

# Moment of Inertia

## Moment of inertia

We will leave it to your physics class to really explain what moment of inertia means. Very briefly it measures an object's resistance (inertia) to a change in its rotational motion. It is analogous to the way mass measures the resistance to changes in the object's linear motion.

Because it has to do with rotational motion the moment of inertia is always measured about a reference line, which is thought of as the axis of rotation.

For a point mass,  $m$ , the moment of inertia about the line is

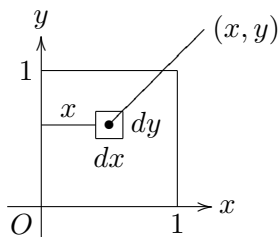
$$I = m d^2,$$

where  $d$  is the distance from the mass to the line. (The letter  $I$  is a standard notation for moment of inertia.)

If we have a distributed mass we compute the moment of inertia by summing the contributions of each of its parts. If the mass has a continuous distribution, this sum is, of course, an integral.

**Example 1:** Suppose the unit square,  $R$ , has density  $\delta = xy$ .

Find its moment of inertia about the  $y$ -axis.



**Answer:** The distance from the small piece of the square (shown in the figure) to the  $y$ -axis is  $x$ . If the piece has mass  $dm$  then its moment of inertia is

$$dI = x^2 dm = x^2 \delta(x, y) dA = x^3 y dx dy.$$

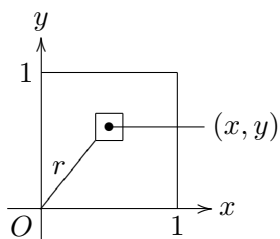
We took a shortcut here: we went straight to the notation for infinitesimal pieces,  $dI$ ,  $dA$  and used equalities rather than using more formal notation  $\Delta I$ ,  $\Delta A$ , using approximations and then taking limits.

In the equation above, we used the notation  $dI$  to indicate it is just a small bit of moment of inertia. We also used that the mass of a piece is density times area. Now it's a simple matter to sum up all the bits of moment of inertia using an integral

$$I = \int \int_R dI = \int_0^1 \int_0^1 x^3 y dy dx = \frac{1}{8}.$$

(Note: this integral is so easy to compute that we don't give the details.)

**Example 2:** For the same square as in example 1, find the *polar moment of inertia*.



**Answer:** The polar moment of inertia of a planar region is the moment of inertia about the origin (the axis of rotation is the  $z$ -axis). Finding this is exactly the same as in example 1, except the distance to the axis is now the polar distance  $r$ . We get,

$$dI = r^2 dm = (x^2 + y^2)\delta(x, y) dA = (x^3y + xy^3) dx dy.$$

Summing, using an integral gives

$$I = \int \int_R dI = \int_0^1 \int_0^1 x^3y + xy^3 dy dx = \frac{1}{4}.$$

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