Geometric Complexity Theory III: On deciding nonvanishing of a Littlewood-Richardson coefficient

Dedicated to Sri Ramakrishna

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Abstract

We point out that the positivity of a Littlewood-Richardson coefficient $c^{\gamma}_{\alpha,\beta}$ for GL_n can be decided in strongly polynomial time. This means that the number of arithmetic operations is polynomial in n and independent of the bit lengths of the specifications of the partitions α, β and γ, and each operation involves numbers whose bitlength is polynomial in n and the bit lengths α, β and γ.

Secondly, we observe that non-vanishing of a generalized Littlewood-Richardson coefficient of any type can be decided in strongly polynomial time assuming an analogue of the saturation conjecture for other types, and that for weights α, β, γ the positivity of $c^{2\gamma}_{2\alpha,2\beta}$ can (unconditionally) be decided in strongly polynomial time.

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1 Introduction

The fundamental Littlewood-Richardson rule in the representation theory of $GL_n(\mathbb{C})$ [3] states that the tensor product of two irreducible representations (Weyl modules) V_α and V_β of a semisimple Lie algebra $\mathcal{G}$ decomposes as follows:

$$V_\alpha \otimes V_\beta = \oplus_\gamma C^\gamma_{\alpha,\beta} V_\gamma, \tag{1}$$

where $C^\gamma_{\alpha,\beta}$ are generalized Littlewood-Richardson coefficients. Here α, β and γ denote the highest weights of $\mathcal{G}$. When $\mathcal{G} = sl_n(\mathbb{C})$ (type A), α and β are partitions (Young diagrams) with at most n rows, and the sum is over all Young diagrams γ of height at most n, and size equal to the sum of the sizes of α and β.

We are interested in finding an efficient algorithm to decide if $C^\gamma_{\alpha,\beta}$ is nonvanishing (positive). This problem arises naturally in the geometric complexity theory approach [15, 14, 16] towards the P vs. NP and related problems.

It has been observed in [12, 11, 17] independently that, when $\mathcal{G}$ is simple of type A, nonvanishing of $C^\gamma_{\alpha,\beta}$ can be decided in polynomial time; i.e., in time that is polynomial in the bitlengths of the specifications of the partitions α, β and γ. Furthermore, the algorithm in [17] is strongly polynomial– i.e., the number of arithmetic steps in the algorithm depends only on the total number of parts of α, β and γ, but not their bitlengths. One crucial ingredient in this algorithm is the saturation theorem in [10], which does not hold for simple Lie algebras of type B, C or D [24]. The result in [17] was extended to other types in [18] assuming a positivity conjecture in [12]. This article combines the results of [17] and [18].

We now state these results in more detail.

First, we consider type A; i.e., $\mathcal{G} = sl_n(\mathbb{C})$. Let $\lambda = (\lambda_1, \cdots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > 0$, be a partition (Young diagram). By its bit length $\langle \lambda \rangle$, we mean the bit length of its specification, which is $\sum_i \log_2(\lambda_i)$. Observe that the dimension of the Weyl module V_λ can be exponential in n, k and the bit lengths of λ_i's. Because the dimension of V_λ is the total number of semistandard tableau of shape λ with entries in $[1, n]$ [3].

Theorem 1.1 *Given partitions α, β and γ, deciding if V_γ exists within $V_\alpha \otimes V_\beta$–i.e. if $C_{\alpha,\beta,\gamma}$ is positive–can be done in polynomial time; i.e., in time that is polynomial in n and the bit lengths of α, β, and γ[1] Furthermore, the*

[1] If we assume that a partition λ is specified as $(\lambda_1, \cdots, \lambda_n)$, with $\lambda_1 \geq \cdots \geq \lambda_n$, where

algorithm is strongly polynomial in the sense of [13].

Strong polynomiality stated in the theorem means that [13]: (1) The number of arithmetic steps in the algorithm is polynomial in n. It does not depend on the bit lengths of α_i, β_j, and γ_k's. (3) The bit length of every intermediate operand that arises in the algorithm is polynomial in the total bit length of α, β and γ.

For general types, we have:

Theorem 1.2 *The positivity of a generalized Littlewood-Richardson coefficient $C^{\nu}_{\lambda,\mu}$ for any semi-simple Lie algebra $\mathcal{G}$ can be decided in strongly polynomial time, assuming the following positivity conjecture made in [12].*

Let $\tilde{C}(n) = \tilde{C}^{\mu}_{\lambda,\mu}(n) = C^{n\nu}_{n\lambda,n\mu}$ denote the stretching function associated with $C^{\nu}_{\lambda,\mu}$. Assume that the type of $\mathcal{G}$ is B, C or D. Then $\tilde{C}(n)$ is a quasi-polynomial of period at most two [12]. That is, there exist polynomials $\tilde{C}_1(n)$ and $\tilde{C}_2(n)$ such that

$$C^{n\nu}_{n\lambda,n\mu} = \begin{cases} \tilde{C}_1(n), & \text{if n is odd;} \\ \tilde{C}_2(n), & \text{if n is even.} \end{cases}$$

Conjecture 1.3 (Positivity conjecture) *[12]*

The quasi-polynomial $\tilde{C}(n) = \tilde{C}^{\nu}_{\lambda,\mu}(n)$ is positive–i.e., the coefficients of $\tilde{C}_i(n)$, $i = 1, 2$, are nonnegative.

This is an extension of an analogous earlier conjecture in [9] for type A. Considerable experimental evidence for these conjectures has been given in these papers.

Here it is assumed that each highest weight is specified by giving its coordinates in the basis of fundamental weights. The bitlength $\langle\lambda\rangle$ is defined to the total bitlength of all coordinates. Strongly polynomial means the number of arithmetic steps is polynomial in the rank of $\mathcal{G}$, and the bit length of every intermediate operand is polynomial in the bitlengths $\langle\lambda\rangle, \langle\mu\rangle$ and $\langle\nu\rangle$ of λ, μ and ν and the rank of $\mathcal{G}$.

$\lambda_i = 0$ for i higher than the height of λ, then the term n can be subsumed in the bit length of the input.

Remark 1.4 *For Theorem 1.2 to hold, we do not need the full statement of the Positivity Conjecture, but only the following analogue of saturation for Lie groups of types B, C, D:*

$$C^{\nu}_{\lambda\mu} = 0 \implies \forall \text{ odd } n, C^{n\nu}_{n\lambda,n\mu} = 0.$$

In fact the following weaker hypothesis suffices: A generalized Littlewood-Richardson coefficient is non-zero if the affine span of the corresponding BZ-polytope [1] contains an integer point.

Finally, we observe that the proof of Theorem 1.1 can be extended to general types using the recent results in [2] and [20]:

Theorem 1.5 *The positivity of a generalized Littlewood-Richardson coefficient $C^{2\nu}_{2\lambda,2\mu}$ for any semi-simple Lie algebra $\mathcal{G}$ can be decided in strongly polynomial time.*

Theorem 1.1 is proved in Section 2, Theorem 1.2 in Section 3 and Theorem 1.5 in Section 4.

2 Littlewood-Richardson coefficient of type A

Here we prove Theorem 1.1. The proof follows easily from the following three results:

1. Littlewood-Richardson rule: specifically, a polyhedral interpretation of the Littlewood-Richardson coefficients. The polytope we use here is more elementary than BZ-polytope [1] and the Hive polytope [10]–the latter two have some stronger properties not used here.

2. Saturation Theorem [10].

3. Polynomial time algorithm for linear programming: e.g. the ellipsoid or the interior point method, and the related strongly polynomial time algorithm for combinatorial linear programming in [13].

Let us begin with a polyhedral interpretation; this should be well known. Recall that the Littlewood-Richardson coefficient $c^{\gamma}_{\alpha,\beta}$ has the following combinatorial interpretation [4] .

Let us say that a word $w = w_1 \cdots w_r$ is a *reverse lattice word* if, when read backwards from the end to any letter w_s, $s < r$, the sequence $w_r \cdots w_s$

contains at least as many 1's as 2's, at least as many 2's as 3's, and so on for all positive integers. The row word $w(T)$ of a skew tableau T is defined to be the word obtained by reading its entries from bottom to top, and left to right. A skew-tableau T of shape γ/α is called a *Littlewood-Richardson skew tableau* if its row word $w(T)$ is a reverse lattice word.

Then $C_{\alpha,\beta}^{\gamma}$ is the number of Littlewood-Richardson skew tableaux of shape γ/α of content β.

Let $r_j^i(T)$, $i \leq n$, $j \leq n$, denote the number of j's in the i-th row of T. These are integers satisfying the constraints:

1. Nonnegativity: $r_j^i \geq 0$.
2. Shape constraints: For $i \leq n$,
$$\alpha_i + \sum_j r_j^i = \gamma_i.$$
3. Content constraints: For $j \leq n$:
$$\sum_i r_j^i = \beta_j.$$
4. Tableau constraints: No $k \leq j$ occurs in the row $i+1$ of T below a j or a higher integer in the row i of T:
$$\alpha_{i+1} + \sum_{k \leq j} r_k^{i+1} \leq \alpha_i + \sum_{k' < j} r_{k'}^i.$$
5. Reverse lattice word constraints: $r_j^i = 0$ for $i < j$, and for $i \leq n$, $1 < j \leq n$:
$$\sum_{i' lei} r_j^{i'} \leq \sum_{i' < i} r_{j-1}^{i'}.$$

Let r denote the vector with the entries $r_j^i(T)$. These constraints can be written in the form of a linear program:

$$Ar \leq b, \tag{2}$$

where the entries of A are $0, 1$ or -1, and the entries of b are homogeneous, integral, linear forms in α_i, β_j, and γ_k's. Thus $C_{\alpha,\beta}^{\gamma}$ is the number of integer points in the polytope P determined by these constraints.

Claim 2.1 *The polytope P contains an integer point iff it is nonempty.*

Proof: One direction is trivial.

Suppose P is nonempty. Since b is homogeneous in α, β and γ, it follows that, for any positive integer q, $C^{q\gamma}_{q\alpha,q\beta}$ is the number of integer points in the scaled polytope qP. All vertices of P have rational coefficients. Hence, for some positive integer q, the scaled polytope qP has an integer point. It follows that, for this q, $C^{q\gamma}_{q\alpha,q\beta}$ is positive. Saturation Theorem [10] says that, in this case, $C^{\gamma}_{\alpha,\beta}$ is positive. Hence, P contains an integer point. Q.E.D.

Whether P is nonempty can be determined in polynomial time using either the ellipsoid or the interior point algorithm for linear programming. Since the linear program (2) is combinatorial [13], this can also be done in strongly polynomial time using the algorithm in [13]. This proves Theorem 1.1.

3 Generalized Littlewood-Richardson Coefficients

In this section we prove Theorem 1.2.

Let $P = P^{\nu}_{\lambda,\mu}$ denote the BZ-polytope [1] whose Ehrhart quasi-polynomial coincides with $\tilde{C}^{\nu}_{\lambda,\mu}(n)$.

Definition 3.1 *For any subset B of $\mathbb{Q}^n$, its affine span over rationals,* $\mathrm{Aff}(B)$ *is*

$$\left\{ v \middle| \ \exists(\{v_i\} \in B, \{\alpha_i\} \in \mathbb{Q}), \ \textit{such that} \ \sum_{i=1}^{k} \alpha_i = 1 \ \textit{and} \ v = \sum_{i=1}^{k} \alpha_i v_i \right\}.$$

Let $\mathbb{Z}_{<2>}$ denote the subring of $\mathbb{Q}$ obtained by localizing $\mathbb{Z}$ at 2–i.e., the subring of fractions with odd denominators. We will call a point in $\mathbb{R}^d$ rational if all its co-ordinates are rational.

Lemma 3.2 *Assume that $\mathcal{G}$ is simple of type B, C or D. If the positivity conjecture is true, the following are equivalent:*

(1) $C^{\nu}_{\lambda\mu} \geq 1$.

(2) There exists an odd integer n such that $C^{n\nu}_{n\lambda\, n\mu} \geq 1$.

(3) P contains a point in $\mathbb{Z}^d_{<2>}$.

(4) Aff(P) *contains a point in* $\mathbb{Z}^d_{<2>}$.

Proof: Clearly, the first three statements are equivalent, and (3) implies (4). It remains to show that (4) implies (3). Let $z \in \mathbb{Z}^d_{<2>} \cap \mathrm{Aff}(P)$.
The $0-$dimensional case is trivial since $\{z\} = P$. Suppose that the dimension of P is greater or equal to 1. Since z has rational coordinates and is contained in Aff(P), $z = ax + (1-a)y$ for some distinct rational points $x, y \in P$, and $a \in \mathbb{Q}$. Let q be a positive integer such that $2^q(x-y) \in \mathbb{Z}^d_{<2>}$.

$\{z + \lambda 2^q(x-y) \big| \lambda \in \mathbb{Z}_{<2>}\}$ is a dense subset of Aff$(\{x,y\})$ in the topology induced by the standard topology of $\mathbb{R}^n$, and is therefore nonempty. Thus $P \cap \mathbb{Z}^d_{<2>} \supseteq \{z + \lambda 2^q(x-y) \big| \lambda \in \mathbb{Z}_{<2>}\} \neq \emptyset$.

Q.E.D.

Now we turn to the proof of Theorem 1.2. First, let us assume that $\mathcal{G}$ is simple of type B, C or D.

The specification of an explicit linear program of the form $Ax \leq b$ defining the BZ-polytope $P = P^{\nu}_{\lambda,\mu}$ can be computed in strongly polynomial time using its description in [1]. It is also clear from [1] that the entries of A here have constant bit lengths. In the terminology of [23], this linear program is combinatorial. Hence, we can determine if P is nonempty in strongly polynomial time by the combinatorial linear programming algorithm in [23]. If P is nonempty, this algorithm can also be extended to find an integral matrix C and an integral vector D so that Aff(P) is defined by the linear system $Cx = D$. One way of achieving this is the following. Find, for every constraint hyperplane h of P, a vertex v_h of P that is the farthest to h. The affine span is the intersection of all constraint hyperplanes h such that $v_h \in h$. Usual linear programming algorithms [7, 8] here, in place of the algorithm in [23], will yield a polynomial time algorithm, instead of a strongly polynomial time algorithm.

By Lemma 3.2 (4), it remains to check if Aff(P) contains a point in $\mathbb{Z}^d_{<2>}$. This can be done as follows. By padding, if necessary, we can assume that C is square. Using [5], we find the Smith normal form S of C and unimodular matrices U and V such that $C = USV$; here S is a diagonal integer matrix, whose i-th diagonal entry divides the $(i+1)$-st diagonal entry. Since the entries of C have constant bit lengths, the algorithm in [5] works in strongly polynomial time. The question now reduces to checking if $USVx = D$ has a solution $x \in \mathbb{Z}^d_{<2>}$. This is so iff $Sy = U^{-1}D$ has a solution $y \in \mathbb{Z}^d_{<2>}$. Since S is diagonal, this can be verified in (strongly) polynomial time by checking each coordinate.

This proves Theorem 1.2 for types B, C, D.

Now let $\mathcal{G}$ be any semisimple algebra. A generalized Littlewood-Richardson coefficient for $\mathcal{G}$ is the product of corresponding generalized Littlewood-Richardson coefficients for each of its simple factors. Hence, without loss of generality, we can assume that $\mathcal{G}$ is simple. If it is of type A, then Theorem 1.2 holds unconditionally by Theorem 1.1. If it is an exceptional simple Lie algebra, then a Littlewood-Richardson coefficient can be computed in $O(1)$ arithmetic steps. This is because, when the rank of $\mathcal{G}$ is constant, the chambers of quasi-polynomiality [22] of the generalized Littlewood-Richardson coefficient, considered as a vector partition function, are generated by $O(1)$ constraints.

This proves Theorem 1.2.

4 Proof of Theorem 1.5

Suppose first that $\mathcal{G}$ is of type B, C, or D. By [2] and [20], it follows that if there exists an integer n such that $C^{n\nu}_{n\lambda,n\mu} \geq 1$, then, $C^{2\nu}_{2\lambda,2\mu} \geq 1$. A weaker form of this result (with 4 in place of 2) was proven in [6]. By the argument in subsection 3, $C^{2\nu}_{2\lambda,2\mu} \geq 1$ if and only if the BZ polytope $P^{\nu}_{\lambda,\mu}$ is nonempty, which can be checked in strongly polynomial time. The argument towards the end of subsection 3 allows the algorithm to be extended to arbitrary semisimple groups. Q.E.D.

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