

## Distances to Planes and Lines

In this note we will look at distances to planes and lines. Our approach is geometric. Very broadly, we will draw a sketch and use vector techniques.

Please note that our sketches are not oriented, drawn to scale or drawn in perspective. Rather they are simple 'cartoons' which show the important features of the problem.

1. *Distance: point to plane:*

Ingredients: i) A point  $P$ , ii) A plane with normal  $\vec{N}$  and containing a point  $Q$ .

The distance from  $P$  to the plane is  $d = |\vec{PQ}| \cos \theta = \left| \vec{PQ} \cdot \frac{\vec{N}}{|\vec{N}|} \right|$ .

We will explain this formula by way of the following example.

**Example 1:** Let  $P = (1, 3, 2)$ . Find the distance from  $P$  to the plane  $x + 2y = 3$ .

**Answer:** First we gather our ingredients.

$Q = (3, 0, 0)$  is a point on the plane (it is easy to find such a point).

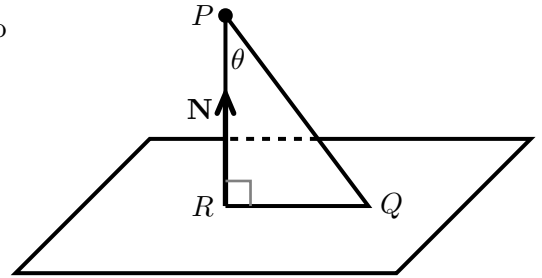
$\vec{N} =$  normal to plane  $= \mathbf{i} + 2\mathbf{j}$ .

$R =$  point on plane closest to  $P$  (this is point unknown and we do not need to find it to find the distance). The figure shows that

$$\text{distance} = |PR| = |\vec{PQ}| \cos \theta = \left| \vec{PQ} \cdot \frac{\vec{N}}{|\vec{N}|} \right|.$$

Computing  $\vec{PQ} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$  gives

$$\text{distance} = \left| \vec{PQ} \cdot \frac{\vec{N}}{|\vec{N}|} \right| = \left| \langle 2, -3, -2 \rangle \cdot \frac{\langle 1, 2, 0 \rangle}{\sqrt{5}} \right| = \frac{4}{\sqrt{5}}.$$



2. *Distance: point to line:*

Ingredients: i) A point  $P$ , ii) A line with direction vector  $\mathbf{v}$  and containing a point  $Q$ .

The distance from  $P$  to the line is  $d = |\mathbf{QP}| \sin \theta = \left| \vec{QP} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right|$ .

We will explain this formula by way of the following example.

**Example 2:** Let  $P = (1, 3, 2)$ , find the distance from the point  $P$  to the line through  $(1, 0, 0)$  and  $(1, 2, 0)$ .

**Answer:** First we gather our ingredients.

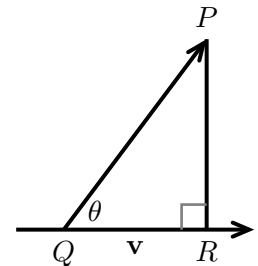
$Q = (1, 0, 0)$  (this is easy to find).

$\mathbf{v} = \langle 1, 2, 0 \rangle - \langle 1, 0, 0 \rangle = 2\mathbf{j}$  is parallel to the line.

$R =$  point on line closest to  $P$  (this is point is unknown).

Using the relation  $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin \theta$ , the figure shows that

$$\text{distance} = |PR| = |\vec{PQ}| \sin \theta = \left| \vec{QP} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right|.$$



Computing:  $\vec{PQ} = 3\mathbf{j} + 2\mathbf{k}$ , which implies  $\left| \vec{QP} \times \frac{\mathbf{v}}{|\mathbf{v}|} \right| = |(3\mathbf{j} + 2\mathbf{k}) \times \mathbf{j}| = | -2\mathbf{i} | = 2$ .

3. *Distance between parallel planes:*

The trick here is to reduce it to the distance from a point to a plane.

**Example 3:** Find the distance between the planes  $x + 2y - z = 4$  and  $x + 2y - z = 3$ .

Both planes have normal  $\mathbf{N} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$  so they are parallel.

Take any point on the first plane, say,  $P = (4, 0, 0)$ .

Distance between planes = distance from  $P$  to second plane.

Choose  $Q = (3, 0, 0)$  = point on second plane

$$\Rightarrow d = |\overrightarrow{\mathbf{QP}} \cdot \frac{\mathbf{N}}{|\mathbf{N}|}| = |\mathbf{i} \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})|/\sqrt{6} = 1/\sqrt{6}.$$

4. *Distance between skew lines:*

We place the lines in parallel planes and find the distance between the planes as in the previous example

As usual it's easy to find a point on each line. Thus, to find the parallel planes we only need to find the normal.

$$\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2,$$

where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the direction vectors of the lines.

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