Examples of Averages

Example: Find the average height of a point on a unit semicircle.

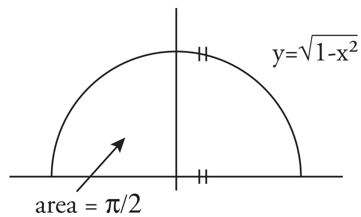

Figure 1: The unit semicircle and an interval dx.

Here $f(x) = \sqrt{1-x^2}$ for $-1 \le x \le 1$, so $a = -1$ and $b = 1$. The average value of $f(x)$ is:

$$\begin{aligned} \text{Avg}(f) &= \frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{1}{2}\int_{-1}^{1}\sqrt{1-x^2}dx \\ &= \frac{1}{2}(\text{Area of a unit seimicircle}) \\ &= \frac{1}{2}\left(\frac{\pi}{2}\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

(We will eventually learn how to find the antiderivative of $\sqrt{1-x^2}$ in the unit on techniques of integration.)

Example: Find the average height of a point on a unit semicircle *with respect to the arclength* θ.

When taking averages, it's extremely important to specify the variable with respect to which the average is taking place. The answer may be different depending on the variable!

As you can see from Figure 2, equal distances along the arc of the semicircle overshadow different lengths on the x-axis. Taking the average with respect to θ will weight the lower parts of the semicircle more heavily than the higher ones. We expect the average with respect to arclength be less than $\frac{\pi}{4}$.

Then average is still given by $\frac{1}{b-a}\int_a^b f(\theta)d\theta$. This time, $a = 0$ and $b = \pi$. The integrand is $y = \sin\theta$, which is the height of the semicircle in terms of θ.

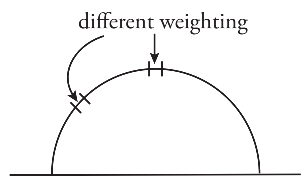

Figure 2: Equal arclengths correspond to different distances on the x-axis.

So our average height is:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(\theta)d\theta &= \frac{1}{\pi}\int_0^\pi \sin\theta d\theta \\
&= \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi \\
&= \frac{1}{\pi}\left(-\cos\pi-(-\cos 0)\right) \\
&= \frac{2}{\pi}
\end{aligned}$$

Can we check our work? Is $\frac{2}{\pi} < \frac{\pi}{4}$? Yes, because $8 < \pi^2$.

Question: How do we interpret the result of the average with respect to arclength?

Answer: One way of thinking of it anticipates our next subject, which is probability. Suppose you picked a point at random along the base of the semicircle (with equal likelihood between -1 and 1) and checked the height above that point. The expected value of that height is given by the first calculation: $\frac{\pi}{4}$.

The second calculation tells you the expected value of the height of a point picked at random on the semicircle, if you were equally likely to pick any point on the semicircle.

Those two average heights are different because distance along the semicircle is different from distance along the x-axis.

Question: Shouldn't the average with respect to arclength have a *bigger* value because the arclength is *longer*?

Answer: Never with averages; when we multiply by $\frac{1}{b-a}$ we are dividing by the total length.

However, the average of a constant is that same constant regardless of what variable we use because we compensate by dividing by $b - a$. The difference between an integral and an average is that we're dividing by that total.