilon(\pi)).$$

PROPOSITION 5 (LAN of Unstable Parametric Submodels) *Suppose Assumptions 2 – 3 hold. For any  $g \in \mathcal{G}(\mathbb{Q}_0)$  and  $b \in \mathcal{B}$ , the corresponding locally unstable data-generating process with distribution  $\mathbb{P}_{1/\sqrt{n},g,b}$  for  $\mathbf{y}^n = \{\mathbf{y}_0, \dots, \mathbf{y}_n\}$  satisfies*

$$\ln \frac{d\mathbb{P}_{1/\sqrt{n},g,b}}{d\mathbb{P}_0} = \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{g}(\mathbf{y}_{t-1}, \mathbf{y}_t)^T \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix} - \frac{1}{2} \Upsilon(1) + o_p(1)$$

where  $\tilde{g}$  and  $\Upsilon(\cdot)$  are defined in Proposition 4, and  $o_p(1)$  denotes a sequence of random variables that converge to zero in probability  $\mathbb{P}_0$ .

COROLLARY 2 (Contiguity) *Suppose Assumptions 2 – 3 hold. The locally unstable generating process with distribution  $\mathbb{P}_{1/\sqrt{n},g,b}$  is contiguous to the stable data-generating process with distribution  $\mathbb{P}_0$ . More precisely,  $X_n \xrightarrow{p} 0$  under  $\mathbb{P}_0$  implies  $X_n \xrightarrow{p} 0$  under  $\mathbb{P}_{1/\sqrt{n},g,b}$  for all  $\mathcal{F}^n$ -measurable random variables  $X_n : \mathcal{Y}^n \rightarrow \mathbb{R}$ .*

## A.2. Auxiliary Results on Moment Functions

Let us now introduce the following basic results (Proposition 6) that extend the standard moment function approximations (Hansen, 1982). Similar results on (functional) central limit theorem with local instability are developed and used in Andrews (1993), Sowell (1996), and Li and Müller (2009).

Define  $\lambda(g^T) \equiv [\lambda(g_1), \lambda(g_2)]$  for all  $g = [g_1, g_2]^T$  with  $g \in \mathcal{G}(\mathbb{Q}_0)$ . We denote

$$(79) \quad \nu_e(g, b, \pi) \equiv \frac{\lambda(g^T)}{\sqrt{\pi}} \left[ \int_0^{\pi} b(u) du \right] \quad \text{and} \quad \nu_o(g, b, \pi) \equiv \frac{\lambda(g^T)}{\sqrt{1-\pi}} \left[ \int_{\pi}^1 b(u) du \right].$$

PROPOSITION 6 *Suppose Assumptions 1 – 5 hold. Then, under  $\mathbb{P}_{1/\sqrt{n},g,b}$ ,*

(i)  $\left[ \begin{array}{c} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) \end{array} \right] \xrightarrow{d} \left[ \begin{array}{c} \frac{1}{\sqrt{\pi}} W(\pi) \\ \frac{1}{\sqrt{1-\pi}} (W(1) - W(\pi)) \end{array} \right] + \left[ \begin{array}{c} \nu_e(g, b, \pi) \\ \nu_o(g, b, \pi) \end{array} \right]$  on  $\mathcal{D}([0, 1])$  for all  $\pi \in [0, 1]$ , where  $W(\pi)$  is a  $d_m$ -dimensional Wiener process and  $\mathcal{D}([0, 1])$  is the space of right continuous functions on  $[0, 1]$  endowed with the Skorohod  $J_1$  topology;

(ii)  $\left[ \begin{array}{c} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_{n,t}) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_{n,t}) \end{array} \right] = \left[ \begin{array}{c} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) \end{array} \right] - \left[ \begin{array}{c} \nu_e(g, b, \pi) \\ \nu_o(g, b, \pi) \end{array} \right] + o_p(1), \forall g_1, g_2 \in \mathcal{T}(\mathbf{Q}_0);$

(iii)  $\left[ \begin{array}{c} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\hat{\theta}_{e,n}) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\hat{\theta}_{e,n}) \end{array} \right] = \left[ \begin{array}{c} [I - D(D^T D)^{-1} D^T] \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) - D(D^T D)^{-1} D^T \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \end{array} \right] + o_p(1),$   
where  $\hat{\theta}_{e,n}$  is the efficient GMM estimator based on estimation sample  $\mathbf{y}_e^n$ .

(iv)  $\left[ \begin{array}{c} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\tilde{\theta}_{e,n}) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\tilde{\theta}_{e,n}) \end{array} \right] = \left[ \begin{array}{c} [I - D(A^T D)^{-1} A^T] \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \\ \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) - D(A^T D)^{-1} A^T \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \end{array} \right] + o_p(1),$   
where  $\tilde{\theta}_{e,n}$  is the recursive GMM estimator based on estimation sample  $\mathbf{y}_e^n$ .

We construct the martingale difference array  $h(\mathbf{y}, \mathbf{y}', \theta_0)$ , inspired by the martingale difference approximation for the temporal-dependent moment function in Hansen (1985). The martingale difference approximation plays a key role in analyzing the semiparametric efficiency bound of estimation based on moment restrictions. To guarantee that  $h(\mathbf{y}, \mathbf{y}', \theta_0)$  is well defined in (80), we postulate the condition of asymptotic negligibility of innovations (Assumption 5 (iii)), which has been used to establish Gordin's CLT (Gordin, 1969).

PROPOSITION 7 Suppose Assumptions 1 – 5 hold. Then  $h(\cdot, \theta_0)$  is defined as follows:

$$(80) \quad h(\mathbf{y}, \mathbf{y}', \theta_0) = m(\mathbf{y}, \mathbf{y}', \theta_0) - \mathbb{E}^{\mathbf{Q}_0} [m_1(\theta_0) | \mathbf{y}_0 = \mathbf{y}] + \sum_{t=1}^{\infty} \{ \mathbb{E}^{\mathbf{Q}_0} [m_{t+1}(\theta_0) | \mathbf{y}_1 = \mathbf{y}'] - \mathbb{E}^{\mathbf{Q}_0} [m_{t+1}(\theta_0) | \mathbf{y}_0 = \mathbf{y}] \}.$$

Moreover,  $h(\cdot, \theta_0)$  satisfies  $\mathbb{E}^{\mathbf{Q}_0} [h(\mathbf{y}, \mathbf{y}', \theta_0) | \mathbf{y}] = 0$  and  $\mathbb{E}^{\mathbf{Q}_0} [h(\mathbf{y}, \mathbf{y}', \theta_0) h(\mathbf{y}, \mathbf{y}', \theta_0)^T] = I$  and

$$(81) \quad \mathbb{E}^{\mathbf{Q}_0} [m(\cdot, \theta_0) f] = \mathbb{E}^{\mathbf{Q}_0} [h(\cdot, \theta_0) f] \text{ for all } f \in L_0^2(\mathbf{Q}_0).$$

Therefore, the tangent set of  $\mathcal{Q}$  at the distribution  $\mathbf{Q}_0$  can be represented by

$$(82) \quad \mathcal{T}(\mathbf{Q}_0) = \{ f \in L_0^2(\mathbf{Q}_0) : \lambda(f) \in \text{lin}(D) \},$$

where operator  $\lambda(f) \equiv \mathbb{E}^{\mathbf{Q}_0} [h(\cdot, \theta_0) f]$  is a linear operator on  $L_0^2(\mathbf{Q}_0)$  and linear space  $\text{lin}(D)$  is spanned by columns of  $D$ , which is defined in (26).

## A.3. Auxiliary Results on GMM Estimators Based on the Estimation Sample

Let us now introduce the following basic results that extend the standard GMM approximations (Hansen, 1982) in Proposition 8. Then, we introduce a new set of GMM approximations in Proposition 9, which are unique to our paper.

PROPOSITION 8 Suppose Assumptions 1 – 5 hold. Let  $\tilde{\theta}_{e,n}$  and  $\hat{\theta}_{e,n}$  be the recursive GMM and the efficient GMM estimators based on the estimation sample  $\mathbf{y}_e^n = \{\mathbf{y}_1, \dots, \mathbf{y}_{[\pi n]}\}$ , respectively. Then, under  $\mathbb{P}_{1/\sqrt{n},g,b}$ ,

$$(i) \quad \sqrt{\pi n} (\tilde{\theta}_{e,n} - \theta_0) = -(A^T D)^{-1} A^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + o_p(1),$$

$$\text{with } A = \begin{bmatrix} D_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad \text{and } A_{22} = [D_{21}(D_{11}^T D_{11})^{-1} D_{21}^T + I]^{-1} D_{22};$$

$$(ii) \quad \sqrt{\pi n} (\hat{\theta}_{e,n} - \theta_0) = -(D^T D)^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + o_p(1).$$

PROPOSITION 9 Suppose Assumptions 1 – 6 hold and  $g \in \mathcal{G}_B(Q_0)$ . Let  $\tilde{\theta}_{e,n}$  and  $\hat{\theta}_{e,n}$  be the recursive GMM estimator and efficient GMM estimator based on the estimation sample  $\mathbf{y}_e^n = \{\mathbf{y}_1, \dots, \mathbf{y}_{[\pi n]}\}$ , respectively. Then, under  $\mathbb{P}_{1/\sqrt{n},g,b}$ , it holds that

$$(i) \quad \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_s(\tilde{\theta}_{e,n}^{(1)}) - \psi_s(\theta_{e,n}^{(1)}) \end{bmatrix} = -\mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T \mathbf{I}_F \mathbf{I}_B^{-1} D_{11}^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t^{(1)}(\theta_{n,t}^{(1)}) \right] + o_p(1);$$

$$(ii) \quad \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_s(\hat{\theta}_{e,n}^{(1)}) - \psi_s(\theta_{e,n}^{(1)}) \end{bmatrix} = -\mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T \mathbf{I}_F \left\{ L_F \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] - L_B \nu_e(g, b, \pi) \right\} + o_p(1).$$

Here the matrices  $L_B$  and  $L_F$  are

$$(83) \quad L_B \equiv \mathbf{I}_B^{-1} D_{11}^T \Gamma_{m,1} \quad \text{and} \quad L_F \equiv \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T,$$

and  $D_{11}$  and  $D$  are Jacobian matrices defined in (26), the information matrices  $\mathbf{I}_B$  and  $\mathbf{I}_Q$  are defined in (27) – (28), and the selection matrices  $\Gamma_{m,1}$  and  $\Gamma_{\theta,1}$  are defined in (16) and (28), respectively.

PROPOSITION 10 Suppose Assumptions 1 – 6 hold and  $g \in \mathcal{G}_B(Q_0)$ . Let  $\mathcal{L}(\theta^{(1)}, \cdot)$  be the loss function for assessing the goodness of fit of the baseline parameter  $\theta^{(1)}$  to the data as defined in (53) – (54). Let  $\tilde{\theta}_{e,n}$  and  $\hat{\theta}_{e,n}$  be the recursive GMM estimator and efficient GMM estimator based on the estimation sample  $\mathbf{y}_e^n = \{\mathbf{y}_1, \dots, \mathbf{y}_{[\pi n]}\}$ , respectively. Let  $\mathbf{y}_o^n = \{\mathbf{y}_{[\pi n]+1}, \dots, \mathbf{y}_n\}$  be the holdout sample. Then, under  $\mathbb{P}_{1/\sqrt{n},g,b}$ , it holds that

$$(i) \quad \begin{bmatrix} \mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n) \\ \mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n) \end{bmatrix} = \begin{bmatrix} ((L_B - 2L_F)\zeta_{e,n} - 2L_\Delta \nu_e)^T \mathbf{I}_F (L_B \zeta_{e,n}) \\ (L_B \zeta_{e,n} - 2L_F \zeta_{o,n} - 2L_\Delta \nu_o)^T \mathbf{I}_F (L_B \zeta_{e,n}) \end{bmatrix} + o_p(1), \text{ and}$$

$$(ii) \quad \begin{bmatrix} \mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n) \\ \mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n) \end{bmatrix} = \begin{bmatrix} -(L_F \zeta_{e,n} + L_\Delta \nu_e)^T \mathbf{I}_F (L_F \zeta_{e,n} + L_\Delta \nu_e) \\ (L_F (\zeta_{e,n} - 2\zeta_{o,n}) + L_\Delta (\nu_e - 2\nu_o))^T \mathbf{I}_F (L_F \zeta_{e,n} + L_\Delta \nu_e) \end{bmatrix} + o_p(1),$$



where  $\nu_e(g, b, \pi)$  and  $\nu_o(g, b, \pi)$  are defined in (79), and the random vectors  $\zeta_{e,n}$  and  $\zeta_{o,n}$  are

$$(84) \quad \zeta_{e,n} \equiv \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) - \nu_e(g, b, \pi) \text{ and } \zeta_{o,n} \equiv \frac{1}{\sqrt{\pi n}} \sum_{t > \pi n} m_t(\theta_0) - \nu_o(g, b, \pi),$$

and the matrices  $L_F$ ,  $L_B$  are defined in (83) and  $L_\Delta \equiv L_F - L_B$ . Further, using Proposition 6, it holds that

$$(85) \quad \begin{bmatrix} \zeta_{e,n} \\ \zeta_{o,n} \end{bmatrix} \xrightarrow{d} \begin{bmatrix} \frac{1}{\sqrt{\pi}} W(\pi) \\ \frac{1}{\sqrt{1-\pi}} (W(1) - W(\pi)) \end{bmatrix}.$$

## APPENDIX B: SEMIPARAMETRIC MINIMAX EFFICIENCY BOUNDS

Given the LAN for the Markov processes with potential local instability, the local asymptotic minimax (LAM) justification for the efficiency bounds can be established using the asymptotic equivalence argument.<sup>28</sup> For the local data-generating process that is described by a locally unstable distribution  $\mathbb{P}_{1/\sqrt{n},g,b}$ , the goal is to estimate the average model parameter value:

$$(86) \quad \vartheta(\mathbb{P}_{1/\sqrt{n},g,b}) \equiv \frac{1}{n} \sum_{t=1}^n \vartheta(Q_{1/\sqrt{n},f_{n,t}}), \text{ with } f_{n,t} = g_1(\mathbf{y}_{t-1}, \mathbf{y}_t) + g_2(\mathbf{y}_{t-1}, \mathbf{y}_t)b(t/n).$$

We formalize the precise meaning of semiparametric efficiency bounds based on local asymptotic minimax risk, which is stated in the following theorem.

**THEOREM 4 (LAM Lower Bounds)** *Suppose assumptions 1 – 5 hold and  $\vartheta(\mathbb{P}_{1/\sqrt{n},g,b})$  exists. Thus, for any  $v \in \mathbb{R}^{d_\theta}$ , any arbitrary estimator sequence  $\check{\theta}_n$  satisfies*

$$\lim_{l \rightarrow \infty} \liminf_{n \rightarrow \infty} \sup_{g \in \mathcal{G}(Q_0), b \in \mathcal{B}} \int l \wedge [\sqrt{n}v^T (\check{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n},g,b}))]^2 d\mathbb{P}_{1/\sqrt{n},g,b} \geq v^T (D^T D)^{-1} v.$$

The method of first calculating the truncated mean squared error (MSE), then letting the ceiling  $l$  increase to infinity, is widely adopted in the literature (e.g., Bickel, 1981; Le Cam and Yang, 2000; Kitamura, Otsu, and Evdokimov, 2013).

**THEOREM 5 (LAM Upper Bounds)** *Suppose assumptions 1 – 5 hold and  $\vartheta(\mathbb{P}_{1/\sqrt{n},g,b})$  exists. Then, for any  $v \in \mathbb{R}^{d_\theta}$ , there exists an estimator sequence  $\hat{\theta}_n$  such that*

$$\lim_{l \rightarrow \infty} \liminf_{n \rightarrow \infty} \sup_{g \in \mathcal{G}(Q_0), b \in \mathcal{B}} \int l \wedge [\sqrt{n}v^T (\hat{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n},g,b}))]^2 d\mathbb{P}_{1/\sqrt{n},g,b} \leq v^T (D^T D)^{-1} v.$$

In our proof, we show that the efficient GMM estimator (Hansen, 1982) can achieve the semiparametric efficiency bound. Importantly, the proof is similar to that of Theorem 1 in Li and Müller (2009) through using

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<sup>28</sup>Dou, Pollard, and Zhou (2010) also appeal to the asymptotic equivalence argument to establish the global minimax upper bound for a non-parametric estimation problem.

Le Cam's theory of asymptotic equivalence. Therefore, Theorems 4 and 5 extend the results on the minimax efficiency bounds for unconditional moment restrictions developed in Chamberlain (1987) to general Markov processes with local instability.

## APPENDIX C: PROOFS OF MAIN THEOREMS

### 1. Proof of Theorem 1.

The test statistic based on the  $C$  statistic is  $\hat{\varphi}_n \equiv \mathbf{1}_{\{C_n > c_{1-\alpha}\}}$ , where  $c_{1-\alpha}$  is the  $(1-\alpha)$  quantile of a chi-square distribution with  $d_{m,2} - d_{\theta,2}$  degrees of freedom. From Proposition 6, we know that Assumption 3.1 of Chen and Santos (2018) is satisfied. Thus, by Lemma 3.2 of Chen and Santos (2018) and the results of Newey (1985a), it follows that for any GMM specification test  $\check{\varphi}_n$  with an asymptotic level  $\alpha$  and an asymptotic local power function  $(\forall \check{\varphi}_n \in \Phi_\alpha(Q_0))$ , it holds that

$$(87) \quad \inf_{g \in \mathcal{A}_\kappa(Q_0)} \lim_{n \rightarrow \infty} \int \check{\varphi}_n d\mathbb{P}_{1/\sqrt{n},g,0} \leq \inf_{g \in \mathcal{A}_\kappa(Q_0)} \lim_{n \rightarrow \infty} \int \hat{\varphi}_n d\mathbb{P}_{1/\sqrt{n},g,0} \quad (\text{i.e., } C \text{ test is asymptotically optimal})$$

$$(88) \quad = \inf_{g \in \mathcal{A}_\kappa(Q_0)} \lim_{n \rightarrow \infty} \mathbb{P}_{1/\sqrt{n},g,0} \left\{ \left| \hat{\mathbb{G}}_n \right|^2 > c_{1-\alpha} \right\},$$

where  $\mathcal{A}_\kappa(Q_0) \equiv \{g \in \mathcal{G}_B(Q_0) : |\lambda^{(2)}(g_1)| \geq \kappa \text{ and } \lambda^{(2)}(g_1) \perp \text{lin}(D_{22})\}$ , and

$$(89) \quad \hat{\mathbb{G}}_n = (\Lambda_2 - \Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2)^{-1/2} \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^n m_t^{(2)}(\hat{\theta}_n) \right];$$

see page 243 of Newey (1985a) and Appendix G.3 of this paper. Here  $\Lambda_2 = I - D_{22}(D_{22}^T D_{22})^{-1} D_{22}^T$ .

Now, we can obtain that (e.g., Newey, 1985a; Chen and Santos, 2018, or Proposition 6 of this paper)

$$(90) \quad \left| \hat{\mathbb{G}}_n \right|^2 \xrightarrow{d} \chi_{d_{m,2}-d_{\theta,2}}^2(\mu_g)$$

where  $\chi_{d_{m,2}-d_{\theta,2}}^2(\mu_g)$  is a noncentral chi-squared random variable with degrees of freedom  $d_{m,2} - d_{\theta,2}$  and noncentrality parameter  $\mu_g = \lambda^{(2)}(g_1)^T (\Lambda_2 - \Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2) \lambda^{(2)}(g_1)$ .

From (47) and (87) – (88), it holds that

$$(91) \quad \inf_{g \in \mathcal{A}_\kappa(Q_0)} q(g, \check{\varphi}) \leq \inf_{g \in \mathcal{A}_\kappa(Q_0)} \lim_{n \rightarrow \infty} \mathbb{P}_{1/\sqrt{n},g} \left\{ \left| \hat{\mathbb{G}}_n \right|^2 > c_{1-\alpha} \right\} = \inf_{g \in \mathcal{A}_\kappa(Q_0)} P \left\{ \chi_{d_{m,2}-d_{\theta,2}}^2(\mu_g) > c_{1-\alpha} \right\},$$

Note that  $\mu_g > 0$  for all  $g \in \mathcal{A}_\kappa(Q_0)$  since  $\Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2$  does not have unit eigenvalues.

Therefore, the local asymptotic maximin power is bounded from above by

$$(92) \quad \inf_{g \in \mathcal{A}_\kappa(Q_0)} q(g, \check{\varphi}) \leq \inf_{g \in \mathcal{A}_\kappa(Q_0)} M_{\frac{d_{m,2}-d_{\theta,2}}{2}}(\sqrt{\mu_g}, \sqrt{c_{1-\alpha}}) = M_{\frac{d_{m,2}-d_{\theta,2}}{2}} \left( \inf_{g \in \mathcal{A}_\kappa(Q_0)} \sqrt{\mu_g}, \sqrt{c_{1-\alpha}} \right),$$

where the equality above is due to the continuity and monotonicity of the Marcum Q-function  $M_\gamma(x_1, x_2)$ .

It holds that

$$\inf_{g \in \mathcal{A}_\kappa(Q_0)} \mu_g = \kappa^2 \times \text{the smallest eigenvalue of } I - \Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2.$$

We shall now show that  $1/(1 + \varrho(\theta_0))$  is an eigenvalue of  $I - \Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2$ , and thus  $\inf_{g \in \mathcal{A}_\kappa(Q_0)} \sqrt{\mu_g} \leq \sqrt{\kappa^2/(1 + \varrho(\theta_0))}$ . In fact,  $1 - 1/(1 + \varrho(\theta_0))$  is an eigenvalue of  $\mathbf{I}_F^{-1/2} (\mathbf{I}_F - \mathbf{I}_B) \mathbf{I}_F^{-1/2} = \mathbf{I}_F^{-1/2} (D_{21}^T \Lambda_2 D_{21}) \mathbf{I}_F^{-1/2}$  and thus an eigenvalue of  $\Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2$ . Therefore,  $1/(1 + \varrho(\theta_0))$  is an eigenvalue of  $I - \Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2$ .

Due to the monotonicity of the generalized Marcum Q-function, the local asymptotic maximin power is upper bounded by

$$(93) \quad \inf_{g \in \mathcal{A}_\kappa(Q_0)} q(g, \tilde{\varphi}) \leq M_{\frac{d_m, 2 - d_{\theta, 2}}{2}} \left( \sqrt{\frac{\kappa^2}{1 + \varrho(\theta_0)}}, \sqrt{c_{1-\alpha}} \right).$$

### ***Proof of Theorem 2.***

According to Proposition 10 (ii), we can show that

$$(94) \quad \mathbb{E} \left[ \text{wlim}_{n \rightarrow \infty} \frac{1}{2} \left( \mathcal{L}(\hat{\theta}_{e,n}; \mathbf{y}_e^n) - \mathcal{L}(\hat{\theta}_{e,n}; \mathbf{y}_e^n) \right) \right] = \pi^{-1} \mathbb{E} [W(\pi)^T L_F^T \mathbf{I}_F L_F W(\pi)] \\ + [\nu_e(g, b, \pi) - \nu_o(g, b, \pi)]^T L_\Delta^T \mathbf{I}_F L_\Delta \nu_e(g, b, \pi),$$

where  $\text{wlim}_{n \rightarrow \infty}$  is the weak convergence limit and  $W(\cdot)$  is a  $d_m$ -dimensional Wiener process. The first term above is

$$(95) \quad \pi^{-1} \mathbb{E} [W(\pi)^T L_F^T \mathbf{I}_F L_F W(\pi)] = \pi^{-1} \mathbb{E} \left[ \text{tr} \left( \mathbf{I}_F^{1/2} L_F W(\pi) W(\pi)^T L_F^T \mathbf{I}_F^{1/2} \right) \right] \\ = \text{tr} \left( \mathbf{I}_F^{1/2} L_F L_F^T \mathbf{I}_F^{1/2} \right).$$

According to the definition of  $L_F$  in (83), it holds that

$$(96) \quad L_F L_F^T = \Gamma_{\theta, 1} \mathbf{I}_\Omega^{-1} \Gamma_{\theta, 1}^T = \mathbf{I}_F^{-1}.$$

Combining (95) and (96) yields

$$(97) \quad \pi^{-1} \mathbb{E} [W(\pi)^T L_F^T \mathbf{I}_F L_F W(\pi)] = d_{\theta, 1}.$$

Since  $\lambda(g_1) \in \text{lin}(D)$ , it holds that  $L_\Delta \lambda(g_1) = 0$  and thus

$$(98) \quad [\nu_e - \nu_o]^T L_\Delta^T \mathbf{I}_F L_\Delta \nu_e = 2 \left( \int_0^\pi b(u) du \right)^2 \lambda(g_2)^T L_\Delta^T \mathbf{I}_F L_\Delta \lambda(g_2).$$

The left hand side of (98) is upper bounded by

$$(99) \quad 2 \left( \int_0^\pi b(u) du \right)^2 \lambda(g_2)^T L_\Delta^T \mathbf{I}_F L_\Delta \lambda(g_2) \leq 2\pi^2 |\lambda(g_2)|^2 \times \text{the largest eigenvalue of } L_\Delta^T \mathbf{I}_F L_\Delta.$$

The largest eigenvalue of  $L_\Delta^T \mathbf{I}_F L_\Delta$  is that of  $\Pi = \mathbf{I}_F^{1/2} L_\Delta L_\Delta^T \mathbf{I}_F^{1/2}$ , which is the dark matter measure  $\varrho(\theta_0)$ .

***Proof of Theorem 3.***

According to Proposition 10 (i), we can show that

$$(100) \quad \mathbb{E} \left[ \text{wlim}_{n \rightarrow \infty} \frac{1}{2} \left( \mathcal{L}(\tilde{\theta}_{e,n}; \mathbf{y}_o^n) - \mathcal{L}(\tilde{\theta}_{e,n}; \mathbf{y}_e^n) \right) \right] = \pi^{-1} \mathbb{E} [W(\pi)^T L_F^T \mathbf{I}_F L_B W(\pi)],$$

where  $\text{wlim}_{n \rightarrow \infty}$  is the weak convergence limit and  $W(\cdot)$  is a  $d_m$ -dimensional Wiener process. Further,

$$(101) \quad \pi^{-1} \mathbb{E} [W(\pi)^T L_F^T \mathbf{I}_F L_B W(\pi)] = \text{tr}(\mathbf{I}_F^{1/2} L_B L_F^T \mathbf{I}_F^{1/2}).$$

Because  $L_B L_F^T = \mathbf{I}_B^{-1} D_{11}^T [D_{11}, 0_{d_{m,1} \times (d_\theta - d_{\theta,1})}] \mathbf{I}_\Omega^{-1} \Gamma_{\theta,1}^T = \Gamma_{\theta,1} \mathbf{I}_\Omega^{-1} \Gamma_{\theta,1}^T = \mathbf{I}_F^{-1}$ , the equality (101) can further be rewritten as

$$(102) \quad \pi^{-1} \mathbb{E} [W(\pi)^T L_F^T \mathbf{I}_F L_B W(\pi)] = d_{\theta,1}.$$

***Proof of Theorem 4.***

The local asymptotic normality (LAN) (see Proposition 5), as well as the implied contiguity, and Le Cam's first and third lemmas play crucial roles in the proof as in the standard proof of semiparametric minimax lower bounds (e.g. van der Vaart, 1998, Theorem 8.11 and Theorem 25.21). Our results are new in the sense that they apply to Markov processes with local instability, which is more general than the i.i.d. case.

Following the literature (e.g. Bickel, Klaassen, Ritov, and Wellner, 1993; van der Vaart, 1998), the functional  $\vartheta(Q)$  is defined to be pathwise differentiable at  $Q_0$  relative to the parametric submodels  $s \mapsto Q_{s,f}$ , if there exists a measurable function  $\dot{\vartheta}: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}^{d_\theta}$  with  $\dot{\vartheta} \in L_0^2(Q_0)$  such that

$$(103) \quad \lim_{s \rightarrow 0} \frac{1}{s} [\vartheta(Q_{s,f}) - \vartheta(Q_0)] = \mathbb{E}^{Q_0} [\dot{\vartheta} f],$$

where  $\dot{\vartheta}(\mathbf{y}_{t-1}, \mathbf{y}_t) \equiv (D^T D)^{-1} D^T h(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta_0)$  with  $h(\mathbf{y}_{t-1}, \mathbf{y}_t, \theta_0)$  defined in Proposition 7 (e.g., Greenwood and Wefelmeyer, 1995). According to Proposition 7,  $h(\cdot, \theta_0)$  satisfies the conditions:  $\mathbb{E}^{Q_0} [h(\mathbf{y}, \mathbf{y}', \theta_0) | \mathbf{y}] = 0$  and  $\mathbb{E}^{Q_0} [h(\mathbf{y}, \mathbf{y}', \theta_0) h(\mathbf{y}, \mathbf{y}', \theta_0)^T] = I$ .

First, we only need to consider the case  $g_1(\mathbf{y}, \mathbf{y}') = v^T \dot{\vartheta}(\mathbf{y}, \mathbf{y}')$ ,  $g_2(\mathbf{y}_{t-1}, \mathbf{y}_t) \equiv 0$ , and  $b(u) \equiv 0$  for establishing the lower bound. In such case,  $f(\mathbf{y}_{t-1}, \mathbf{y}_t) \equiv g_1(\mathbf{y}_{t-1}, \mathbf{y}_t)$  for all  $1 \leq t \leq n$ . Second, we further focus on the estimators  $\check{\theta}_n$  such that  $\sqrt{n}(\check{\theta}_n - \theta_0)$  is uniformly tight under the distribution  $\mathbb{P}_0$ , similar to van der Vaart (1998). The tightness assumption can be dropped by a compactification argument (e.g. van der Vaart, 1988;

van der Vaart and Wellner, 1996, Chapter 3.11). Moreover, without loss of generality, due to Prohorov's theorem, we can assume that

$$(104) \quad \left( \sqrt{n} (v^T \check{\theta}_n - v^T \theta_0), \frac{1}{\sqrt{n}} \sum_{t=1}^n g_1(\mathbf{y}_{t-1}, \mathbf{y}_t) \right) \xrightarrow{d} (\Xi_0, U_0),$$

where  $U_0 \sim N(0, v^T (D^T D)^{-1} v)$  (see Proposition 4). Using the contiguity between  $\mathbb{P}_{1/\sqrt{n}, g, 0}$  and  $\mathbb{P}_0$ , Le Cam's third lemma (e.g. van der Vaart, 1998, Theorem 6.6), and differentiability of  $\vartheta(Q_{s,f})$  with respect to  $s$ , we know that under the sequence of distributions  $\mathbb{P}_{1/\sqrt{n}, g, 0}$ ,

$$(105) \quad \sqrt{n} (v^T \check{\theta}_n - v^T \vartheta(\mathbb{P}_{1/\sqrt{n}, g, 0})) \xrightarrow{d} \Xi_g,$$

where, appealing to Theorem 8.3 of van der Vaart (1998), the limiting random variable  $\Xi_g$  has the following representation with certain measurable function  $\tau : \mathbb{R}^{d_\theta} \rightarrow \mathbb{R}$ :

$$(106) \quad \begin{aligned} \Xi_g &= \tau(X_g) - v^T \xi \\ &= \tau(X_g) - \mathbb{E}^{Q_0} [v^T \dot{\vartheta} f] \\ &= \tau(X_g) - [v^T (D^T D)^{-1} v]. \end{aligned}$$

Here the local estimation bias  $\xi \equiv (D^T D)^{-1} D^T \lambda(g_1) = (D^T D)^{-1} v$  (similar to Corollary 1 or the proof of Proposition 6 (ii)) and  $X_g \sim N(\xi, (D^T D)^{-1})$ . Based on Theorem 8.6 of van der Vaart (1998) for estimating normal means, it holds that for all measurable function  $\tau$ ,

$$(107) \quad \mathbb{E}^{Q_{1/\sqrt{n}, f}} [\Xi_g^2] \geq \mathbb{E}^{Q_0} [(v^T X_0)^2] = v^T (D^T D)^{-1} v.$$

The key idea of (104) – (106) is a change-of-measure argument, inspired by Le Cam's theory of asymptotic equivalence, whose stronger form has also been developed and used in the minimax inference of Dou, Pollard, and Zhou (2010).

Consequently, it suffices to show that the left-hand side of (107) is a lower bound for the minimax risk  $R$ :

$$(108) \quad R \equiv \lim_{l \rightarrow \infty} \liminf_{n \rightarrow \infty} \int l \wedge [\sqrt{n} v^T (\check{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n}, g, 0}))]^2 d\mathbb{P}_{1/\sqrt{n}, g, 0}.$$

In fact, it holds that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int l \wedge [\sqrt{n} v^T (\check{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n}, g, 0}))]^2 d\mathbb{P}_{1/\sqrt{n}, g, 0} &\geq \liminf_{n \rightarrow \infty} \int l \wedge [\sqrt{n} v^T (\check{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n}, g, 0}))]^2 d\mathbb{P}_{1/\sqrt{n}, g, 0} \\ &= \mathbb{E}^{Q_{1/\sqrt{n}, g, 0}} [l \wedge \Xi_g^2]. \end{aligned}$$

Thus, the minimax risk can be bounded from below by

$$(109) \quad R \geq \lim_{l \rightarrow \infty} \mathbb{E}^{Q_{1/\sqrt{n}, f}} [l \wedge \Xi_g^2] \geq \lim_{l \rightarrow \infty} \mathbb{E}^{Q_{1/\sqrt{n}, f}} [l \wedge \Xi_g^2].$$

According to the monotone convergence theorem, it follow that

$$(110) \quad R \geq \mathbb{E}^{\mathbb{Q}_1/\sqrt{n},f} [\Xi_g^2].$$

Combining (107) and (110), the local asymptotic minimax lower bound result holds:  $R \geq v^T(D^T D)^{-1}v$ .

***Proof of Theorem 5.***

It holds that

$$(111) \quad \sqrt{n} \left[ \hat{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n},g,b}) \right] = \sqrt{n} (\hat{\theta}_n - \theta_0) - \sqrt{n} [\vartheta(\mathbb{P}_{1/\sqrt{n},g,b}) - \theta_0]$$

According to Proposition 8 (ii), it follows that

$$(112) \quad \sqrt{n} (\hat{\theta}_n - \theta_0) = -(D^T D)^{-1} D^T \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^n m_t(\theta_0) \right] + o_p(1),$$

Consequently, similar to Corollary 1 or the proof of Proposition 6 (ii), it holds that

$$(113) \quad \sqrt{n} [\vartheta(\mathbb{P}_{1/\sqrt{n},g,b}) - \theta_0] = -(D^T D)^{-1} D^T \lambda(g_1) + o(1).$$

Thus, appealing to Proposition 6 (i), we can show that

$$(114) \quad \sqrt{n} \left[ \hat{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n},g,b}) \right] \xrightarrow{d} -(D^T D)^{-1} D^T W(1),$$

where  $W(\cdot)$  is a  $d_m$ -dimensional Wiener process. Therefore, for any  $v \in \mathbb{R}^{d_\theta}$ ,

$$(115) \quad \liminf_{n \rightarrow \infty} \int l \wedge \left[ \sqrt{n} v^T (\hat{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n},g,b})) \right]^2 d\mathbb{P}_{1/\sqrt{n},g,b} = \mathbb{E} [l \wedge X^2], \quad \text{with } X \sim N(0, v^T (D^T D)^{-1} v).$$

Let  $l$  increase monotonically to infinity, and using the monotonic convergence theorem, we can obtain

$$(116) \quad \lim_{l \rightarrow \infty} \liminf_{n \rightarrow \infty} \int l \wedge \left[ \sqrt{n} v^T (\hat{\theta}_n - \vartheta(\mathbb{P}_{1/\sqrt{n},g,b})) \right]^2 d\mathbb{P}_{1/\sqrt{n},g,b} = \mathbb{E} [X^2] = v^T (D^T D)^{-1} v.$$

## APPENDIX D: PROOFS OF PROPOSITIONS

### ***1. Proof of Proposition 1***

Following the standard argument such as in the proof of Theorem 7.2 of [van der Vaart \(1998\)](#), we can show that  $\mathbb{E}^{\mathbb{Q}_0} [f] = 0$ . Thus, it must hold that

$$(117) \quad \mathbb{E}^{\mathbb{Q}_0} [\Delta(s)] = \mathbb{E}^{\mathbb{Q}_0} \left[ \frac{dQ_{s,f}}{dQ_0} - 1 \right] = \int dQ_{s,f} - \int dQ_0 = 0.$$

According to Proposition 3, the conditional expectations denoted by  $\bar{f}(\mathbf{y}_{t-1}) = \mathbb{E}^{\mathbb{Q}_0} [f(\mathbf{y}_{t-1}, \mathbf{y}_t) | \mathbf{y}_{t-1}]$  and  $\bar{f}(\mathbf{y}_t) = \mathbb{E}^{\mathbb{Q}_0} [f(\mathbf{y}_{t-1}, \mathbf{y}_t) | \mathbf{y}_t]$  are the scores for the marginal distributions of  $\mathbf{y}_{t-1}$  and  $\mathbf{y}_t$ , respectively. Because the marginal distributions are constant over time, it must hold that

$$(118) \quad \mathbb{E}^{\mathbb{Q}_0} [f(\mathbf{y}, \mathbf{y}') | \mathbf{y}] = \mathbb{E}^{\mathbb{Q}_0} [f(\mathbf{y}', \mathbf{y}) | \mathbf{y}].$$

## 2. Proof of Proposition 2

According to Definition 4, it follows that

$$(119) \quad \mathbb{E}^{\mathbb{Q}_{1/\sqrt{n}, f_{n,t}}} [m_t(\theta_0)] = \int m_t(\theta_0) [1 + f_{n,t}/\sqrt{n} + \Delta_n] d\mathbb{Q}_0.$$

Because  $\mathbb{E}^{\mathbb{Q}_0} [m_t(\theta_0)] = 0$ , the equality (119) above leads to

$$(120) \quad \mathbb{E}^{\mathbb{Q}_{1/\sqrt{n}, f_{n,t}}} [m_t(\theta_0)] = \frac{\lambda(g_1) + \lambda(g_2)b(t/n)}{\sqrt{n}} + \int m_t(\theta_0) \Delta_n d\mathbb{Q}_0.$$

Based on Assumption 5 and Definition 4, the Cauchy-Schwarz inequality leads to

$$(121) \quad \left| \int m_t(\theta_0) \Delta_n d\mathbb{Q}_0 \right| \leq \mathbb{E}^{\mathbb{Q}_0} [|m_t(\theta_0)|^2]^{1/2} \mathbb{E}^{\mathbb{Q}_0} [|\Delta_n|^2]^{1/2} = o\left(\frac{1}{\sqrt{n}}\right).$$

## 3. Proof of Proposition 3

By the definition of marginal distribution, it holds that

$$\begin{aligned} d\mu_{s,f}(\mathbf{y}) &= \int_{\mathbf{y}' \in \mathcal{Y}} d\mathbb{Q}_{s,f}(\mathbf{y}, \mathbf{y}') = \int_{\mathbf{y}' \in \mathcal{Y}} [1 + sf(\mathbf{y}, \mathbf{y}') + s\Delta_{\mathbb{Q}}(s)] d\mathbb{Q}(\mathbf{y}, \mathbf{y}') \\ &= \left[ 1 + s \int_{\mathbf{y}' \in \mathcal{Y}} f(\mathbf{y}, \mathbf{y}') dK_{s,f}(\mathbf{y}' | \mathbf{y}) + s \int_{\mathbf{y}' \in \mathcal{Y}} \Delta_{\mathbb{Q}}(s) dK_{s,f}(\mathbf{y}' | \mathbf{y}) \right] d\mu(\mathbf{y}). \end{aligned}$$

By the definition of  $\bar{f}(\mathbf{y})$ , we know that

$$(122) \quad d\mu_{s,f}(\mathbf{y}) = [1 + s\bar{f}(\mathbf{y}) + s\Delta_{\mu}(s)] d\mu(\mathbf{y}),$$

where  $\Delta_{\mu}(s) \equiv \mathbb{E}^{\mathbb{Q}} [\Delta_{\mathbb{Q}}(s) | \mathbf{y}]$  and it converges to zero in quadratic mean under  $\mu$  as  $s \rightarrow 0$ . Further, by definition, it holds that

$$\begin{aligned} dK_{s,f}(\mathbf{y}' | \mathbf{y}) &= \frac{d\mathbb{Q}_{s,f}(\mathbf{y}, \mathbf{y}')}{d\mu_{s,f}(\mathbf{y})} = \frac{1 + sf(\mathbf{y}, \mathbf{y}') + s\Delta_{\mathbb{Q}}(s)}{1 + s\bar{f}(\mathbf{y}) + s\Delta_{\mu}(s)} \frac{d\mathbb{Q}(\mathbf{y}, \mathbf{y}')}{d\mu(\mathbf{y})} \\ &= \frac{1 + sf(\mathbf{y}, \mathbf{y}') + s\Delta_{\mathbb{Q}}(s)}{1 + s\bar{f}(\mathbf{y}) + s\Delta_{\mu}(s)} dK(\mathbf{y}' | \mathbf{y}). \end{aligned}$$

Rearranging and combining terms leads to

$$(123) \quad dK_{s,f}(\mathbf{y}' | \mathbf{y}) = \{1 + s[f(\mathbf{y}, \mathbf{y}') - \bar{f}(\mathbf{y})] + s\Delta_K(\mathbf{y}, s)\} dK(\mathbf{y}' | \mathbf{y})$$

where  $\Delta_K(\mathbf{y}, s)$  converges to zero in quadratic mean under  $K(\mathbf{y}'|\mathbf{y})$  as  $s \rightarrow 0$  for all  $\mathbf{y} \in \mathcal{Y}$ . By definition of  $\tilde{f}(\mathbf{y}, \mathbf{y}')$ , it holds that  $\mathbb{E}^Q [\tilde{f}(\mathbf{y}, \mathbf{y}')|\mathbf{y}] = 0$ . And thus, similar to the proof of Proposition 1, we can show that  $\mathbb{E}^Q [\Delta_K(\mathbf{y}, s)|\mathbf{y}] = 0$ .

#### 4. Proof of Proposition 4.

According to Assumption 3 (i), it holds that

$$(124) \quad n^{-1} \max_{1 \leq t \leq n} |g(\mathbf{y}_{t-1}, \mathbf{y}_t)|^2 \xrightarrow{p} 0.$$

According to simple algebra, we can show that

$$(125) \quad n^{-1} \sum_{t \leq \pi n} \tilde{f}_{n,t}^2 = n^{-1} \sum_{t \leq \pi n} [\tilde{g}_1(\mathbf{y}_{t-1}, \mathbf{y}_t)^2 + 2\tilde{g}_1(\mathbf{y}_{t-1}, \mathbf{y}_t)\tilde{g}_2(\mathbf{y}_{t-1}, \mathbf{y}_t)b(t/n) + \tilde{g}_2(\mathbf{y}_{t-1}, \mathbf{y}_t)^2b(t/n)^2].$$

Therefore, by Lemma 4 of Li and Müller (2009), it follows that

$$n^{-1} \sum_{t \leq \pi n} \tilde{f}_{n,t}^2 \xrightarrow{p} \mathbb{E}^{Q_0} [\tilde{g}_1^2] \pi + 2\mathbb{E}^{Q_0} [\tilde{g}_1\tilde{g}_2] \int_0^\pi b(u)du + \mathbb{E}^{Q_0} [\tilde{g}_1^2] \int_0^\pi b(u)^2du.$$

And hence, it holds that

$$(126) \quad n^{-1} \sum_{t \leq \pi n} \tilde{f}_{n,t}^2 \rightarrow \Upsilon(\pi) \equiv \mathbb{E}^{Q_0} [\tilde{g}^T B_\pi \tilde{g}].$$

Using the same argument, we can show that

$$(127) \quad n^{-1} \sum_{t \leq \pi n} \mathbb{E}_{t-1}^Q [\tilde{f}_{n,t}^2] \xrightarrow{p} \Upsilon(\pi) \equiv \mathbb{E}^{Q_0} [\tilde{g}^T B_\pi \tilde{g}].$$

The results above and Assumption 3 (i) together lead to a Lindeberg-type condition. Thus, according to the mixing condition implied by the Doeblin condition for the Markov process, we can obtain the following CLT result for martingale difference sequences:

$$(128) \quad \frac{1}{\sqrt{n}} \sum_{t \leq \pi n} \tilde{f}_{n,t} \xrightarrow{d} N(0, \Upsilon(\pi)).$$

#### 5. Proof of Proposition 5.

The proof is similar to that of Theorem 7.2 in van der Vaart (1998), except that we allow for non-IID time series and local instability. For brevity, we denote  $K_{n,t} \equiv K_{1/\sqrt{n}, g_{n,t}}$ . The random variable  $W_{n,t} \equiv \frac{dK_{n,t}}{dK_0} - 1$  is well defined with probability one. According to (123), it follows that

$$(129) \quad \sum_{t \leq n} W_{n,t} = \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{f}_{n,t} + \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{\Delta}_{n,t}.$$



where  $\tilde{f}_{n,t} \equiv f_{n,t} - E_{t-1}^{Q_0}[f_{n,t}]$ . Because  $E_{t-1}^{Q_0}[\tilde{\Delta}_{n,t}] = 0$  and  $E^{Q_0}[\tilde{\Delta}_{n,t}^2] \rightarrow 0$  as  $n \rightarrow \infty$  for all  $t = 1, \dots, n$ , it follows that

$$(130) \quad E^{Q_0} \left[ \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{\Delta}_{n,t} \right] = 0 \quad \text{and} \quad \text{var}^{Q_0} \left[ \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{\Delta}_{n,t} \right] \leq \frac{1}{n} \sum_{t \leq n} E^{Q_0}[\tilde{\Delta}_{n,t}^2] \rightarrow 0.$$

Thus,  $\frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{\Delta}_{n,t} = o_p(1)$  under  $Q_0$ . And hence, the following approximation holds:

$$(131) \quad \sum_{t \leq n} W_{n,t} = \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{f}_{n,t} + o_p(1).$$

By Taylor expansion, we have

$$(132) \quad \ln(1+x) = x - \frac{1}{2}x^2 + x^2 R(x),$$

where  $R(x)$  is a continuous function such that  $R(x) \rightarrow 0$  as  $x \rightarrow 0$ . Therefore, it follows that

$$(133) \quad \ln \prod_{t \leq n} \frac{dK_{n,t}}{dK_0} = \sum_{t \leq n} \ln(1 + W_{n,t}) = \sum_{t \leq n} \left[ W_{n,t} - \frac{1}{2}W_{n,t}^2 + W_{n,t}^2 R(W_{n,t}) \right]$$

$$(134) \quad = \sum_{t \leq n} W_{n,t} - \frac{1}{2} \sum_{t \leq n} W_{n,t}^2 + \sum_{t \leq n} W_{n,t}^2 R(W_{n,t}).$$

Combining (131) and (134) yields

$$(135) \quad \ln \prod_{t \leq n} \frac{dK_{n,t}}{dK_0} = \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{f}_{n,t} - \frac{1}{2} \sum_{t \leq n} W_{n,t}^2 + \sum_{t \leq n} W_{n,t}^2 R(W_{n,t}) + o_p(1).$$

We shall first show that

$$(136) \quad \sum_{t \leq n} W_{n,t}^2 = \frac{1}{n} \sum_{t \leq n} \tilde{f}_{n,t}^2 + o_p(1).$$

In fact, by the triangular inequality and the Cauchy-Schwarz inequality, it follows that

$$(137) \quad \left| \sum_{t \leq n} W_{n,t}^2 - \frac{1}{n} \sum_{t \leq n} \tilde{f}_{n,t}^2 \right| \leq \sum_{t \leq n} \left| \frac{1}{\sqrt{n}} \tilde{\Delta}_{n,t} \left( \frac{2}{\sqrt{n}} \tilde{f}_{n,t} + \frac{1}{\sqrt{n}} \tilde{\Delta}_{n,t} \right) \right|$$

$$(138) \quad \leq \left( \frac{1}{n} \sum_{t \leq n} \tilde{\Delta}_{n,t}^2 \right)^{1/2} \left[ \frac{1}{n} \sum_{t \leq n} (2\tilde{f}_{n,t} + \tilde{\Delta}_{n,t})^2 \right]^{1/2}.$$

Based on (123), it is straightforward to show that  $\frac{1}{n} \sum_{t \leq n} \tilde{\Delta}_{n,t}^2 = o_p(1)$ . Further, according to Assumption 3 (ii), it follows that  $\frac{1}{n} \sum_{t \leq n} (2\tilde{f}_{n,t} + \tilde{\Delta}_{n,t})^2 \leq \frac{1}{n} \sum_{t \leq n} 4\tilde{f}_{n,t}^2 + 2\tilde{\Delta}_{n,t}^2 = O_p(1)$ . Substituting them into (138)

leads to  $\sum_{t \leq n} W_{n,t}^2 - \frac{1}{n} \sum_{t \leq n} \tilde{f}_{n,t}^2 = o_p(1)$ . Therefore, the equality (135) can be rewritten as

$$(139) \quad \ln \prod_{t \leq n} \frac{dK_{n,t}}{dK_0} = \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{f}_{n,t} - \frac{1}{2n} \sum_{t \leq n} \tilde{f}_{n,t}^2 + \sum_{t \leq n} W_{n,t}^2 R(W_{n,t}) + o_p(1)$$

$$(140) \quad = \frac{1}{\sqrt{n}} \sum_{t \leq n} \tilde{f}_{n,t} - \frac{1}{2} \int_0^1 \Upsilon(u) du + \sum_{t \leq n} W_{n,t}^2 R(W_{n,t}) + o_p(1).$$

Finally, we shall show that  $\sum_{t \leq n} W_{n,t}^2 R(W_{n,t}) = o_p(1)$ . In fact, it holds that

$$(141) \quad \sum_{t \leq n} W_{n,t}^2 |R(W_{n,t})| \leq \max_{1 \leq t \leq n} |R(W_{n,t})| \sum_{t \leq n} W_{n,t}^2.$$

It suffices to show that  $\max_{1 \leq t \leq n} |R(W_{n,t})| = o_p(1)$ , as we have shown that  $\sum_{t \leq n} W_{n,t}^2 = O_p(1)$ . In fact, for any  $\epsilon > 0$ , there exists  $\epsilon_R > 0$  such that

$$(142) \quad \mathbb{P}_0 \left( \max_{1 \leq t \leq n} |R(W_{n,t})| > \epsilon \right) \leq \sum_{t \leq n} \mathbb{P}_0 (|R(W_{n,t})| > \epsilon) \leq \sum_{t \leq n} \mathbb{P}_0 (W_{n,t}^2 > \epsilon_R)$$

$$(143) \quad \leq \sum_{t \leq n} \mathbb{P}_0 (\tilde{f}_{n,t}^2 > n\epsilon_R/4) + \sum_{t \leq n} \mathbb{P}_0 (\tilde{\Delta}_{n,t}^2 > n\epsilon_R/4).$$

By Markov's inequality, we can further show that

$$(144) \quad \mathbb{P}_0 \left( \max_{1 \leq t \leq n} |R(W_{n,t})| > \epsilon \right) \leq \frac{4}{n\epsilon_R} \sum_{t \leq n} \mathbb{E}^{\mathbb{Q}_0} [\tilde{f}_{n,t}^2 \mathbf{1}\{\tilde{f}_{n,t}^2 > n\epsilon_R/4\}] + \frac{4}{n\epsilon_R} \sum_{t \leq n} \mathbb{E}^{\mathbb{Q}_0} [\tilde{\Delta}_{n,t}^2].$$

According to Assumption 3 (ii), the squared conditional scores  $\tilde{f}_{n,t}^2$  are uniformly integrable, and thus

$$(145) \quad \frac{1}{n} \sum_{t \leq n} \mathbb{E}^{\mathbb{Q}_0} [\tilde{f}_{n,t}^2 \mathbf{1}\{\tilde{f}_{n,t}^2 > n\epsilon_R/4\}] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Further, according to (123), it holds that

$$(146) \quad \frac{1}{n} \sum_{t \leq n} \mathbb{E}^{\mathbb{Q}_0} [\tilde{\Delta}_{n,t}^2] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore,  $\mathbb{P}_0 (\max_{1 \leq t \leq n} |R(W_{n,t})| > \epsilon) \rightarrow 0$  as  $n \rightarrow \infty$ .

## 6. Proof of Proposition 6

We first prove part (i). According to Proposition 2, if defining  $\tilde{m}_t(\theta_0) \equiv m_t(\theta_0) - \frac{1}{\sqrt{n}} \lambda(g^T) \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix}$  for  $t = 1, \dots, n$ , we have

$$(147) \quad \mathbb{E}^{\mathbb{Q}_{1/\sqrt{n}, f_{n,t}}} [\tilde{m}_t(\theta_0)] = o\left(\frac{1}{\sqrt{n}}\right), \text{ with } f_{n,t} = g(\mathbf{y}_{t-1}, \mathbf{y}_t)^T \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix}.$$

Further, for  $m_t(\theta_0)$  which satisfies Assumption 5, we know that the corresponding  $\tilde{m}_t(\theta_0)$  also satisfies Assumption 5. Therefore, appealing to the functional central limit theorem (invariance principle) of McLeish (1975a) and Phillips and Durlauf (1986), we know that

$$(148) \quad \frac{1}{\sqrt{n}} \sum_{t \leq \pi n} \tilde{m}_t(\theta_0) \xrightarrow{d} W(\pi), \quad \text{for all } \pi \in [0, 1].$$

Thus, it holds that

$$(149) \quad \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) = \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} \tilde{m}_t(\theta_0) + \frac{1}{n} \sum_{t \leq \pi n} \frac{\lambda(g^T)}{\sqrt{\pi}} \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix} \xrightarrow{d} \frac{W(\pi)}{\sqrt{\pi}} + \frac{\lambda(g^T)}{\sqrt{\pi}} \begin{bmatrix} \pi \\ \int_0^\pi b(u) du \end{bmatrix}.$$

Similarly, we can show that

$$(150) \quad \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) \xrightarrow{d} \frac{W(1) - W(\pi)}{\sqrt{1-\pi}} + \frac{\lambda(g^T)}{\sqrt{1-\pi}} \begin{bmatrix} 1-\pi \\ \int_\pi^1 b(u) du \end{bmatrix}.$$

Now, we prove part (ii). Because  $g_1, g_2 \in \mathcal{T}(\mathbf{Q}_0)$ , by the definition of  $\theta_{n,t}$ , we know that

$$(151) \quad 0 = \int m_t(\theta_{n,t}) d\mathbf{Q}_{1/\sqrt{n}, f_{n,t}}, \quad \text{for all } t, n.$$

Using Taylor expansion, we can obtain

$$(152) \quad 0 = \int \left[ m_t(\theta_0) + \nabla_\theta m_t(\dot{\theta}_{n,t})(\theta_{n,t} - \theta_0) \right] [1 + f_{n,t}/\sqrt{n} + \Delta_{n,t}/\sqrt{n}] d\mathbf{Q}_0, \quad \text{for all } t, n,$$

where  $\dot{\theta}_{n,t}$  lies between  $\theta_0$  and  $\theta_{n,t}$  for all  $t$  and  $n$ . Suppose  $\theta_{n,t}$  converges  $\theta_0$  at the rate of  $\sqrt{n}$  (it will be verified later). According to Assumption 5, it follows that

$$(153) \quad 0 = \frac{1}{\sqrt{n}} \lambda(g^T) \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix} + D(\theta_{n,t} - \theta_0) + o\left(\frac{1}{\sqrt{n}}\right), \quad \text{for all } t, n.$$

Therefore, the parameter sequence  $\theta_{n,t}$  can be specified as follows

$$(154) \quad \theta_{n,t} - \theta_0 = -(D^T D)^{-1} D^T \frac{1}{\sqrt{n}} \lambda(g^T) \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix} + o\left(\frac{1}{\sqrt{n}}\right), \quad \text{for all } t, n.$$

And hence, using Taylor expansion again leads to

$$(155) \quad \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_{n,t}) = \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) - \frac{1}{n} \sum_{t \leq \pi n} \nabla_\theta m_t(\dot{\theta}_{n,t})(D^T D)^{-1} D^T \frac{\lambda(g^T)}{\sqrt{\pi}} \begin{bmatrix} 1 \\ b(t/n) \end{bmatrix} + o(1).$$

Due to Assumption 5, appealing to Lemma 4 of [Li and Müller \(2009\)](#) leads to

$$(156) \quad \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_{n,t}) = \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) - D(D^T D)^{-1} D^T \frac{\lambda(g^T)}{\sqrt{\pi}} \left[ \begin{array}{c} \pi \\ \int_0^\pi b(u) du \end{array} \right] + o(1).$$

Because  $g_1, g_2 \in \mathcal{T}(\mathbb{Q}_0)$ , it holds that  $\lambda(g_1), \lambda(g_2) \in \text{lin}(D)$ , and thus

$$(157) \quad \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_{n,t}) = \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) - \frac{\lambda(g^T)}{\sqrt{\pi}} \left[ \begin{array}{c} \pi \\ \int_0^\pi b(u) du \end{array} \right] + o(1).$$

Similarly, we can show that

$$(158) \quad \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_{n,t}) = \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) - \frac{\lambda(g^T)}{\sqrt{1-\pi}} \left[ \begin{array}{c} 1-\pi \\ \int_\pi^1 b(u) du \end{array} \right] + o(1).$$

Finally, we prove parts (iii) and (iv). Using Taylor expansion, we can obtain

$$(159) \quad \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\hat{\theta}_{e,n}) = \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) + \frac{1}{\pi n} \sum_{t \leq \pi n} \nabla_\theta m_t(\hat{\theta}_{e,n}) \left[ \sqrt{\pi n}(\hat{\theta}_{e,n} - \theta_0) \right] + o(1),$$

where  $\hat{\theta}_{e,n}$  lies between  $\hat{\theta}_{e,n}$  and  $\theta_0$ . According to Proposition 8 (ii), it holds that

$$\begin{aligned} \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\hat{\theta}_{e,n}) &= \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) - D(D^T D)^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + o_p(1) \\ &= [I - D(D^T D)^{-1} D^T] \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + o_p(1). \end{aligned}$$

Similarly, we can show that

$$(160) \quad \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\hat{\theta}_{e,n}) = \frac{1}{\sqrt{(1-\pi)n}} \sum_{t > \pi n} m_t(\theta_0) - D(D^T D)^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + o_p(1).$$

Part (iv) can be proved in the same way. We shall not repeat it.

## 7. Proof of Proposition 7

The tangent set  $\mathcal{T}(\mathbb{Q}_0)$  can be characterized as follows (e.g., [Severini and Tripathi, 2013](#); [Chen and Santos, 2018](#)):

$$(161) \quad \mathcal{T}(\mathbb{Q}_0) = \{f \in L_0^2(\mathbb{Q}_0) : \mathbb{E}^{\mathbb{Q}_0} [m(\cdot, \theta_0) f] \in \text{lin}(D)\},$$

where  $\text{lin}(D)$  is the linear space spanned by the column vectors of  $D$ . Therefore, it suffices to show that

$$(162) \quad \mathbb{E}^{\mathbb{Q}_0} [m(\cdot, \theta_0)f] = \mathbb{E}^{\mathbb{Q}_0} [h(\cdot, \theta_0)f] \text{ for all } f \in L_0^2(\mathbb{Q}_0).$$

Under the assumption, the following identity holds:

$$(163) \quad \mathbb{E}^{\mathbb{Q}_0} [h(\mathbf{y}, \mathbf{y}', \theta_0)f(\mathbf{y}, \mathbf{y}')] = \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}, \mathbf{y}', \theta_0)f(\mathbf{y}, \mathbf{y}')] - \sum_{k=1}^{\infty} A_k,$$

where

$$(164) \quad A_k = \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}] f(\mathbf{y}, \mathbf{y}') \} - \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_k, \mathbf{y}_{k+1}, \theta_0) | \mathbf{y}_1 = \mathbf{y}'] f(\mathbf{y}, \mathbf{y}') \}.$$

Further, for each  $k \geq 1$ , the Markov property implies that

$$(165) \quad \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_k, \mathbf{y}_{k+1}, \theta_0) | \mathbf{y}_1 = \mathbf{y}'] f(\mathbf{y}, \mathbf{y}') = \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}'] f(\mathbf{y}, \mathbf{y}').$$

Thus, the equation (164) can be rewritten as

$$(166) \quad A_k = \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}] f(\mathbf{y}, \mathbf{y}') \} - \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}'] f(\mathbf{y}, \mathbf{y}') \}.$$

It suffices to show that  $A_k = 0$  for all  $k$ . In fact, the following equalities hold:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}'] f(\mathbf{y}, \mathbf{y}') \} \\ &= \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}'] \mathbb{E}^{\mathbb{Q}_0} [f(\mathbf{y}, \mathbf{y}') | \mathbf{y}'] \} \quad (\text{Law of Iterated Projections}) \\ &= \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}'] \mathbb{E}^{\mathbb{Q}_0} [f(\mathbf{y}', \mathbf{y}) | \mathbf{y}'] \} \quad (\text{Proposition 3}) \\ &= \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_{k-1}, \mathbf{y}_k, \theta_0) | \mathbf{y}_0 = \mathbf{y}'] f(\mathbf{y}', \mathbf{y}) \} \quad (\text{Law of Iterated Projections}) \end{aligned}$$

Therefore,  $A_k = 0$  for all  $k \geq 1$ , and hence from (163), it follows that

$$(167) \quad \mathbb{E}^{\mathbb{Q}_0} [h(\mathbf{y}, \mathbf{y}', \theta_0)f(\mathbf{y}, \mathbf{y}')] = \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}, \mathbf{y}', \theta_0)f(\mathbf{y}, \mathbf{y}')].$$

According to [Greenwood and Wefelmeyer \(1995\)](#), we know that

$$(168) \quad \mathbb{E}^{\mathbb{Q}_0} [h(\mathbf{y}_0, \mathbf{y}_1, \theta_0)h(\mathbf{y}_0, \mathbf{y}_1, \theta_0)^T] = \sum_{\tau=-\infty}^{\infty} \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_0, \mathbf{y}_1, \theta_0)m(\mathbf{y}_\tau, \mathbf{y}_{\tau+1}, \theta_0)^T] = I.$$

By Markov's property and the law of iterated projections, for all  $k \geq 0$ ,

$$(169) \quad \mathbb{E}^{\mathbb{Q}_0} \{ \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_k, \mathbf{y}_{k+1}, \theta_0) | \mathbf{y}_1] | \mathbf{y}_0 \} = \mathbb{E}^{\mathbb{Q}_0} [m(\mathbf{y}_k, \mathbf{y}_{k+1}, \theta_0) | \mathbf{y}_0].$$

Therefore,  $\mathbb{E}^{\mathbb{Q}_0} [h(\mathbf{y}, \mathbf{y}', \theta_0) | \mathbf{y}] = 0$ .

### 8. Proof of Proposition 8.

The proof follows the standard GMM approximations in Hansen (1982), Hansen (2007b), and Hansen (2012).

### 9. Proof of Proposition 9.

The cases of  $\psi_s$  with  $s \in \{e, o\}$  follow the same derivations, and so we only show the case  $s = e$ . We first prove part (i). Given the parameter value  $\theta_{e,n}^{(1)}$ , the constrained efficient GMM estimator  $(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}))^T$  for the full model satisfies the first-order condition

$$(170) \quad \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) = \Gamma_{\theta,1}^T \Lambda_{e,n}, \quad \text{with } \Gamma_{\theta,1} = [I, 0_{d_{\theta,1} \times d_{\theta,2}}],$$

and  $\Lambda_{e,n}$  is a  $d_{\theta,1} \times 1$  vector of Lagrangian multipliers for the constraints  $\Gamma_{\theta,1}\theta = \theta_n^{(1)}$  in search of the constrained GMM estimator  $(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}))^T$ . The Taylor expansion of  $\nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n)$  around  $\theta_0$  leads to

$$(171) \quad \frac{1}{\sqrt{\pi n}} \Gamma_{\theta,1}^T \Lambda_{e,n} = 2D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + 2\mathbf{I}_Q \sqrt{\pi n} \begin{bmatrix} \theta_{e,n}^{(1)} - \theta_0^{(1)} \\ \psi_e(\theta_{e,n}^{(1)}) - \theta_0^{(2)} \end{bmatrix} + o_p(1)$$

We first multiply both sides of (171) by  $\Gamma_{\theta,1} \mathbf{I}_Q^{-1}$ , and then by  $(\Gamma_{\theta,1} \mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T)^{-1}$ . The optimal Lagrangian multipliers can be represented as

$$(172) \quad \frac{1}{\sqrt{\pi n}} \Lambda_{e,n} = 2(\Gamma_{\theta,1} \mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T)^{-1} \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + 2(\Gamma_{\theta,1} \mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T)^{-1} \sqrt{\pi n} (\theta_{e,n}^{(1)} - \theta_0^{(1)}) + o_p(1).$$

Substituting (171) and (172) into (170) yields

$$(173) \quad \begin{aligned} \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) &= 2\Gamma_{\theta,1}^T (\Gamma_{\theta,1} \mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T)^{-1} \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] \\ &\quad + 2\Gamma_{\theta,1}^T (\Gamma_{\theta,1} \mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T)^{-1} \sqrt{\pi n} (\theta_{e,n}^{(1)} - \theta_0^{(1)}) + o_p(1). \end{aligned}$$

According to Proposition 1, we substitute (44) into (173) and obtain

$$(174) \quad \begin{aligned} \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) &= 2\Gamma_{\theta,1}^T \mathbf{I}_F \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] \\ &\quad - 2\Gamma_{\theta,1}^T \mathbf{I}_F \mathbf{I}_B^{-1} D_{11}^T \left[ \sqrt{\pi} \lambda^{(1)}(g_1) + \lambda^{(1)}(g_2) \int_0^\pi b(u) du / \sqrt{\pi} \right] + o_p(1). \end{aligned}$$

Based on (79), we have

$$(175) \quad \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) = 2\Gamma_{\theta,1}^T \mathbf{I}_F \left[ \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] - \mathbf{I}_B^{-1} D_{11}^T \Gamma_{m,1} \nu(g, b, \pi) \right] + o_p(1).$$

Given the baseline efficient GMM estimator  $\tilde{\theta}_{e,n}^{(1)}$  based on the estimation sample, the constrained GMM estimator  $(\tilde{\theta}_{e,n}^{(1)}, \psi_e(\tilde{\theta}_{e,n}^{(1)}))^T$  for the full model satisfies the first-order condition

$$(176) \quad \nabla J(\tilde{\theta}_{e,n}^{(1)}, \psi_e(\tilde{\theta}_{e,n}^{(1)}); \mathbf{y}_e^n) = \Gamma_{\theta,1}^T \Lambda_{e,n}^{(1)}, \quad \text{with } \Gamma_{\theta,1} = [I, 0_{d_{\theta,1} \times d_{\theta,2}}],$$

and  $\Lambda_{e,n}^{(1)}$  is a  $d_{\theta,1} \times 1$  vector of Lagrangian multipliers for the constraints  $\Gamma_{\theta,1}\theta = \tilde{\theta}_{e,n}^{(1)}$  in search of the constrained GMM estimator  $(\tilde{\theta}_{e,n}^{(1)}, \psi_e(\tilde{\theta}_{e,n}^{(1)}))^T$ . The Taylor expansion of  $\nabla J(\tilde{\theta}_{e,n}^{(1)}, \psi_e(\tilde{\theta}_{e,n}^{(1)}); \mathbf{y}_e^n)$  around  $(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}))^T$ , together with (176), leads to

$$(177) \quad \frac{1}{\sqrt{\pi n}} \Gamma_{\theta,1}^T \Lambda_{e,n}^{(1)} = \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) + 2\mathbf{I}_Q \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\tilde{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} + o_p(1).$$

We first multiply both sides of (177) by  $\Gamma_{\theta,1} \mathbf{I}_Q^{-1}$ , and then by  $(\Gamma_{\theta,1} \mathbf{I}_Q \Gamma_{\theta,1}^T)^{-1}$ . The optimal Lagrangian multipliers can be represented as

$$(178) \quad \frac{1}{\sqrt{\pi n}} \Lambda_{e,n}^{(1)} = \mathbf{I}_F \Gamma_{\theta,1} \mathbf{I}_Q^{-1} \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) + 2\mathbf{I}_F \sqrt{\pi n} (\tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)}) + o_p(1).$$

Further substituting (173) into equation (178) above yields

$$(179) \quad \frac{1}{\sqrt{\pi n}} \Lambda_{e,n}^{(1)} = 2\mathbf{I}_F \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] + 2\mathbf{I}_F \sqrt{\pi n} (\tilde{\theta}_{e,n}^{(1)} - \theta_0^{(1)}) + o_p(1).$$

Based on Proposition 8, we can obtain

$$(180) \quad \sqrt{\pi n} (\tilde{\theta}_{e,n}^{(1)} - \theta_0^{(1)}) = -\mathbf{I}_B^{-1} D_{11}^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t^{(1)}(\theta_0^{(1)}) \right] + o_p(1).$$

Substituting (180) into (179) gives the following asymptotic representation of  $\frac{1}{\sqrt{\pi n}} \Lambda_{e,n}^{(1)}$ :

$$(181) \quad \frac{1}{\sqrt{\pi n}} \Lambda_{e,n}^{(1)} = 2\mathbf{I}_F \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] - 2\mathbf{I}_F \mathbf{I}_B^{-1} D_{11}^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t^{(1)}(\theta_0^{(1)}) \right] + o_p(1).$$

We substitute (173) and (181) into (177) and multiply the both sides by  $\mathbf{I}_Q^{-1}/2$ . The estimator can be represented by

$$(182) \quad \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\tilde{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} = -\mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T \mathbf{I}_F \mathbf{I}_B^{-1} D_{11}^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t^{(1)}(\theta_0^{(1)}) - \nu_e^{(1)}(g, b, \pi) \right] + o_p(1) \\ = -\mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T \mathbf{I}_F \mathbf{I}_B^{-1} D_{11}^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t^{(1)}(\theta_{n,t}^{(1)}) \right] + o_p(1).$$

Now we prove part (ii). The estimators  $\psi_e(\theta_{e,n}^{(1)})$  and  $\hat{\theta}_{e,n}^{(2)} = \psi_e(\hat{\theta}_{e,n}^{(1)})$  are the constrained efficient GMM estimators for the nuisance parameter  $\theta^{(2)}$  when controlling for  $\Gamma_{\theta,1}\theta = \theta_{e,n}^{(1)}$  and  $\Gamma_{\theta,1}\theta = \hat{\theta}_{e,n}^{(1)}$ , respectively. Due to the first order condition  $\nabla J(\hat{\theta}_n^{(1)}, \psi_e(\hat{\theta}_n^{(1)}); \mathbf{y}_e^n) = 0$ , the Taylor expansion of  $\nabla J(\hat{\theta}_n^{(1)}, \psi_e(\hat{\theta}_n^{(1)}); \mathbf{y}_e^n)$  around  $(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}))^T$  leads to

$$(183) \quad 0 = \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) + 2\mathbf{I}_Q \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\hat{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} + o_p(1).$$

Substituting (173) into (183) and multiplying the both sides by  $\mathbf{I}_Q^{-1}/2$ , we have

$$(184) \quad \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\hat{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} = -\mathbf{I}_Q^{-1} \Gamma_{\theta,1}^T \mathbf{I}_F \left\{ \Gamma_{\theta,1} \mathbf{I}_Q^{-1} D^T \left[ \frac{1}{\sqrt{\pi n}} \sum_{t \leq \pi n} m_t(\theta_0) \right] - \mathbf{I}_B^{-1} D_{11}^T \Gamma_{m,1} \nu_e \right\} + o_p(1).$$

### 10. Proof of Proposition 10.

We first approximate  $\mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n)$ . According to the second-order Taylor expansion around  $(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}))$ , it follows that

$$(185) \quad \begin{aligned} \mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n) &= \left[ \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) \right]^T \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\tilde{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} \\ &\quad + \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\tilde{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix}^T \mathbf{I}_Q \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\tilde{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} + o_p(1). \end{aligned}$$

Thus, following (175) and (182), it holds that

$$(186) \quad \mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n) = -2[L_F \zeta_{e,n} + L_\Delta \nu_e]^T \mathbf{I}_F L_B \zeta_{e,n} + \zeta_{e,n}^T L_B^T \mathbf{I}_F L_B \zeta_{e,n} + o_p(1).$$

We now approximate  $\mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n)$ . According to the second-order Taylor expansion around  $(\theta_{o,n}^{(1)}, \psi_o(\theta_{o,n}^{(1)}))$ , it follows that

$$(187) \quad \begin{aligned} \mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n) &= \left[ \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{o,n}^{(1)}, \psi_o(\theta_{o,n}^{(1)}); \mathbf{y}_o^n) \right]^T \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{o,n}^{(1)} \\ \psi_o(\tilde{\theta}_{e,n}^{(1)}) - \psi_o(\theta_{o,n}^{(1)}) \end{bmatrix} \\ &\quad + \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{o,n}^{(1)} \\ \psi_o(\tilde{\theta}_{e,n}^{(1)}) - \psi_o(\theta_{o,n}^{(1)}) \end{bmatrix}^T \mathbf{I}_Q \sqrt{\pi n} \begin{bmatrix} \tilde{\theta}_{e,n}^{(1)} - \theta_{o,n}^{(1)} \\ \psi_o(\tilde{\theta}_{e,n}^{(1)}) - \psi_o(\theta_{o,n}^{(1)}) \end{bmatrix} + o_p(1). \end{aligned}$$

Similarly, we can show that

$$(188) \quad \mathcal{L}(\tilde{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n) = -2[L_F \zeta_{o,n} + L_\Delta \nu_o]^T \mathbf{I}_F L_B \zeta_{e,n} + \zeta_{e,n}^T L_B^T \mathbf{I}_F L_B \zeta_{e,n} + o_p(1).$$

We now approximate  $\mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n)$ . According to the second-order Taylor expansion around  $(\theta_{e,n}^{(1)}, \psi_o(\theta_{e,n}^{(1)}))$ ,



it follows that

$$(189) \quad \mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n) = \left[ \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{e,n}^{(1)}, \psi_e(\theta_{e,n}^{(1)}); \mathbf{y}_e^n) \right]^T \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\hat{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} \\ + \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\hat{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix}^T \mathbf{I}_\Omega \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{e,n}^{(1)} \\ \psi_e(\hat{\theta}_{e,n}^{(1)}) - \psi_e(\theta_{e,n}^{(1)}) \end{bmatrix} + o_p(1).$$

Similarly, we can show that

$$(190) \quad \mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_e^n) = -[L_F \zeta_{e,n} + L_\Delta \nu_e]^T \mathbf{I}_F [L_F \zeta_{e,n} + L_\Delta \nu_e] + o_p(1).$$

We now approximate  $\mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n)$ . According to the second-order Taylor expansion around  $(\theta_{o,n}^{(1)}, \psi_o(\theta_{o,n}^{(1)}))$ , it follows that

$$(191) \quad \mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n) = \left[ \frac{1}{\sqrt{\pi n}} \nabla J(\theta_{o,n}^{(1)}, \psi_o(\theta_{o,n}^{(1)}); \mathbf{y}_o^n) \right]^T \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{o,n}^{(1)} \\ \psi_o(\hat{\theta}_{e,n}^{(1)}) - \psi_o(\theta_{o,n}^{(1)}) \end{bmatrix} \\ + \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{o,n}^{(1)} \\ \psi_o(\hat{\theta}_{e,n}^{(1)}) - \psi_o(\theta_{o,n}^{(1)}) \end{bmatrix}^T \mathbf{I}_\Omega \sqrt{\pi n} \begin{bmatrix} \hat{\theta}_{e,n}^{(1)} - \theta_{o,n}^{(1)} \\ \psi_o(\hat{\theta}_{e,n}^{(1)}) - \psi_o(\theta_{o,n}^{(1)}) \end{bmatrix} + o_p(1).$$

Similarly, we can show that

$$(192) \quad \mathcal{L}(\hat{\theta}_{e,n}^{(1)}; \mathbf{y}_o^n) = -2[L_F \zeta_{o,n} + L_\Delta \nu_o]^T \mathbf{I}_F [L_F \zeta_{e,n} + L_\Delta \nu_e] + [L_F \zeta_{e,n} + L_\Delta \nu_e]^T \mathbf{I}_F [L_F \zeta_{e,n} + L_\Delta \nu_e] + o_p(1).$$

## APPENDIX E: PROOFS OF COROLLARIES

### 1. Proof of Corollary 1.

We can derive the result following the same derivations for (154) under the baseline GMM model  $\mathcal{Q}^{(1)}$ .

### 2. Proof of Corollary 2.

The proof is similar to that of Lemma 1 of [Li and Müller \(2009\)](#), which is based on Le Cam's first lemma (see, e.g., [van der Vaart, 1998](#), Page 88).

## APPENDIX F: DERIVATION OF THE DISASTER RISK MODEL

We first show how to derive the Euler equation, and then how to obtain the dark matter measure  $\varrho(p, \xi)$ . The total return of market equity from  $t$  to  $t+1$  is  $e^{r_{M,t+1}}$ , which is unknown at  $t$ , and the total return of the risk-free bond from  $t$  to  $t+1$  is  $e^{r_{f,t}}$ , which is known at  $t$ . Thus, the excess log return of equity is  $r_{t+1} = r_{M,t+1} - r_{f,t}$ . The inter-temporal marginal rate of substitution is  $\mathcal{M}_{t,t+1} = \delta_D e^{-\gamma_D g_{t+1}}$ . The Euler equations for the risk-free

rate and the market equity return are

$$(193) \quad 1 = \mathbb{E}_t [\mathcal{M}_{t,t+1} e^{r_{M,t+1}}] \quad \text{and} \quad e^{-r_{f,t}} = \mathbb{E}_t [\mathcal{M}_{t,t+1}].$$

Thus, we obtain the following Euler equation for the excess log return:

$$(194) \quad \mathbb{E}_t [\mathcal{M}_{t,t+1}] = \mathbb{E}_t [\mathcal{M}_{t,t+1} e^{r_{t+1}}].$$

The left-hand side of (194) is equal to

$$\mathbb{E}_t [\mathcal{M}_{t,t+1}] = \mathbb{E}_t [e^{-\gamma_D g_{t+1}}] = (1-p)e^{-\gamma_D \mu + \frac{1}{2} \gamma_D^2 \sigma^2} + p\xi \frac{e^{\gamma_D v}}{\xi - \gamma_D},$$

and the right-hand side of (194) is equal to

$$\mathbb{E}_t [\mathcal{M}_{t,t+1} e^{r_{t+1}}] = \mathbb{E}_t [e^{-\gamma_D g_{t+1} + r_{t+1}}] = (1-p)e^{-\gamma_D \mu + \eta + \frac{1}{2}(\gamma_D^2 \xi^2 + \tau^2 - 2\gamma_D \rho \sigma \tau)} + p\xi \frac{e^{\frac{\xi^2}{2} + (\gamma_D - b)v}}{\xi + b - \gamma_D}.$$

Thus, the Euler equation (194) can be rewritten as

$$(195) \quad (1-p)e^{-\gamma_D \mu + \frac{1}{2} \gamma_D^2 \sigma^2} [e^{\eta + \frac{\tau^2}{2} - \gamma_D \rho \sigma \tau} - 1] = p\Delta(\xi), \quad \text{where } \Delta(\xi) = \xi \left( \frac{e^{\gamma_D v}}{\xi - \gamma_D} - \frac{e^{\frac{\xi^2}{2} + (\gamma_D - b)v}}{\xi + b - \gamma_D} \right).$$

Using the Taylor expansion, we have the following approximation:

$$(196) \quad e^{\eta + \frac{\tau^2}{2} - \gamma_D \rho \sigma \tau} - 1 \approx \eta + \frac{\tau^2}{2} - \gamma_D \rho \sigma \tau,$$

which, combined with (195), gives the approximated Euler equation in (67).

Now, we show how to derive the dark matter measure. The Jacobian matrix of the moment restrictions and the asymptotic variance-covariance matrix are

$$(197) \quad D_{11} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{p}{\xi^2} \end{bmatrix} \quad \text{and} \quad \Omega_{11} = \begin{bmatrix} p(1-p) & 0 \\ 0 & (1-p)\sigma^2 + \frac{p}{\xi^2} \end{bmatrix} \approx \begin{bmatrix} p(1-p) & 0 \\ 0 & \frac{p}{\xi^2} \end{bmatrix}, \quad \text{respectively.}$$

The approximation above is simply due to the tiny magnitude of  $\sigma^2 \approx 0$ . The information matrix for the baseline model is

$$(198) \quad \Sigma_1 = D_{11}^T \Omega_{11}^{-1} D_{11} \approx \begin{bmatrix} \frac{1}{p(1-p)} & 0 \\ 0 & \frac{p}{\xi^2} \end{bmatrix}.$$

Next, the Jacobian matrix of moments restrictions and the asymptotic variance-covariance matrix for the full

model are

$$(199) \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{p}{\xi^2} \\ -(1-p)\frac{\partial\eta(p,\xi)}{\partial p} & -(1-p)\frac{\partial\eta(p,\xi)}{\partial\xi} - \frac{pb}{\xi^2} \end{bmatrix},$$

and

$$(200) \quad \Omega = \begin{bmatrix} p(1-p) & 0 & 0 \\ 0 & (1-p)\sigma^2 + \frac{p}{\xi^2} & (1-p)\rho\sigma\tau + bp/\xi^2 \\ 0 & (1-p)\rho\sigma\tau + bp/\xi^2 & (1-p)\tau^2 + pb^2/\xi^2 \end{bmatrix},$$

where

$$(201) \quad \eta(p, \xi) \equiv \gamma_D \rho \sigma \tau - \frac{\tau^2}{2} + \ln \left[ 1 + e^{\gamma_D \mu - \frac{\gamma_D^2 \sigma^2}{2}} \xi \left( \frac{e^{\gamma_D \underline{v}}}{\xi - \gamma_D} - e^{\frac{1}{2}\xi^2} \frac{e^{(\gamma_D - b)\underline{v}}}{\xi + b - \gamma_D} \right) \frac{p}{1-p} \right].$$

We can also derive the closed-form solution for the dark matter measure in (70) if we use the approximated Euler equation in (67). In this case, using the notation introduced in (68) and (71), we can express the information matrix for  $(p, \xi)$  under the full GMM model as

$$(202) \quad \Sigma \approx \begin{bmatrix} \frac{1}{p(1-p)} + \frac{\Delta(\xi)^2}{(1-\rho^2)\tau^2} \frac{e^{2\gamma_D\mu - \gamma_D^2\sigma^2}}{(1-p)^3} & \frac{p}{(1-\rho^2)\tau^2} \frac{e^{2\gamma_D\mu - \gamma_D^2\sigma^2}}{(1-p)^2} \Delta(\xi) \dot{\Delta}(\xi) \\ \frac{p}{(1-\rho^2)\tau^2} \frac{e^{2\gamma_D\mu - \gamma_D^2\sigma^2}}{(1-p)^2} \Delta(\xi) \dot{\Delta}(\xi) & \frac{p}{\xi^2} + \frac{\dot{\Delta}(\xi)^2}{(1-\rho^2)\tau^2} e^{2\gamma_D\mu - \gamma_D^2\sigma^2} \frac{p^2}{1-p} \end{bmatrix}.$$

The largest eigenvalue of the matrix  $\Sigma^{1/2}\Sigma_1^{-1}\Sigma^{1/2}$  is also the largest eigenvalue of  $\Sigma_1^{-1/2}\Sigma\Sigma_1^{-1/2}$ . In this case, the eigenvalues and eigenvectors are available in closed form. This gives us the formula for  $\varrho(\theta)$  in (70).

## APPENDIX G: MISCELLANEOUS PROOFS AND DERIVATIONS

### G.1. *Moment Rotations*

Construct a lower block triangular matrix  $L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$  such that

$$(203) \quad \Omega^{-1} = L^T L.$$

It is most straightforward to analyze a rotated system of moment restrictions. Let

$$(204) \quad \tilde{m}_t(\theta) = L m_t(\theta) = \begin{bmatrix} L_{11} m_t^{(1)}(\theta^{(1)}) \\ L_{21} m_t^{(1)}(\theta^{(1)}) + L_{22} m_t^{(2)}(\theta) \end{bmatrix} = \begin{bmatrix} \tilde{m}_t^{(1)}(\theta^{(1)}) \\ \tilde{m}_t^{(2)}(\theta) \end{bmatrix}.$$

Further, we let

$$(205) \quad \tilde{D} = LD = \begin{bmatrix} L_{11}D_{11} & 0 \\ L_{21}D_{11} + L_{22}D_{21} & L_{22}D_{22} \end{bmatrix} = \begin{bmatrix} \tilde{D}_{11} & 0 \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix}.$$

For notational simplicity, we drop the  $\tilde{\cdot}$  but use the transformed system.

### G.2. Hellinger-differentiability Condition

The condition (33) is equivalent to the following condition:

$$(206) \quad \left( \frac{dQ_{s,g}}{dQ} \right)^{1/2} = 1 + \frac{1}{2}sg + s\varepsilon(s)$$

where  $\varepsilon(s)$  converges to zero in  $L^2(Q)$  as  $s \rightarrow 0$ . Equation (206) is equivalent to

$$(207) \quad \lim_{s \rightarrow 0} \int \left[ \frac{1}{s} \left( \left( \frac{dQ_{s,g}}{dQ} \right)^{1/2} - 1 \right) - \frac{1}{2}g \right]^2 dQ = \lim_{s \rightarrow 0} \int \varepsilon(s)^2 dQ = 0.$$

### G.3. The Expression of $\Lambda$

Let  $D = [D_1, D_2]$  where  $D_1^T = [D_{11}^T, D_{21}^T]$  and  $D_2^T = [0, D_{22}^T]$ . Thus, we have<sup>29</sup>

$$(208) \quad P_2 = I - D_2 (D_2^T D_2)^{-1} D_2^T = \begin{bmatrix} I & 0 \\ 0 & \Lambda_2 \end{bmatrix}.$$

Using the rules for the inversion of partitioned matrices, we have

$$(D^T D)^{-1} = \begin{bmatrix} (D_1^T P_2 D_1)^{-1} & - (D_1^T P_2 D_1)^{-1} D_1^T D_2 (D_2^T D_2)^{-1} \\ - (D_2^T D_2)^{-1} D_2^T D_1 (D_1^T P_2 D_1)^{-1} & (D_2^T D_2)^{-1} + (D_2^T D_2)^{-1} D_2^T D_1 (D_1^T P_2 D_1)^{-1} D_1^T D_2 (D_2^T D_2)^{-1} \end{bmatrix}.$$

Thus, we can show that

$$(209) \quad \begin{aligned} D (D^T D)^{-1} D^T &= D_1 (D_1^T P_2 D_1)^{-1} D_1^T - D_1 (D_1^T P_2 D_1)^{-1} D_1^T (I - P_2) \\ &\quad - (I - P_2) D_1 (D_1^T P_2 D_1)^{-1} D_1^T \\ &\quad + (I - P_2) + (I - P_2) D_1 (D_1^T P_2 D_1)^{-1} D_1^T (I - P_2) \\ &= I - P_2 + P_2 D_1 (D_1^T P_2 D_1)^{-1} D_1^T P_2. \end{aligned}$$

Therefore, it holds that

$$(210) \quad \Lambda = I - D (D^T D)^{-1} D^T = P_2 - P_2 D_1 (D_1^T P_2 D_1)^{-1} D_1^T P_2.$$

---

<sup>29</sup>The matrix inversion is the generalized inversion.

Recall that  $\mathbf{I}_F = D_1^T P_2 D_1$  (from Equation (29)). The matrix  $\Lambda$  can be rewritten as

$$(211) \quad \Lambda = \begin{bmatrix} I - D_{11} \mathbf{I}_F^{-1} D_{11}^T & D_{11} \mathbf{I}_F^{-1} D_{11}^T \Lambda_2 \\ \Lambda_2 D_{11} \mathbf{I}_F^{-1} D_{11}^T & \Lambda_2 - \Lambda_2 D_{21} \mathbf{I}_F^{-1} D_{21}^T \Lambda_2 \end{bmatrix}.$$

## APPENDIX H: DARK MATTER OF LONG-RUN RISK MODELS

In the second example, we consider a long-run risk model similar to [Bansal and Yaron \(2004\)](#) and [Bansal, Kiku, and Yaron \(2012\)](#). In the model, the representative agent has recursive preferences as in [Epstein and Zin \(1989\)](#) and [Weil \(1989\)](#) and maximizes her lifetime utility,

$$(212) \quad V_t = \left[ (1 - \delta_L) C_t^{1-1/\psi_L} + \delta_L \left( \mathbb{E}_t \left[ V_{t+1}^{1-\gamma_L} \right] \right)^{\frac{1-1/\psi_L}{1-\gamma_L}} \right]^{\frac{1}{1-1/\psi_L}},$$

where  $C_t$  is consumption at time  $t$ ,  $\delta_L$  is the rate of time preference,  $\gamma_L$  is the coefficient of risk aversion for timeless gambles, and  $\psi_L$  is the elasticity of intertemporal substitution when there is perfect certainty. The log growth rate of consumption  $\Delta c_t$ , the expected consumption growth  $x_t$ , and the conditional volatility of consumption growth  $\sigma_t$  evolve as follows:

$$(213a) \quad \Delta c_{t+1} = \mu_c + x_t + \sigma_t \epsilon_{c,t+1}$$

$$(213b) \quad x_{t+1} = \rho x_t + \varphi_x \sigma_t \epsilon_{x,t+1}$$

$$(213c) \quad \tilde{\sigma}_{t+1}^2 = \bar{\sigma}^2 + \nu(\tilde{\sigma}_t^2 - \bar{\sigma}^2) + \sigma_w \epsilon_{\sigma,t+1}$$

$$(213d) \quad \sigma_{t+1}^2 = \max(\underline{\sigma}^2, \tilde{\sigma}_{t+1}^2)$$

where the shocks  $\epsilon_{c,t}$ ,  $\epsilon_{x,t}$ , and  $\epsilon_{\sigma,t}$  are i.i.d.  $N(0, 1)$  and mutually independent. The volatility process (213c) potentially allows for negative values of  $\tilde{\sigma}_t^2$ . Following the literature, we impose a small positive lower bound  $\underline{\sigma}$  ( $= 1$  bps) on variance  $\sigma_t$  in solutions and simulations. Negative values of conditional variance can also be avoided by changing the specification. For example, the process of  $\sigma_t^2$  can be specified as a discrete-time version of the square root process.<sup>30</sup>

Next, the log dividend growth  $\Delta d_t$  follows

$$(214) \quad \Delta d_{t+1} = \mu_d + \phi_d x_t + \varphi_{d,c} \sigma_t \epsilon_{c,t+1} + \varphi_{d,d} \sigma_t \epsilon_{d,t+1},$$

---

<sup>30</sup>To ensure that our analysis applies as closely as possible to the model as formulated in the literature, we deliberately choose to follow [Bansal, Kiku, and Yaron \(2012, 2016a\)](#). In particular, following these papers, we also solve the model using a local log-linear expansion around the steady state. Thus, the approximate price-dividend ratio is not affected by the presence of the lower bound on the conditional variance process. As [Bansal, Kiku, and Yaron \(2016a\)](#) show, the resulting approximation error, when compared to the global numerical solution, is negligible. When computing our asymptotic Fisher fragility measure, we impose the lower bound on conditional variance. Thus, our asymptotic measure reflects the specification of the conditional variance process with the lower bound.

TABLE I  
PARAMETERS OF THE BENCHMARK LONG-RUN RISK MODEL (M1).

Preferences	$\delta_L$	$\gamma_L$	$\psi_L$			
	0.9989	10	1.5			
Consumption	$\mu_c$	$\rho$	$\varphi_x$	$\bar{\sigma}$	$\nu$	$\sigma_w$
	0.0015	0.975	0.038	0.0072	0.999	$2.8e-6$
Dividends	$\mu_d$	$\phi_d$	$\varphi_{d,c}$	$\varphi_{d,d}$		
	0.0015	2.5	2.6	5.96		
Returns	$\varphi_r$					
	3.0					

Note: Model M2 has  $\nu = 0.98$  and  $\gamma_L = 27$ , while the other parameters are the same as in Model M1.

where the shocks  $\epsilon_{d,t}$  are *i.i.d.*  $N(0, 1)$  and independent of the other shocks in (213a–213c). The equilibrium excess log return follows

$$(215) \quad r_{t+1}^e = \mu_{r,t}^e + \beta_c \sigma_t \epsilon_{c,t+1} + \beta_x \sigma_t \epsilon_{x,t+1} + \beta_\sigma \sigma_w \epsilon_{\sigma,t+1} + \varphi_{d,d} \sigma_t \epsilon_{d,t+1},$$

where the conditional average log excess return is

$$(216) \quad \mu_{r,t}^e = \lambda_c \beta_c \sigma_t^2 + \lambda_x \beta_x \varphi_x \sigma_t^2 + \lambda_\sigma \beta_\sigma \sigma_w^2 - \frac{1}{2} \sigma_{r_m,t}^2,$$

$$(217) \quad \text{where } \sigma_{r_m,t}^2 = \beta_c^2 \sigma_t^2 + \beta_x^2 \sigma_t^2 + \beta_\sigma^2 \sigma_w^2 + \varphi_{d,d}^2 \sigma_t^2.$$

The expressions for  $\lambda_c$ ,  $\lambda_x$ ,  $\lambda_\sigma$ ,  $\beta_c$ ,  $\beta_x$ ,  $\beta_\sigma$ , and  $A_{m,j}$  with  $j = 0, 1, 2$  are presented in Online Appendix.

The model contains stochastic singularities. For instance, the excess log market return  $r_{t+1}^e$  is a deterministic function of  $\Delta c_{t+1}, \Delta d_{t+1}, x_{t+1}, x_t, \sigma_{t+1}^2$ , and  $\sigma_t^2$ . The log price-dividend ratio  $z_{m,t}$  is a deterministic function of  $x_t$  and  $\sigma_t^2$ . To avoid the problems posed by stochastic singularities, we add noise shocks  $\varphi_r \sigma_t \epsilon_{r,t+1}$  to stock returns, with  $\epsilon_{r,t}$  being *i.i.d.* standard normal variables and mutually independent of other variables. This is a standard approach in the dynamic stochastic general equilibrium (DSGE) literature for dealing with stochastic singularity. The stochastic singularity is one of the main reasons why we adopt the moment-based method, rather than the likelihood-based method, to evaluate and characterize the structural models.

### *Quantitative analysis.*

In our computation of the dark matter measure, we consider a system of moment restrictions based on the joint dynamics of time series  $(\Delta c_{t+1}, x_t, \sigma_t^2, \Delta d_{t+1}, r_{t+1}^e)$ .<sup>31</sup>

We choose the model of consumption (213a)–(213d) and dividend (214) as the baseline model  $\mathcal{Q}^{(1)}$  with variables  $(\Delta c_{t+1}, x_t, \sigma_t^2, \Delta d_{t+1})$ . The moment restrictions associated with the baseline model constitute the set of baseline moment restrictions. We assume that the econometrician observes the processes of consumption and dividends, including the conditional mean and volatility  $x_t$  and  $\sigma_t^2$ , and the process of asset returns.

The simulated moments and sample moments are listed in Table II. The sample moments are based on

<sup>31</sup>Details can be found in the Online Appendix.

annual data from 1930 to 2008, and the simulated moments are 80-year annual data aggregated from monthly simulated data.

TABLE II  
SIMULATED AND SAMPLE MOMENTS.

Moment	Data	Model 1			Model 2		
	Estimate	5%	Median	95%	5%	Median	95%
$\mathbb{E}[r_M - r_f]$	7.09	2.33	5.88	10.58	3.65	6.78	10.05
$\mathbb{E}[r_M]$	7.66	2.91	6.66	11.20	4.42	7.75	11.20
$\sigma(r_M)$	20.28	12.10	20.99	29.11	15.01	17.55	20.33
$\mathbb{E}[r_f]$	0.57	-0.20	0.77	1.45	0.47	0.96	1.46
$\sigma(r_f)$	2.86	0.64	1.07	1.62	0.73	0.94	1.23
$\mathbb{E}[p - d]$	3.36	2.69	2.99	3.30	2.77	2.81	2.85
$\sigma(p - d)$	0.45	0.13	0.18	0.28	0.09	0.11	0.13

Accordingly, the baseline parameters are  $\theta^{(1)} = (\mu_c, \rho, \varphi_x, \bar{\sigma}^2, \nu, \sigma_w, \mu_d, \phi_d, \varphi_{d,c}, \varphi_{d,d})$  with  $d_{\theta,1} = 10$ . By measuring the fragility of the long-run risk model relative to this particular baseline, we can interpret the fragility measure as quantifying the information that asset pricing restrictions provide for the consumption and dividend dynamics above and beyond the information contained in consumption and dividend data. We explicitly account for uncertainty about preference parameters  $\gamma_L$  and  $\psi_L$  by including them in the nuisance parameter vector  $\theta^{(2)}$ . Thus,  $\theta^{(2)} = (\gamma_L, \psi_L)$ . The extra data investigated by the full structural model  $\mathcal{Q}$  are the excess log market returns  $r_{t+1}^e$ . Other parameters are included in the auxiliary parameter vector  $(\delta_L, \varphi_r)$  which are fixed at known values, and these values are part of the imposed functional-form specification of the structural component that is under fragility assessment. Note that the baseline model covers the joint dynamics of consumption growth and dividend growth. The structural model adds the description of the distribution of stock returns in relation to the consumption and dividend growth processes.

The parameter values of Model M1 follow [Bansal, Kiku, and Yaron \(2012\)](#) and are summarized in [Table I](#). As [Bansal, Kiku, and Yaron \(2012\)](#) (Table 2, page 194) show, the simulated first and second moments, based on the parametrization of Model M1, match the set of key asset pricing moments in the data reasonably well. The same is true for Model M2 whose parameter values are also reported in [Table III](#) (see [Table II](#) for the asset pricing moment matching). Our main purpose of presenting [Table III](#) is to compare the fragility of M1 and M2, two different calibrations of the LRR model, while the two panels are meant to further illustrate the fact that different treatments of the nuisance parameters can also affect the fragility measure.

First, consider panel I of [Table III](#). This panel contains fragility measures computed under the specification that treats preference parameters as unknown nuisance parameters whose uncertainty needs to be taken into account when computing the dark matter measure. The row (M1) of panel I reports fragility measures for Model M1 when the unknown nuisance parameters are  $\gamma_L$  and  $\psi_L$ . The dark matter measure  $\varrho(\theta_0) = 196.3$  is large. This implies that to match the precision of the estimator for the full structural model in all directions, the estimator based on the baseline model would require a time-series sample that is 196.3 times as long.

The values of  $\varrho(\theta_0)$  suggest that the asset pricing implications of the structural model are highly sensitive to plausible perturbations of parameter values. We compute the fragility measure for each individual parameter

TABLE III  
DARK MATTER MEASURES FOR THE LONG-RUN RISK MODELS

Model	$\varrho(\theta_0)$	$\theta^{(1)}$									
		$\mu_c$	$\rho$	$\varphi_x$	$\overline{\sigma}^2$	$\nu$	$\sigma_w$	$\mu_d$	$\phi_d$	$\varphi_{d,c}$	$\varphi_{d,d}$
I. Nuisance parameter vector $\psi$ : $(\gamma_{\text{L}}, \psi_{\text{L}})$											
(M1)	196.3	1.0	1.1	1.0	48.9	97.8	1.0	1.0	3.4	1.0	1.0
(M2)	21.1	1.0	1.1	1.0	1.0	3.4	1.0	1.4	4.2	1.0	1.0
II. Nuisance parameter vector $\psi$ : empty											
(M1)	$3.57 \cdot 10^5$	1.0	2.1	1.1	115.6	117.5	1.3	1.1	7.1	1.0	1.0
(M2)	287.7	1.0	2.5	1.0	1.0	6.3	1.0	1.9	31.3	1.0	1.0

Note: The direction corresponding to the worst-case one-dimensional fragility measure  $\varrho(\theta_0)$  for Model M1 is given by  $v_{\max}^* = [0.000, 0.000, -0.000, 0.020, -0.001, 0.999, -0.001, 0.000, -0.000, 0.000]$ . Model M2 has  $\nu = 0.98$  and  $\gamma_L = 27$  with other parameters unchanged. In panel I, we account for the uncertainty of preference parameters  $\theta^{(2)} = (\gamma_L, \psi_L)$  to compute the dark matter measure of the baseline parameters  $\theta^{(1)}$ . That is,  $\theta^{(1)}$  and  $\theta^{(2)}$  are treated as unknown parameters, while  $(\delta_L, \varphi_r)$  are treated as auxiliary parameters with known fixed values in panel I. In panel II, these preference parameters are fixed as auxiliary parameters with the nuisance parameter vector  $\theta^{(2)}$  empty. That is,  $\theta^{(1)}$  is treated as unknown parameters, while  $(\gamma_L, \psi_L)$  and  $(\delta_L, \varphi_r)$  are treated as auxiliary parameters with known fixed values in panel II.

in the vector  $\theta^{(1)}$ . All of the univariate measures are much lower than the worst-case one-dimensional fragility measure (i.e. the dark matter measure)  $\varrho(\theta_0)$ , with a larger fragility measure for  $\bar{\sigma}^2$  (the long-run variance of consumption growth) and  $\nu$  (the persistence of conditional variance of consumption growth) than for the other individual parameters. This shows that it is not sufficient to consider perturbations of parameters one at a time to quantify model fragility; Müller (2012) highlighted a similar insight on sensitivity analysis.

In comparison, in panel II of Table III we show fragility measures under the specification that ignores the uncertainty about preference parameters. This type of analysis is sensible if the model is not fully estimated, but rather the preference parameters are fixed at certain values. For instance, one may specifically design a model to capture the moments of asset returns with a low value of risk aversion. In that case, the choice of the preference parameters is effectively subsumed by the specification of the functional form of the model, and treating them as auxiliary parameters is in line with the logic of the model construction. The fragility measures in Panel II are higher. In particular, the worst-case one-dimensional fragility (i.e. the dark matter measure)  $\varrho(\theta_0)$  increases dramatically from 196.3 to  $3.57 \cdot 10^5$ .

In our model we have assumed that the conditional mean and volatility of consumption growth,  $x_t$  and  $\sigma_t$ , are observable. An interesting question is whether the model becomes more or less fragile when agents observe  $x_t$  and  $\sigma_t$  but the econometrician does not (e.g., Schorfheide, Song, and Yaron, 2018). While a formal analysis is beyond the scope of this paper, the lack of observability of the state variables is expected to add to the effective degrees of freedom and increase model fragility. When the agents themselves need to learn about the latent states and potentially deal with model uncertainty (e.g., Collin-Dufresne, Johannes, and Lochstoer,



2016; Hansen and Sargent, 2010), the cross-equation restrictions implied by asset prices will differ, and it is difficult to anticipate the effect on model fragility. Numerically, the assumption that  $x_t$  and  $\sigma_t^2$  are observable means that we do not need to integrate out  $x_t$  or  $\sigma_t^2$  in the moment restrictions when computing the model fragility measure. Furthermore, since we are examining the fragility of a specific calibration of the model, we can compute the fragility measure under the set of calibrated parameter values, instead of having to first filter out the values of  $x_t$  and  $\sigma_t^2$  from the data, as in Constantinides and Ghosh (2011), Bansal, Kiku, and Yaron (2016b), and Schorfheide, Song, and Yaron (2018), and then estimate the corresponding parameter values.

### *Monte Carlo experiments.*

We use simulations to illustrate the connection among the dark matter measure, internal refutability, and external validity of long-run risk models in finite samples. In the simulation experiment, we assume that all the parameters except  $\nu$  are treated as auxiliary parameters fixed at known constant values, subsumed into the functional form of the moment function (i.e. model specifications). From the dark matter evaluation in Table III, we learn that the assumed identification of  $\nu$  (i.e., the uncertainty of  $\nu$ ) is a major source of model fragility for long-run risk models. Focusing on  $\nu$  simplifies our simulation illustration and increases the transparency by allowing us to consider a few key (transformed) moment restrictions (i.e. a small yet essential subset of the moment restrictions used in constructing Table III):

$$(218) \quad m_t(\theta) = \Omega(\theta)^{-1/2} \begin{bmatrix} (\tilde{\sigma}_t^2 - \bar{\sigma}^2)\epsilon_{\sigma,t+1} \\ r_{t+1}^e - \mu_{r,t}^e - \beta_c \sigma_t \epsilon_{c,t+1} - \beta_x \sigma_t \epsilon_{x,t+1} - \beta_\sigma \sigma_w \epsilon_{\sigma,t+1} \end{bmatrix} \text{ and } \theta = \nu,$$

where variables  $\epsilon_{c,t+1}$ ,  $\epsilon_{x,t+1}$ , and  $\epsilon_{\sigma,t+1}$  are the residuals in (213a) – (213d) depending on observed data and unknown parameters in  $\theta$ , and  $\mu_{r,t}^e$  is defined in (216) and also dependent on observed data and unknown parameters in  $\theta$ . Here  $\Omega(\theta)$  is the asymptotic covariance matrix of the untransformed moments, and it is a diagonal matrix  $\Omega(\theta) = \text{diag}\{\sigma_w^2/(1 - \nu^2), \varphi_{d,d}^2 \bar{\sigma}^2\}$ . In equation (218), the first matrix element is the baseline moment, and the second is the asset pricing moment. Clearly, the nuisance parameter vector  $\theta^{(2)}$  is empty in this simulation example.

We assume that the true local data-generating process has a time-varying relation between the expected log excess return and the dynamic parameters:

$$(219) \quad r_{t,n}^e = r_t^e + \frac{\iota_t \delta_r}{\sqrt{n}}, \text{ with } \iota_t = \begin{cases} 1, & \text{when } 1 \leq t \leq \lfloor \pi n \rfloor \\ -1, & \text{when } \lfloor \pi n \rfloor < t \leq n, \end{cases}$$

where the time series  $\iota_t$  captures the structural breaks. The corresponding moment biases, evaluated at  $\theta_0$ , are

$$(220) \quad \mathbb{E}^{\mathbb{Q}_0} [m_t(\theta_0)] = \begin{bmatrix} 0 \\ \frac{\lambda_t^{(2)}}{\sqrt{n}} \end{bmatrix} \text{ with } \lambda_t^{(2)} \equiv \frac{\iota_t \delta_r}{\varphi_{d,d} \bar{\sigma}},$$

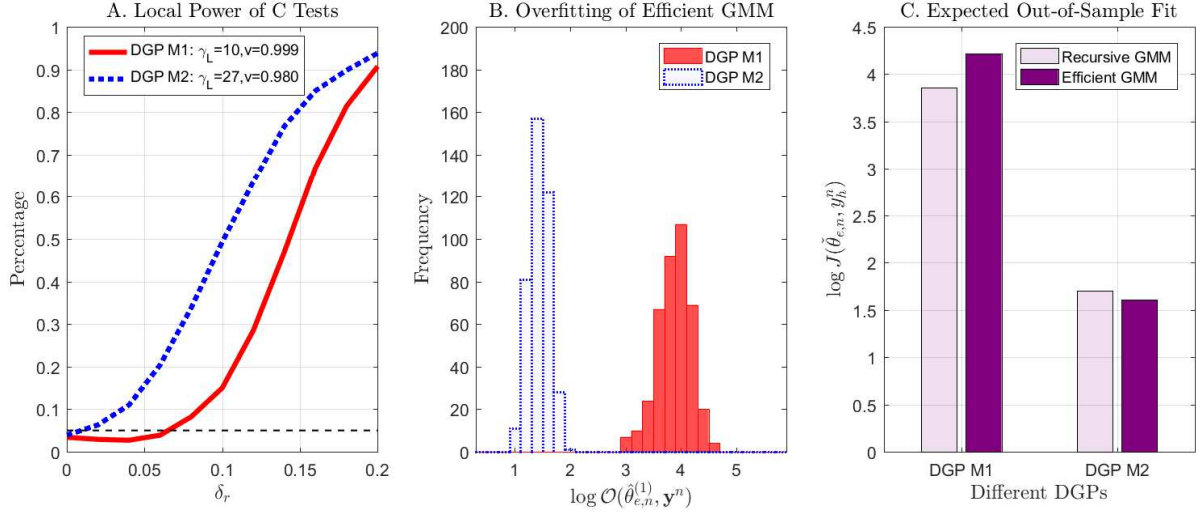


FIGURE 5.— Monte Carlo experiments for long-run risk models. In panel A, we simulate 1000 independent monthly time series with length  $n = 1200$  (i.e. 100 years). In panels B and C, we simulate 400 independent monthly time series with length  $n = 1200$  (i.e. 100 years) and break point  $\pi = 1/2$ . We set  $\delta_r = 0.02$  for panels B and C. In the simulation experiment, we assume that all the parameters except  $\nu$  are treated as auxiliary parameters fixed at known constant values, subsumed into the functional form of the moment function (i.e. model specifications).

Therefore, under the data-generating processes M1 and M2 in Table I, the moments have identical local biases  $\lambda_t^{(2)}$  after substituting the calibrated parameter values into (220). This guarantees that the comparisons across models in panels A and B of Figure 5 are valid.

Figure 5 shows three different simulation experiments. Panel A displays the local power functions of  $C$  tests. The solid and dotted curves reflect the test powers when the data-generating processes are characterized by calibrations M1 and M2 in Table I, respectively. In this experiment, we vary the local misspecification  $\delta_r$  in the risk premium. The data-generating process under calibration M1 features an excessively large amount of dark matter according to Table I, and thus it has low internal refutability (i.e. little test power) consistent with Theorem 1.

Panel B of Figure 5 displays the histograms of logged overfitting measures  $\log \mathcal{O}(\hat{\theta}_{e,n}^{(1)}, \mathbf{y}^n)$  of efficient GMM estimators for two data-generating processes under calibrations M1 and M2 in Table I. In this experiment, we specify a structural break in the risk premium in the middle of the time-series sample with  $\delta_r = 0.02$ . Panel B shows that the calibrated structural model with too much dark matter (model M1) is likely to have more severe overfitting concerns for the efficient GMM estimator, which is consistent with Theorem 2.

Panel C of Figure 5 compares the expected out-of-sample fits between recursive GMM estimators  $\tilde{\theta}_{e,n}$  and efficient GMM estimators  $\hat{\theta}_{e,n}$ . The two types of estimators are defined in Section 5.2. Consistent with the conventional intuition, efficient GMM estimators outperform their recursive counterparts in terms of the expected out-of-sample fit under the data-generating process M2. This is because the additional identification information is reliable and meaningful when the amount of dark matter is not excessively large. On the contrary, recursive GMM estimators outperform their efficient counterparts in terms of the expected out-of-sample fit

under the data-generating process M1 with too much dark matter. This means that the concern of misspecification and instability entirely offsets– and even reverses – the efficiency gain from the additional moment restrictions. Again, this experiment suggests that the econometrician should back off from efficiency to gain more robustness for the estimation results when the model contains a large amount of dark matter.