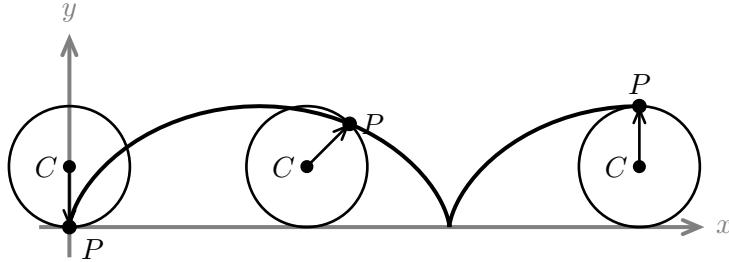


## Cusp on the Cycloid

The graph of the cycloid has points where the graph touches the  $x$ -axis. These points are usually called *cusps*.

What you saw in the previous video was an analysis of the behavior of the trajectory near the cusps. We will go through that analysis again and discuss what's happening physically on a rolling wheel.



In order to simplify the way our equations look, let's take the radius of the wheel to be  $a = 1$ . Then the parametric equations for the cycloid are

$$x(\theta) = \theta - \sin \theta, \quad y(\theta) = 1 - \cos \theta.$$

Taking derivatives we get  $\frac{dx}{d\theta} = 1 - \cos \theta$  and  $\frac{dy}{d\theta} = \sin \theta$ .

Thus the slope of the curve is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin \theta}{1 - \cos \theta}.$$

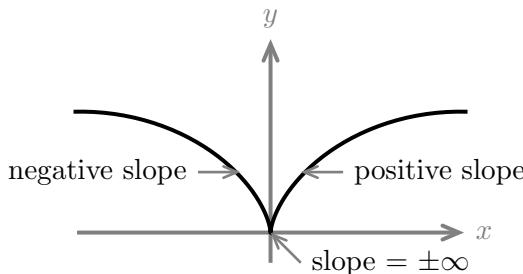
As  $\theta \rightarrow 0$  this is of indeterminate form  $0/0$ . Using L'Hospital's rule we get

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{1 - \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\cos \theta}{\sin \theta}$$

Since  $\cos \theta$  goes to 1 and  $\sin \theta$  goes to 0 this limit does not exist.

But looking at it more carefully we see that as  $\theta \rightarrow 0^-$  the limit goes to  $-\infty$  and as  $\theta \rightarrow 0^+$  it goes to  $+\infty$ . That is, right at the cusp the slope of the curve is  $-\infty$  to the left and  $+\infty$  to the right.

This mirrors what we see in the graph



Later, when we learn about velocity we'll see that, at  $\theta = 0$ ,  $\frac{dx}{d\theta} = 0$  and  $\frac{dy}{d\theta} = 0$  means the velocity is 0. At the cusp, the point changes abruptly from moving down to moving up. Physically this can only happen if the velocity is 0 at the changeover point.

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18.02SC Multivariable Calculus

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