Area and determinants in 2D

We've used vectors to compute lengths and angles, but we haven't yet used them to compute areas. Suppose we want to find the area of a pentagon. Can we compute it using vectors? Yes, and that is going to be our goal.

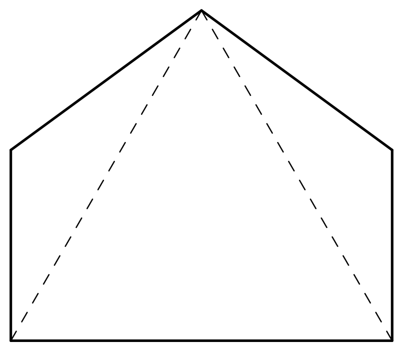

Figure 1: A pentagon can be cut into three triangles.

We start by simplifying the problem; we cut the pentagon into three triangles. If we can find the area of a triangle using vectors then we can find the area of the pentagon.

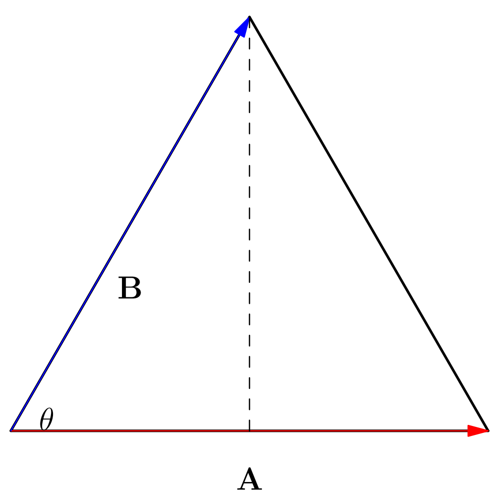

Figure 2: Using vectors to find the area of a triangle.

We start with a triangle in the plane described by vectors **A** and **B**. We know that its area is base times height over 2. The length of the base is just $|\mathbf{A}|$ and its height is $|\mathbf{B}|\sin\theta$, so its area is:

$$\frac{1}{2}|\mathbf{A}||\mathbf{B}|\sin\theta.$$

This is similar to the geometric formula for the dot product, but we have a sine function in place of the cosine function. We could use a dot product to find $\cos\theta$ and then solve for $\sin\theta$ using $\sin^2\theta + \cos^2\theta = 1$, but instead we'll use an easier way to solve the problem.

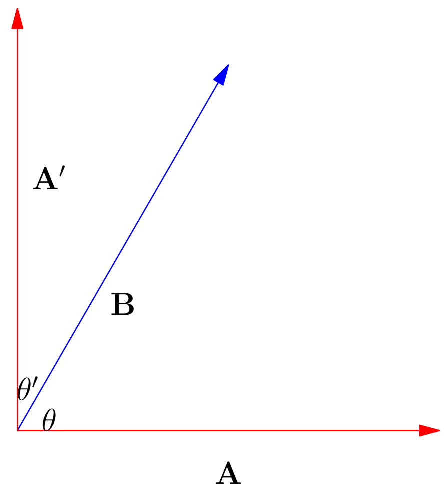

Figure 3: $\cos(\theta') = \sin(\theta)$.

We don't have a simple way to find sines of angles, but the dot product gives us a simple way to find cosines. Let's look for an angle whose cosine is the same as $\sin\theta$. We'll rotate $\mathbf{A}$ by ninety degrees to get a new vector $\mathbf{A}'$. (See Figure 3.) The angle θ' between $\mathbf{A}'$ and $\mathbf{B}$ equals $\frac{\pi}{2} - \theta$, so $\cos(\theta') = \sin(\theta)$. Therefore,

$$\begin{aligned} |\mathbf{A}||\mathbf{B}|\sin\theta &= |\mathbf{A}'||\mathbf{B}|\cos\theta' \\ &= \mathbf{A}'\cdot\mathbf{B}. \end{aligned}$$

This looks like a good idea if we can find $\mathbf{A}'$. It turns out to be relatively easy to find $\mathbf{A}'$.

If $\mathbf{A} = \langle a_1, a_2\rangle$, then $\mathbf{A}' =$

a) $\langle a_2, a_1\rangle$

b) $\langle a_2, -a_1\rangle$

c) $\langle -a_2, a_1\rangle$

d) $\langle -a_1, a_2\rangle$

e) None of the above

The favorite answer seems to be $\langle -a_2, a_1\rangle$. Let's try this with a sample vector $\mathbf{A} = \langle a_1, a_2\rangle$. We start by drawing $\mathbf{A}$ together with the rectangle formed

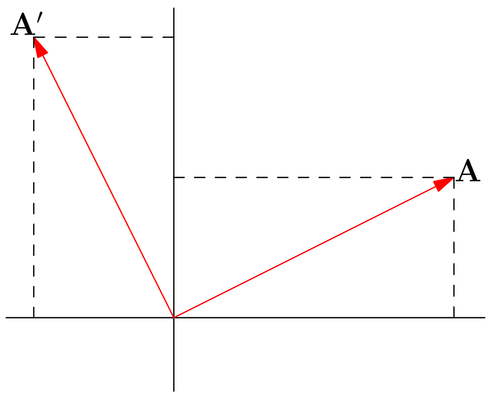

Figure 4: To visualize $\mathbf{A}'$, rotate the rectangle surrounding $\mathbf{A}$.

by the lines $x = 0$, $y = 0$, $x = a_1$ and $y = a_2$. To see what happens when we rotate the vector, we rotate this box about the origin $(0, 0)$.

It turns out that the base of the rotated rectangle has length a_2 and extends along the negative y-axis, and the height of the rotated rectangle is a_1. Therefore $\mathbf{A}' = \langle -a_2, a_1 \rangle$.

We could have instead rotated the rectangle clockwise about the origin to find the vector $\langle a_2, -a_1 \rangle$.

Returning to our original problem, we have $\mathbf{A}' = \langle -a_2, a_1 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$.

$$\begin{aligned} |\mathbf{A}||\mathbf{B}|\sin\theta &= \mathbf{A}' \cdot \mathbf{B} \\ &= \langle -a_2, a_1 \rangle \cdot \langle b_1, b_2 \rangle \\ &= a_1 b_2 - a_2 b_1 \\ &= \det(\mathbf{A}, \mathbf{B}) \\ &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{aligned}$$

Here $\det(\mathbf{A}, \mathbf{B}) = a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ denotes the determinant of the matrix of vectors $\mathbf{A}$ and $\mathbf{B}$. To find the determinant of a two by two matrix we multiply the upper left entry by the lower right entry, then subtract the product of the upper right and lower left entries.

Geometrically, the determinant measures the area of the parallelogram described by $\mathbf{A}$ and $\mathbf{B}$ (not the triangle, because we haven't divided by 2 yet.) Since areas are always positive values we may need to take the absolute value of the determinant to get the area; the sign of the determinant changes depending on the order of the vectors: $\det(\mathbf{A}, \mathbf{B}) = -\det(\mathbf{B}, \mathbf{A})$.

We conclude that the area of the parallelogram described by vectors $\mathbf{A}$ and $\mathbf{B}$ is the absolute value of:

$$|\mathbf{A}||\mathbf{B}|\sin\theta = \det(\mathbf{A}, \mathbf{B}).$$

The area of the triangle (half the parallelogram) described by $\mathbf{A}$ and $\mathbf{B}$ is the absolute value of:

$$\frac{1}{2}|\mathbf{A}||\mathbf{B}|\sin\theta = \frac{1}{2}\det(\mathbf{A}, \mathbf{B}).$$