Random Trees in Electrical networks

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Abstract

This paper contains results relating currents and voltages in resistive networks to appropriate random trees or forests in those networks. Since each resistive network has a reversible Markov chain equivalent, we obtain equivalent results for the latter as well. We describe a way of obtaining a harmonic function on a weighted graph given the boundary values, by choosing random forests of the graph. As applications of the theorems discussed, (which give formulae of the Kirchhoff tree kind), we obtain an expression for the expected transit time from one state to another in a reversible Markov chain in terms of its arborescences.

The methods of this paper can also be used to give alternative proofs of the Kirchhoff tree formula.

1 Introduction

The first formula involving trees in electrical networks is due to Kirchhoff (see [1]). He gave an expression for the equivalent conductance between two nodes of a resistive (or as we shall henceforth say, conductive) network. Kirchoff's formula states that the equivalent conductance between two nodes of a network is $\frac{\sum_i t_i}{\sum_j f_j}$, where, t_i is the product of the conductances of the ith tree and f_j is the product of the conductances of the jth forest that separates the nodes in question.

In recent years,formulae involving trees have been discovered for general Markov chains too (see [2], [3]).

The physical model used to explain Ohm's law involves Brownian motions of charge carriers. So it is not surprising that a conductive network may be mapped onto a Markov chain with very similar properties. Trees seem to figure in Markov chains as "partial histories". In a Markov chain started at time $-\infty$ and ended at time τ, the set of directed edges corresponding to the final visits to each state, form a directed tree rooted at the state reached at time τ. The evolution of these histories as the Markov chain progresses is another Markov chain which has arborescences (i.e rooted trees) as states. This Markov chain has interesting properties, and can be regarded as underlying the results that appear in this paper.

The formulae presented here are multiterminal extensions of the Kirchhoff tree formula. They all have equivalent forms in reversible Markov chains, which can be written out by simply substituting for injected current, branch current and node voltage, their Markov chain equivalents given in Section **??** .

We obtain Kirchhoff's formula for the equivalent conductance between two nodes, from the result we have called the VJ Theorem, when we apply a unit voltage across them. Since the proof of the VJ Theorem is purely graph theoretic, we get a new derivation of the Kirchhoff conductance formula as well, quite different in nature from the usual proof using Binet - Cauchy Theorem.

Although this is not the concern of this paper, we state that these formulae can be implemented in a natural way using randomized algorithms to give fast approximate

solutions to network problems. The essential idea is that we make numerous “observations” of the network, each of which is computationally light, and the “overall picture” that we get is an approximation to the network’s actual behaviour. The more observations we make, the better our approximation is expected to be. The critical step in each observation is that of picking a random tree (forest), the chance of picking a particular tree (forest) being proportional to the product of its branch conductances.
Picking random trees has been found useful in other contexts as well, such as optimizing on server positions in a computer network, and finding euler paths in a graph, and this problem has been well studied (see [4], [5], [10], [13]). There are a variety of efficient algorithms available that perform this task.

Here is a preview of three of the theorems in network form.
The VV Theorem (special case):
In a conductive network with nodes $S_1, \ldots, S_n$, apply an external voltage source of 1 volt across S_1 and S_2 and choose S_2 as the ground. Let v_k be the voltage at S_k. (Thus $v_1 = 1$, $v_2 = 0$).
Then v_k is given as follows :
Consider all those forests f (of the network) which have $n-2$ branches, (consisting of two disjoint trees, whose union touches every vertex of the network), such that S_1 and S_2 are not vertices of the same tree in f.
Call this set F_{12}. Choose a forest randomly out of F_{12}, with probability proportional to the product of its branch conductances. The voltage v_k at node S_k is the probability that in the chosen forest f, S_k is a node of the same tree as S_1.

This theorem is really a result on harmonic functions on weighted graphs, since but for the “poles” S_1 and S_2, the voltage at any vertex is the weighted mean of its neighbour’s voltages.

The JI Theorem:
In the network mentioned above, let currents $J_1, \ldots, J_n$ be injected externally into nodes $S_1, \ldots, S_n$. Choose a random spanning tree t of the network, the probability of choosing t being proportional to the product of its branch conductances. Setting all conductances other than those of t to 0, we get a certain current distribution according to which the current in any branch not in t is 0. The actual current distribution is the expected distribution, under a random choice of tree t, or the average of the distributions taken over all trees (the weight of a distribution being the product of the conductances of the corresponding tree).

The IV Theorem:
Let the J_k’s be as above. Suppose we are given any current distribution in the branches of the graph that is consistent with the injected currents, but which does not necessarily satisfy the Kirchhoff voltage law. Choose a random tree t as in the JI Theorem. Take any vertex as ground (potential zero). Calculate the node potential of a vertex by adding the branch voltages along the unique path in t from the ground to that vertex. A correct node voltage distribution is given by the expected value of node potentials under the choice of a random tree t.

The JI and IV theorems above, involve choosing random trees while the VV Theorem involves choosing a random forest. In each case, the probability of a choice is proportional to the product of its conductances. There is a well-known algorithm using Markov-chains for choosing random trees with this probability distribution (see [4], [5]). As we shall see in the section with Theorem VV, this algorithm can modified to choose random forests too.

2 The network - Markov equivalence

We now talk about the correspondence between electrical networks and reversible Markov chains. This is well-known (see [6], [7], [8], [9]), but has been included for the sake of completeness.
Consider a conductive network with nodes $S_1, \ldots, S_n$. Let $g_{kj} = g_{jk}$ be the branch conductance between the nodes S_j and S_k. If $j = k$ the conductance obviously has no effect on the electrical behavior of the network, however, because in our Markov analogue they do contribute, we allow g_{jj} to be positive. Let v_k be the voltage of S_k, J_k the current injected from outside the network into the node S_k and i_{kl} the branch current flowing from S_k to S_l in the conductance g_{kl} . (Conductance is the inverse of resistance. In this paper, where there is no ambiguity, we shall use the word conductance for a branch as well.)
The Kirchhoff current law is expressed in the following equation :-
$J_k = \sum_{m=1}^{n} g_{mk}\ (v_k - v_m)$. Let the injected current vector be called J, the vector of node voltages be V, and the branch current matrix be I. Thus $J = (J_1, \ldots, J_n)$, $V = (v_1, \ldots, v_n)$, and $I = \{i_{kl}\}_{n\times n}$.

Consider the Markov chain with states $S_1, \ldots, S_n$ (these will correspond to nodes of our network anyway, so there is no clash of notations). Let transition probability p_{kl} be $\frac{g_{kl}}{g_k}$ where $g_k = \sum_{m=1}^{n} g_{km}$. Let us further assume that

$$\sum_{k=1}^{n}\sum_{l=1}^{n} g_{kl} = 1.$$

This is just a scaling of the conductances, but is convenient. We assume that the network is connected, and therefore that the corresponding Markov chain has a path of positive probability from any state to any other - i.e the markov chain is strongly connected. If it is also aperiodic, it has a unique stationary distribution, which can be easily verified to be $\pi = (g_1, \ldots, g_n)$, with the scaled conductances. Let the initial probability distribution be $p^{(0)}$. Suppose we have a stopping rule under which the expected run-time, for some (and hence, since the Markov chain is strongly connected and finite, every) initial probability distribution is finite. Let the probability of the walk terminating at S_k be $p_k^{(u)}$, and the corresponding vector be $p^{(u)}$. Let e_k be the expected number of times the walker visits state S_k, where the last move of the walk at which the walker stops is not counted as a visit. Thus for example if the walk began at S_1 and the stopping rule declares that at the first visit to S_k, the walk ends, then according to us the number of visits to S_k is always 0. With this notation, the following statement can be easily verified for each k:

$$p_k^{(0)} - p_k^{(u)} = e_k - \sum_{m=1}^{n} (p_{mk}\ e_m).$$

This essentially says that "initial probability - final probability = net outflow - net inflow" In our present situation, $p_{mk} = \frac{g_{mk}}{g_m}$ so we have

$$p_k^{(0)} - p_k^{(u)} = e_k - \sum_{m=1}^{n} (\frac{g_{mk}}{g_m})e_m.$$

In anticipation, let $\frac{e_m}{g_m}$ be called v_m (this will actually behave like voltage), and let J_k denote the L.H.S (which will play the role of injected current). Our equation then becomes $J_k = g_k \times v_k - \sum_{m=1}^{n}(g_{mk}\ v_m)$ or $J_k = \sum_{m=1}^{n} g_{mk}(v_k - v_m)$, which is identical to the Kirchhoff current equation we had earlier. This verifies the correspondence. In a network, $i_{kl} = (v_k - v_l)g_{kl}$, which therefore becomes $e_k(p_{kl}) - e_l(p_{lk})$ in the Markov chain. Therefore the Markov analogue of current i_{kl} is the expected difference between the number of transitions from S_k to S_l and the number of transitions from S_l to S_k. In the network results that follow, whenever we use the phrase "A node S_k whose voltage is fixed externally", we mean that the injected current J_k at S_k is not necessarily 0. However, for a node S_l whose voltage is not fixed externally, J_l is necessarily 0.

3 Preliminaries

An arborescence of a directed graph is a tree, in which a node has been singled out as root and all branches are so directed that from any node of the graph to the root, there is a unique directed path. A Markov chain is a process consisting of a succession of events where the probability of an event happening is a function of the preceding event that has occurred in the process. In this paper, we only consider Markov chains which are finite and which are strongly connected. By strongly connected, we mean that if E_1 and E_2 are any events in our sample space, and our process launches itself with E_1, the probability that E_2 occurs n events later is non-zero for some integer n. If a Markov chain is aperiodic and strongly connected, it is a result that whatever be the probability distribution with which the process begins, the probabilities of the various events tend to a "stationary probability distribution" if we wait sufficiently long. We call the events from the sample space of a Markov chain, states, and given that E_i has just occurred, call the probability that E_j occurs next, the transition probability from E_i to E_j, which we denote by p_{ij}. Let the stationary probability distribution of a finite, irreducible Markov chain be $\{\pi_1, \pi_2, ...\}$, corresponding to events $\{E_1, E_2, ...\}$. If $\pi_i \times p_{ij} = \pi_j \times p_{ji}$, the Markov chain is called reversible.

Let R be a subset of $S = \{S_1, \ldots, S_n\}$. We then call by F_R the set of all maximal forests of the network that "separate" states in R. Thus F_R consists of all those subgraphs f of the network for which any S_k in S is connected by a unique (conducting) path in f to some state in R (a state is always regarded to be connected to itself by the null path). It follows that by the condition of connectivity in f, the states of S are partitioned into $|R|$ blocks such that each block has exactly one state of R. If S_k belongs to R and f to F_R, we shall denote by $B_f(k)$, the set of states which are connected to S_k by a path in f. We will often perform algebraic operations with a forest f in F_R. In every such case, f is interpreted as the product of the conductances of the forest f. This is a helpful abuse of notation; it creates no ambiguity, but does simplify our expressions. (For example, $f_1 + f_2$ represents the sum of the products of conductances of the forests f_1 and f_2.)

We define $S_f(k)$, for f in F_R to be the unique state (we henceforth use 'state' synonymously with node and vertex, since nodes become states of the related Markov chain) of R that S_k is connected to by a path in f. $v_f(k)$ is taken to be the voltage of the state $S_f(k)$. It is true that a forest f may belong to both F_R and F_Q, where R and Q are different vertex sets, but in all our expressions, we talk of forests f in a particular F_R, and so $S_f(k)$, and $v_f(k)$ are well defined by the context in which they appear.

4 The VJ Theorem

We now give a formula for the injected currents J_k in terms of the externally fixed voltages of the network. When J is the vector $(-1, 1, 0, \ldots, 0)$, we get Kirchhoff's formula for equivalent conductance.

Theorem VJ:

Let R be a nonempty subset of S not containing S_1. Let $Q = R \cup S_1$. Let the voltages at nodes of Q be the only ones fixed externally.

Then

$$J_1 = \frac{\sum_{h \in F_R} (v_1 - v_h(1))\ h}{\sum_{f \in F_Q} (f)}.$$

Proof :

We say that a forest f_1 is contained in another forest f_2 if every branch of f_1 is also a branch of f_2. If f belongs to F_Q, h belongs to F_R and f is contained in h, then h has exactly one branch that f doesn't, which we shall denote as h/f. If we look at the partition of S induced by the forest f (in which, we recall, every block has exactly one representative of Q), we see that h/f has exactly one endpoint in $B_f(1)$. Let $v(h/f)$ be the voltage of h/f, taken directed outward from $B_f(1)$. We observe that the net current leaving $B_f(1)$, (and entering states outside) is J_1 for any f in F_Q since the only state in $B_f(1)$ that can possibly have non-zero injected current is S_1. Therefore,

$$J_1 = \sum_{(k,l \ \ s.t \ \ S_f(k) = S_1 \neq S_f(l))} g_{kl}(v_k - v_l).$$

i.e

$$J_1 (\sum_{f \in F_Q} f) =$$

$$\sum_{f \in F_Q} (\sum_{(k,l \ \ s.t \ \ S_f(k) = S_1 \neq S_f(l))} (f)(g_{kl})(v_k - v_l)).$$

This may be rewritten as

$$\sum_{(f \in F_Q, h \in F_R \ \ s.t \ \ f \subset h)} h \times v(h/f)$$

$$= \sum_{h \in F_R} h \times (\sum_{(f \in F_Q \ \ s.t \ \ f \subset h)} v(h/f))$$

But for a fixed h in F_R,

$$\sum_{(f \in F_Q \ \ s.t \ \ f \subset h)} v(h/f)$$

is the sum of branch voltages along the path from S_1 to $S_h(1)$ contained in h , and this is just $v_1 - v_h(1)$. Therefore

$$J_1 (\sum_{f \in F_Q} f) = \sum_{h \in F_R} h \times (v_1 - v_h(1)).$$

This proves the theorem.

The VJ Theorem has a probabilistic interpretation. In what follows, whenever we say choose a random forest in F_Q, it is implicit that the probability of choosing f is proportional to the product of its conductances. Similarly by a random branch

we mean that it is chosen with probability proportional to its conductance. Let g be the sum of all conductances of our network. Choose a random forest f in F_Q. Choose a random branch g_{kl} from the network. Then $S_f(k)$ and $S_f(l)$ are states in Q. Let the number of branches in the unique path from S_m to $S_f(m)$ in f be called $d_f(m)$ for each m and each f .
Consider the current vector $J(f,kl)$ given by

$J_m = 0$ if S_m is not $S_f(k)$ or $S_f(l)$.

$J_m = \frac{g\times(v_f(k)-v_f(l))}{1+d_f(k)+d_f(l)}$ if $m = S_f(k)$.

$J_m = \frac{g\times(v_f(l)-v_f(m))}{1+d_f(k)+d_f(l)}$ if $m = S_f(l)$

Then, the theorem says that the expected value of $J(f,kl)$ when f and g_{kl} are random, is the actual J that is injected into the network. $1 + d_f(k) + d_f(l)$ is just the length of the connecting path via g_{lk} (in f) between $S_f(k)$ and $S_f(l)$ when these are distinct ; it was used to simplify the expression.

To see why the two forms of the VJ Theorem are identical, let us find the expected value of $J_1(f,kl)$, the first component of $J(f,kl)$. If S_1 is not in Q, the result is obvious since then $J_1(f,kl)$ is always 0, so we assume that S_1 belongs to Q and let $R = Q - S_1$.

$$E[J_1(f,kl)] =$$
$$\frac{\sum_{f\in F_Q}(\sum_{(k,l\ s.t\ S_k\in B_f(1))}(f)(\frac{g_{kl}}{g})(v_1 - v_f(l))(\frac{g}{1+d_f(k)+d_f(l)}))}{\sum_{f\in F_Q} f}.$$

We observe that if S_l is not in $B_f(1)$, then $f \times g_{kl}$ corresponds to a forest h in F_R, while if it is, $v_1 - v_f(1)$ is 0. Further, the number of pairs of the form (f, g_{kl}) that lead to a forest h in F_R, is the number of branches in the unique path from S_1 to $S_h(1)$ in h. This is the same as $1+d_f(k)+d_f(l)$ for any pair $\{f, g_{kl}\}$ that gives rise to h. Therefore the above expression is equal to

$$\frac{\sum_{h\in F_R}(v_1 - v_h(1)) \times h}{\sum_{f\in F_Q} f},$$

which is J_1.

5 The VV Theorem

Theorem VV :
Let R be a non-empty subset of S, not containing S_1. Let the states of R be the only ones whose voltages are fixed externally, (i.e for all S_l in $S - R$, let J_l be 0). Then,

$$v_1 = \frac{\sum_{h\in F_R}(v_h(1) \times h)}{\sum_{h\in F_R}(h)}$$

Proof :
Let $Q = R \cup S_1$. By the VJ Theorem,

$$J_1 = \frac{\sum_{h\in F_R}(v_1 - v_h(1)) \times h}{\sum_{f\in F_Q} f}.$$

$J_1 = 0$, and so

$$0 = \sum_{h\in F_R}((v_1 - v_h(1)) \times h),$$

which implies the stated theorem.
All that was needed in proving the above VV Theorem is that for any state S_l in $S - R$,

$$v_l = \frac{\sum_{m=1}^{n}((g_{lm})(v_m))}{\sum_{m=1}^{n} g_{lm}}$$

holds, which is a condition of harmonicity outside R.

Here is the equivalent probabilistic version :
Choose a random forest h out of the set F_R, with probability proportional to the product of its conductances. The voltage at S_1, then is the expected value of the voltage of the unique state of R that S_1 is connected to via h.
A Markov chain, having elements of F_R as states and stationary probability of f proportional to the product of its conductances can be obtained from the Markov chain that gives random trees. Take the network, and fuse all states of R. We now have a new conductive network whose branches all come from the original (though some may have become parallel). It is clear that the forests in F_R, consist of precisely those collections of branches that form the trees of the new network, and that this correspondence is bijective. Now, simply use the tree Markov chain to give trees of the new network (this method works perfectly even when we have parallel branches), and pick corresponding forests from F_R. This gives random forests of F_R with the right probabilities.

6 The JI Theorem

Here is a theorem that expresses the current distribution in a network in terms of the injected currents. Let T be the set of all (spanning) trees. Let I be the current matrix $\{i_{kl}\}_{n\times n}$ where i_{kl} is the current flowing in the conductance from k to l. Let J be the vector of injected currents. If we were to set all conductances other than those in a particular tree t to 0, we would have a current distribution that would be 0 in all branches except those of t. Let this current distribution in matrix form be denoted as I_t.
Theorem JI :

$$I = \frac{\sum_{t\in T} t \times (I_t)}{\sum_{t\in T} t}$$

(where as in our previous theorems, when t appears in an algebraic expression, it is taken to be the product of the conductances in t.)
Proof :
The current matrix I is a linear function of the vector of injected currents J. A vector is a valid J if and only if the sum of its components $J_1, \ldots, J_n$ is 0. Such vectors form a vectorspace and it is sufficient to prove the theorem for a basis of this vectorspace. It is clear that vectors of the form z_k, $k = 2$ to n form a basis, where z_k is the vector with $(z_k)_1 = -1$, $(z_k)_k = 1$ and all other entries $(z_k)_l = 0$. For simplicity in notation, we shall prove the theorem for $J = z_2$. Let S_1 be taken to be ground, (i.e $v_1 = 0$). Let R be $\{S_1, S_2\}$. We need to prove that

$$i_{lk} = \frac{\sum_{t\in T}(I_t)_{lk} \times t}{\sum_{t\in T} t}$$

for each pair $\{S_l, S_k\}$. We know that $i_{lk} = g_{lk} \times (v_l - v_k)$ which, using the VV Theorem is

$$g_{lk} \times \frac{(\sum_{h\in F_R} h \times v_h(l)) - (\sum_{h\in F_R} h \times v_h(k))}{\sum_{h\in F_R} h}.$$

This is
$$\frac{g_{lk}\sum_{h\in F_R} h(v_h(l)-v_h(k))}{\sum_{h\in F_R} h}.$$
Now, the possible values of $v_h(l)-v_h(k)$ over varying forests h are $-v_2, v_2$ and 0, which arise respectively from the cases when
(a) S_l is in $B_h(1)$ and S_k is in $B_h(2)$
(b) S_l is in $B_h(2)$ and S_k is in $B_h(1)$
(c) Both belong to the same block with respect to h.
Therefore,
$$i_{lk} = (\frac{g_{lk}}{\sum_{h\in F_R} h}) \times \{ \sum_{(h\in F_R \ s.t\ S_l\in B_h(2)\ and\ S_k\in B_h(1))} h \times v_2 +$$
$$\sum_{(h\in F_R, S_l\in B_h(1),\ s.t\ S_k\in B_h(2))} h \times (-v_2)\}.$$
From Theorem VJ,
$$J_2 = \frac{\sum_{t\in T}(t)\times(v_2-v_1)}{\sum_{h\in F_R} h}.$$
So,
$$1 = \frac{\sum_{t\in T} t\times(v_2)}{\sum_{h\in F_R} h},$$
and therefore
$$v_2 = \frac{\sum_{h\in F_R} h}{\sum_{t\in T} t}.$$
This is actually just Kirchhoff's formula for equivalent resistance. We have it as a special case of Theorem VJ. Substituting this for v_2,
$$i_{lk} = (\frac{g_{lk}}{\sum_{t\in T} t}) \times (\sum_{(h\in F_R \ s.t\ S_l\in B_h(2)\ and\ S_k\in B_h(1))} h$$
$$+ \sum_{(h\in F_R \ s.t\ S_l\in B_h(1)\ and\ S_k\in B_h(2))} -h)$$
Given a tree t, if g_{lk} is in t, call by $\overline{h}$ the forest corresponding to $\frac{t}{g_{lk}}$ (which therefore "separates" S_l and S_k). Then, $(I_t)_{lk}$ is
(a) 0 if $\overline{h}$ is not in F_R.
(b) 1 if $\overline{h}$ is in F_R, $S_{\overline{h}}(l) = S_2$ and $S_{\overline{h}}(k) = S_1$.
(c) -1 if $\overline{h}$ is in F_R, $S_{\overline{h}}(l) = S_1$ and $S_{\overline{h}}(k) = S_2$.
These exhaust all possibilities. Therefore,
$$\frac{\sum_{t\in T}(I_t)_{lk}\times t}{\sum_{t\in T}(t)} = (\frac{g_{lk}}{\sum_{t\in T} t}) \times \{ \sum_{(h\in F_R \ s.t\ S_l\in B_h(2)\ and\ S_k\in B_h(1))} (h)$$
$$+ \sum_{(h\in F_R \ s.t\ S_l\in B_h(1)\ and\ S_k\in B_h(2))} (-h)\},$$
which from what we just saw is i_{lk}. This proves our theorem.

The JI Theorem has a compact probabilistic interpretation. Choose a random tree t (with probability proportional to t), and calculate the branch current matrix after all conductances not in t have been set to 0. Then, the expected value of the current distribution that we would get is the actual distribution.

7 The IV Theorem

This theorem gives node voltages for a particular J. Like the previous theorem, it involves picking a random tree. Let J be the vector of injected currents. Let I be a current matrix $\{i_{lm}\}_{n\times n}$ such that $\sum_{m=1}^{n}(i_{lm}) = J_l$.
We note that this need not be the actual current distribution that results from J in the network we are working with. We could, for example, obtain a valid I of this kind by taking some tree of the network and finding the current distribution if all other conductances were set to 0. This is (in general) a much easier task than solving for the branch currents in our network which could have as many as $\frac{(n)(n-1)}{2}$ branches.
Theorem IV :
Choose a random tree t in T (with notation from Theorem JI), with probability proportional to the product of its conductances. Choose S_1 as ground (this choice is arbitrary). For each S_k, walk along the unique path that is in tree t from S_1 to S_k, find

$$\sum_{all\ g_{lm}\ traversed\ from\ l\ to\ m\ in\ that\ path} \left(\frac{i_{lm}}{g_{lm}}\right),$$

call this $v_t(k)$, and call the voltage vector $(0, v_t(2), \ldots, v_t(n))$ V_t.
Then, the actual voltage vector V (with S_1 as ground) is the expected value of the voltage vectors V_t.
i.e

$$V = \frac{\sum_{t\in T} t \times (V_t)}{\sum_{t\in T} t}$$

Proof:
Consider the $\frac{n(n+1)}{2}$ current matrices $I[k,l]$ corresponding to ordered pairs (k,l), $k > l$, such that in $I[k,l]$, $i_{kl} = 1$, $i_{lk} = -1$ and all other entries are 0. These form a basis of the vectorspace of all possible current matrices. Given a current matrix I, there is a unique injected current vector J corresponding to it which is a linear function of I. Given J, and the values of the conductances, taking S_1 as ground, there is a single (node) voltage vector V. All of these are related linearly, so it is enough to prove the theorem in the case where I has $i_{21} = 1$, $i_{12} = -1$, and all other entries 0. If we prove this case, it proves the theorem for any of our basis matrices; the fact that S_1 has been chosen as ground does not affect generality - it only causes a constant shift in the voltages. The I we have chosen corresponds to $J = (-1, 1, 0, \ldots, 0)$. It clearly suffices to prove that the theorem gives the correct value of v_3 and of v_2.
To prove the v_3 case, we partition the set of all trees into three classes :
T_0, the set of all trees t which do not contain the branch g_{12}.
T_1, the set of all trees t which contain g_{12} and have the property that the path joining S_3 and S_1 in t does not go through S_2.
T_2, the set of all trees t which contain g_{12} and have the property that the path joining S_3 and S_1 in t goes through S_2.
If we choose a random tree t,
$(V_t)_3$ is 0 if t is in T_0 or T_1. If t is in T_2, $(V_t)_3$ is $\frac{1}{g_{12}}$.
Let $R = \{S_1, S_2\}$. Then, t is in T_2 if and only if g_{12} is in t and the forest h obtained by removing g_{12} from t (which always is in F_R) is such that $S_h(3) = S_2$.
Therefore,

$$\sum_{t \in T}(V_t)_3\ t = \sum_{h \in F_R, S_h(3)=S_2} (1/g_{12}) \times (h \times g_{12}) = \sum_{h \in F_R, S_h(3)=S_2} h.$$

Using the VJ Theorem, we have proved that if $J = (-1, 1, 0, \ldots, 0)$, then $v_2 = \frac{\sum_{h \in F_R} h}{\sum_{t \in T}(t)}$, where S_1 is ground. This proves the theorem for v_2. From this,

$$\frac{1}{\sum_{t \in T} t} \times \sum_{h \in F_R\ s.t\ S_h(3)=S_2} (h) = \frac{v_2}{\sum_{h \in F_R} h} \times \sum_{h \in F_R, S_h(3)=S_2} h,$$

which from the VV Theorem is v_3. This completes our proof.

8 Applications to reversible Markov chains

Using the equivalence that was mentioned in the beginning, all of the network theorems can be translated quite literally into theorems for reversible Markov chains. Here however we shall only consider some interesting special cases. We start with a reversible Markov chain given by transition matrix P with states $(S_1, \ldots, S_n)$. Construct the equivalent electrical network in which the nodes are the states S_k and the conductance g_{kl} is $p_{kl}(\pi_k) = p_{lk}(\pi_l)$, π being the stationary distribution. Then $p_{kl} = \frac{g_{kl}}{g_k}$, where

$$g_k = \sum_{m=1}^{n} g_{km}.$$

g_k becomes the stationary probability at S_k. We will need to manipulate clumps of arborescences or "orchards" of the weighted directed graph represented by our Markov chain, so we need some more notation. (here the weight of the directed edge from k to l is taken as p_{kl}). An orchard is a rooted forest; If we take a forest f, choose a root for every connected component of it, and direct all the edges in each connected component, so that there is a directed path from every state to the root of the component of f that it is in, we get an orchard. An orchard is fully determined by the forest that is its imprint in the graph, and the set of nodes or states that are its roots. The orchard corresponding to a forest f, and root set R will be denoted by $[f, R]$. We shall denote the product of its edge transition probabilities by $o[f, R]$. Let R be a subset of S. Assume for convenience that S_1 is not in R, but S_2 is.

Consider a random walk that originates at S_1 at time 0. Impose a stopping rule according to which we stop the walk the first time the walker reaches a state in R. We shall now give formulae for the expected duration of the walk (which we shall henceforth call τ_R), and the probability that the walk terminates at S_2. The electrical equivalent of this problem is as follows (from what we did in Section **??**): The voltages of states in $Q = R \cup \{S_1\}$ are fixed externally, with the voltages of states in R set to 0. We don't know v_1,(which is the expected number of visits to S_1 into g_k) but we do know that $J_1 = 1$, since the walker is known to start from S_1 with probability 1. From the VJ Theorem, we have

$$1 = J_1 = \frac{\sum_{h \in F_R}(h)(v_1 - v_h(1))}{\sum_{f \in F_Q}(f)}.$$

In our problem, $v_h(1)$ is always zero. So

$$v_1 = \frac{\sum_{f \in F_Q}(f)}{\sum_{h \in F_R}(h)}$$

Let e_k be the expected number of departures from the state S_k during the course of the walk.

$$\tau_R = \sum_{k=1}^{n} e_k = \sum_{k=1}^{n} v_k \times g_k.$$

Using Theorem VV, this becomes

$$\frac{\sum_{k=1}^{n}(g_k)\sum_{f\in F_Q} f \times v_f(k)}{\sum_{f\in F_Q}(f)} = \frac{\sum_{f\in F_Q}\sum_{k\ s.t\ S_k\in B_f(1)}(f \times g_k \times v_1)}{\sum_{f\in F_Q}(f)}.$$

($v_f(k)$ is non-zero only if S_k is in $B_f(1)$). Substituting for v_1, this is

$$\frac{\sum_{f\in F_Q}\sum_{k\ s.t\ S_k\in B_f(1)}(f \times g_k)}{\sum_{h\in F_R}(h)}.$$

Dividing numerator and denominator by $\prod_{S_k\in S-R}(g_k)$, we get

$$\tau_R = \frac{1}{\sum_{h\in F_R} o[h,R]} \times \sum_{f\in F_Q}\ \sum_{(k\ s.t\ S_k\in B_f(1))} o[f, R\cup\{S_k\}],$$

which is a formula only in terms of transition probabilities of the Markov chain. The probability that the chain terminates at S_2 is the current that flows out of the network from S_2, which is $-J_2$. We use the VJ Theorem to find this. Let set U be $Q-\{S_2\}$.

$$\begin{aligned}
-J2 &= \frac{1}{\sum_{f\in F_Q} f} \times \sum_{u\in F_U}((v_u(2)-v_2)(u)) \\
&= \frac{1}{\sum_{f\in F_Q} f} \times \sum_{u\in F_U,\ s.t\ S_u(2)=S_1}(v_1)(u) \\
&= \frac{1}{\sum_{h\in F_R} h} \times \sum_{u\in F_U,\ s.t\ S_u(2)=S_1}(u) \\
&= \frac{1}{\sum_{h\in F_R} h} \times \sum_{h\in F_R,\ s.t\ S_h(1)=S_2}(h)
\end{aligned}$$

Dividing numerator and denominator by $\prod_{S_k\in S-R} g_k$, we have

$$-J_2 = \frac{1}{\sum_{h\in F_R} o[h,R]} \times \sum_{h\in F_R\ s.t\ S_1\in B_h(2)}(o[h,R])$$

This is the probability that the walk terminates at S_2. We shall now give an application of Theorem JI. Suppose, in the Markov chain considered above, we started out with an initial probability distribution $p^{(0)}$. Let us denote by $(i^{(m)})_{kl}$ the expected number of transitions from k to l minus those from l to k upto time m. Let $I^{(m)}$ be $\{(i^{(m)})_{kl}\}_{n\times n}$. We shall find a formula for $\lim_{m\to\infty} I^{(m)}$ which we call $I^{(\infty)}$ This is in some sense the net flow that takes place in the Markov chain to reach equilibrium. let $p^{(m)}$ be the resulting probability distribution at time m.Let $J^{(m)}$ be $p^{(0)} - p^{(m)}$. Then, by the equivalence seen earlier, $I^{(m)} = \frac{\sum_{t\in T} I_t\ t}{\sum_{t\in T} t}$, where t and T are from the corresponding electrical network, and have their customary meanings (I_t being calculated using $J^{(m)}$). (We have this formula whenever the expected run-time is finite.) Therefore, to find I in the limit, it is enough to use $J^{(\infty)} = p^{(0)} - \pi$, where π is the stationary probability, since $p^{(m)}$ tends to π as m tends to infinity. With our scaling of conductances, π is simply $(g_1,\ldots,g_n)$. Let vector w_k be a vector of length n, where $(w_k)_l$, the component in its lth position , is $p^{(0)}$ if $l \neq k$, and $p^{(0)} - 1$ if $l = k$. This clearly is a valid J vector, and further,

$J^{(\infty)} = \sum_{m=1}^{n}((g_m)(w_m))$. For an arborescence $[t, S_m]$ (the orchard corresponding to a spanning tree rooted at S_m), given a probability distribution p, we talk about the "flow" $U([t, S_m])$ through it. This is an $n \times n$ matrix whose klth entry,
$u_{kl} = 0$ if neither p_{kl} nor p_{lk} is a directed edge of the arborescence.
If p_{kl} is in $[t, S_m]$, u_{kl} is the sum of probabilities of all states for which the path to S_m in the arborescence goes through p_{kl}.
If p_{lk} is in $[t, S_k]$, u_{kl} is $-u_{lk}$, where u_{lk} is defined using the previous statement. Let A be the set of all arborescences. If $a \in A$, let $o(a)$ be the product of a's transition probabilities.
(All this seems a little elaborate and artificial, but is necessary if we want to get results about the reversible Markov chain from its own parameters, without going to the equivalent electrical network.) Our claim is that

$$I^{(\infty)} = \frac{\sum_{a \in A}(U(a)\ o(a))}{\sum_{a \in A} o(a)},$$

where the flow matrix $U(a)$ is calculated using $p^{(0)}$. We observe that here, $U([t, S_m])$ is just I_t calculated using w_m as the injected current vector. Also, $o[t, S_k] = \frac{(t)(g_k)}{\prod_{m=1}^{n} g_m}$, and therefore $\frac{o[t,S_k]}{g_k} = \frac{o[t,S_l]}{g_l}$. Thus,

$$\frac{\sum_{t \in T}(I_t\ t)}{\sum_{t \in T} t},$$

(where J is w_k)

$$= \frac{\sum_{t \in T}(U([t, S_k])o[t, S_k])}{g_k} \times \frac{\prod_{m=1}^{n} g_m}{\sum_{t \in T}(t)}$$

$$= \frac{\frac{\sum_{t \in T}(U([t,S_k])o[t,S_k])}{g_k}}{\sum_{t \in T}(\sum_{m=1}^{n} o[t, S_m])}$$

(The denominator takes this form when we use $\sum_{k=1}^{n}(g_k) = 1$) Now , using the fact that $J^{(\infty)} = \sum_{m=1}^{n}(g_m\ w_m)$, and combining linearly the corresponding expressions for I, we get the desired result.

9 Conclusion

In this paper we have proved four theorems about electrical networks, namely the theorems VJ, JI, IV and VV. Theorem VJ expresses the currents injected into the network in terms of the voltages of nodes into which they are injected. Theorem JI expresses branch currents in terms of the currents injected externally. In Theorem IV, an arbitrary current distribution is taken which, for the given injected currents, satisfies the Kirchoff current law. The voltages at nodes are expressed in terms of this current distribution. The VV Theorem expresses the voltages of each node in terms of the voltages of the nodes into which currents are injected. The mode of proof used for these results can also be used to give alternative proofs of the Kirchhoff tree formula. We have noted that the formulae give expressions in terms of expected values, and so might lead to efficient methods of solving resistive circuits approximately. Finally, we have obtained three formulae for reversible Markov chains, one of which gives an expression for transit time from one state to another.

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