Differentials and Linear Approximation

Linear approximation allows us to estimate the value of $f(x+\Delta x)$ based on the values of $f(x)$ and $f'(x)$. We replace the change in horizontal position Δx by the differential dx. Similarly, we replace the change in height Δy by dy. (See Figure 1.)

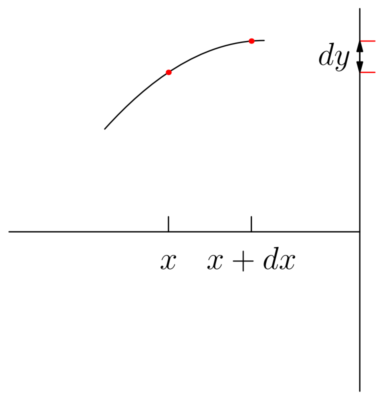

Figure 1: We use dx and dy in place of Δx and Δy.

Example: Find the approximate value of $(64.1)^{\frac{1}{3}}$.

Method 1 (using differentials)

We're going to use a linear approximation of the function $y = f(x) = x^{\frac{1}{3}}$. Our base point will be $x_0 = 64$ because it's easy to compute $y_0 = 64^{\frac{1}{3}} = 4$. By definition, $dy = f'(x)dx = \frac{1}{3}x^{-\frac{2}{3}}dx$.

$$\begin{aligned} dy &= \frac{1}{3}(64)^{-\frac{2}{3}}dx \\ &= \frac{1}{3}\frac{1}{16}dx \\ &= \frac{1}{48}dx \end{aligned}$$

We want to approximate $(64.1)^{\frac{1}{3}}$, so $x + dx = 64.1$ and $dx = 0.1 = \frac{1}{10}$. At the value $64.1 = x_0 + dx$, $f(x)$ is exactly equal to $y_0 + \Delta y$ (because this is how we defined Δy) and is approximately equal to $y_0 + dy$, where dy is is linear in dx as derived above.

In essence, the point $(x_0 + dx, y_0 + dy)$ is an infinitessimaly small step away from (x_0, y_0) along the tangent line. Of course $\frac{1}{10}$ is not infinitessimally small, which is why this is an approximation rather than an exact value.

$$(64.1)^{\frac{1}{3}} \approx y + dy$$

$$
\begin{aligned}
&\approx 4 + \frac{1}{48}dx \\
&\approx 4 + \frac{1}{48}\frac{1}{10} \\
&\approx 4.002
\end{aligned}
$$

Method 2 (review)

When we compare this to our previous notation we discover that the calculations are the same; only the notation has changed.

The basic formual for linear approximation is:

$$f(x) = f(a) + f'(a)(x - a)$$

Here $a = 64$ and $f(x) = x^{\frac{1}{3}}$, so $f(a) = f(64) = 4$ and $f'(a) = \frac{1}{3}a^{-\frac{2}{3}} = \frac{1}{48}$

Our approximation then becomes:

$$
\begin{aligned}
f(x) &\approx f(a) + f'(a)(x - a) \\
x^{\frac{1}{3}} &\approx 4 + \frac{1}{48}(x - 64) \\
(64.1)^{\frac{1}{3}} &\approx 4 + \frac{1}{48}\frac{1}{10} \\
(64.1)^{\frac{1}{3}} &\approx 4.002
\end{aligned}
$$

We get the same answer had before, by doing a nearly identical calculation.