Heat Flow and a Faster Algorithm to Compute the Surface Area of a Convex Body

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Abstract

We draw on the observation that the amount of heat diffusing outside of a heated body in a short period of time is proportional to its surface area, to design a simple algorithm for approximating the surface area of a convex body given by a membership oracle. Our method has a complexity of $O^*(n^4)$, where n is the dimension, compared to $O^*(n^{8.5})$ for the previous best algorithm. We show that our complexity cannot be improved given the current state-of-the-art in volume estimation.

1 Introduction

An important class of algorithmic questions centers around estimating geometric invariants of convex bodies. Arguably, the most basic invariant is the volume. It can be shown ([5], [1]) that any deterministic algorithm to approximate the volume of a convex body within a constant factor in $\mathbb{R}^n$ needs time exponential in the dimension n. Remarkably, randomized algorithms turn out to be more powerful. In their pathbreaking paper [3] Dyer, Frieze and Kannan gave the first randomized polynomial time algorithm to approximate the volume of a convex body to arbitrary accuracy. Since then a considerable body of work has been devoted to improving the complexity of volume computation culminating with the recent best of $O^*(n^4)$ due to Lovász and Vempala [11].

Another fundamental geometric invariant associated with a convex body is surface area. Estimating the surface area was mentioned as an open problem by Grötschel, Lovász, and Schrijver in 1988 ([9]). Dyer, Gritzmann and Hufnagel [4] showed in 1998 that it could be solved in randomized polynomial time. The primary focus of their paper was to establish that the computation of surface area and certain other mixed volumes was possible in randomized polynomial time, and they assumed access to oracles for δ-neighbourhoods of the convex body. They did not discuss the complexity of their algorithm given only a membership oracle for the convex body. Below, we indicate an $O^*(n^{8.5})$ analysis of their algorithm in terms of the more restricted queries.

In this paper we develop a new technique for estimating volumes of boundaries based on ideas from heat propagation. The underlying intuition is that the amount of heat escaping from a heated object in a small interval of time is proportional to the surface area.

It turns out that this intuition lends itself to an efficient randomized algorithm for computing surface areas of convex bodies, given by a membership oracle. In this paper we describe the algorithm and the analysis of the algorithm, proving a complexity bound of $O^*(n^4)$. The $O^*(\cdot)$ notation hides the polynomial dependence on the relative error ϵ, and poly-logarithmic factors in the parameters of the problem. Since, as will be shown below, surface area estimation is at least as hard as volume approximation, this bound is the best possible, given the current state-of-the-art in volume estimation.

We note that this bound cannot be obtained using methods previously proposed in [4] due to a bottleneck in their approach. The method in [4] exploited the fact that $\mathrm{vol}(K + B\delta)$ is a polynomial in δ, where $B\delta$ is a ball of radius δ and the Minkowski sum $K + B\delta$ corresponds to the set of points within a distance δ of K. The surface area is the coefficient of the linear term, which they then estimate by interpolation. However, in a natural setting, we only have access to a membership oracle for K, but not for $K + B\delta$. Therefore a membership oracle for $K + B\delta$ has to be constructed, which as far as we can see, requires solving a quadratic programming problem on a convex set. Given access only to a *membership oracle*, the best known algorithm to handle this task is due to Kalai and Vempala, and makes $O^*(n^{4.5})$ oracle calls ([7]), which gives a bound on the complexity of the algorithm in [4] that is $O^*(n^{8.5})$.

Even with a stronger *separation oracle* the complexity of the method in [4] is $O^*(n^5)$, since the associated quadratic programming problem requires $O^*(n)$ operations ([14], [2].) On the other hand, the complexity of our method is $O^*(n^4)$ using only a *membership oracle*, matching the complexity of the volume computation of Lovász and Vempala [11].

2 Overview of the algorithm

Notation. Throughout this paper, B will denote the unit n-dimensional ball, K will denote an n-dimensional convex body such that $rB \subseteq K \subseteq RB$. $S = \mathrm{vol}(\partial K)$ will denote the surface area of K and $V = \mathrm{vol}(K)$, its volume.

We first observe that problem of estimating the surface area of a convex body is at least as hard as that of estimating the volume. This observation can be stated as

Proposition 1 *If the surface area of any n-dimensional convex body K can be approximated in $O(n^\beta \textsf{polylog}(\frac{nR}{\delta r})\textsf{poly}(\frac{1}{\epsilon}))$ time, the volume can be approximated in $O(n^\beta \textsf{polylog}(\frac{nR}{\delta r})\textsf{poly}(\frac{1}{\epsilon}))$ time, where δ is the probability that the relative error exceeds ϵ.*

The proof of the proposition relies on the fact that given a body K there is a simple relationship between the volume of K and the surface area of the cylinder $K \times [0, h]$. More specifically (see Fig. 2)

$$2\mathrm{vol}(K) = \mathrm{vol}(\partial(K \times [0, h])) - h\mathrm{vol}(\partial K)$$

Thus an efficient algorithm for surface area estimation would also lead to an almost equally efficient algorithm for estimating the volume. The details can be found in the Appendix.

Our approach provides an estimate for the isoperimetric ratio $\frac{S}{V}$. Using the fastest existing algorithm for volume approximation, we obtain a separate estimate for V. Multiplying these two estimates yields the surface area S.

The underlying intuition of our algorithm is that the heat diffuses from a heated body through its boundary. Therefore the amount of heat escaping in a short period of time is proportional to the surface area of the object. Recalling that a point source of heat diffuses at time t according to the Gaussian distribution $\frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|^2}{4t}}$ leads to the following informal description of the algorithm (see details in Section 3):

Step 1. Take $x_1, \ldots, x_N$ to be samples from the uniform distribution on K.

Step 2. For each x_i, let $y_i = x_i + v_i$, where v_i is sampled from the Gaussian distribution with density $\frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|^2}{4t}}$ for some appropriate value of t. Thus y_i is obtained from x_i by taking a random Gaussian step.

Step 3. Let $\hat{N}$ be the number of y's, which land outside of K. $\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}$ is an estimate for $\frac{S}{V}$.

Step 4. Using an existing algorithm, produce an estimate $\hat{V}$ for the volume. Estimate the surface area as $\hat{V}\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}$.

We will show that that each of the Steps 1,3,4 can be done using at most $O^*(n^4)$ calls to the membership oracle[1]. Step 2, of course, does not require any calls to the oracle at all.

The main technical result of this paper is to show how to choose values of t and N, such that

$$(1-\epsilon)\left(\frac{S}{V}\sqrt{\frac{t}{\pi}}\right) < \frac{\hat{N}}{N} < (1+\epsilon)\left(\frac{S}{V}\sqrt{\frac{t}{\pi}}\right)$$

It is not known how to efficiently obtain independent random samples from the uniform distribution on K. We show how to relax this condition and use *almost independent* samples from a *nearly uniform* distribution instead, to derive these estimates.

We then apply certain results from [10] and [11], to generate $O\left(\frac{n}{\epsilon^3}\right)$ such samples making at most $O^*\left(\frac{n^4}{\epsilon^3}\right)$ oracle calls.

Putting these and some additional observations together, we obtain the following theorem which is the main result of this paper:

Theorem 2.1 *The surface area of a convex body K, given by a membership oracle, and parameters r, R such that $rB \subseteq K \subseteq RB$ can be approximated to within a*

[1]It is customary to count the number of oracle calls rather than the number of arithmetic steps in the volume literature, while measuring the complexity.

relative error of ϵ with probability $1-\delta$ using at most

$$O\left(n^4 \log\frac{1}{\delta}\left(\frac{1}{\epsilon^2}\log^9\frac{n}{\epsilon}+\log^8 n\log\frac{R}{r}+\frac{1}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)\right)$$

i. e. $O^*(n^4)$ oracle calls.

The number of arithmetic operations is $O^*(n^6)$, on numbers with a polylogarithmic number of digits. This is the same as that for volume computation in [11].

3 Algorithm to compute the surface area

3.1 Notation and Preliminaries

A body K is said to be in *t-isotropic position* if, for every unit vector u,

$$\frac{1}{t}\leq\int_K (u^T(x-\bar{x}))^2dx\leq t,$$

where $\bar{x}$ is the center of mass of K. Let ρ be the uniform distribution on convex body K. We call a random point x ϵ-uniform if

$$\sup_{\text{measurable } A} P(x\in A)-\rho(A)\leq\frac{\epsilon}{2},$$

Two random variables will be called μ-independent if for any two Borel sets A and B in their ranges,

$$|P(X\in A, Y\in B)-P(X\in A)P(Y\in B)|\leq\mu.$$

A density ρ' is said to have $\mathcal{L}^2$ norm $\int_K V^2\frac{d\rho'^2}{d\rho^2}d\rho$ with respect to the uniform distribution on K.

A consequence of the results on page 4 of ([11]), and Theorem 7.2, [15] that, given a starting point that is ϵ-uniform, and comes from a distribution that has a bounded $\mathcal{L}^2$ norm it takes $O(n^3\ln^7\frac{n}{\epsilon\mu})$ oracle calls per point, to generate N points $x_1,\ldots,x_N$ that are ϵ-uniform and such that each pair is μ-independent from a convex body that is 2-isotropic. This fact plays a crucial role in allowing the surface area algorithm to have a complexity bounded by $O^*(n^4)$.

3.2 Algorithm

We present an algorithm below that outputs an ϵ-approximation to the surface area of a convex body K with probability $>3/4$. Running it $\lceil 36\ln\left(\frac{2}{\delta}\right)\rceil$ times and taking the median of the outputs gives the result with a confidence $>1-\delta$.

Input: Convex body K, given by a membership oracle, and parameters r, R such that $rB\subseteq K\subseteq RB$ and an error parameter $\epsilon<1$.

Output: An estimate $\hat{S}$, that with probability $>3/4$ has a relative error of less than ϵ with respect to S.

Set $\epsilon' := \frac{\epsilon}{8}$, $\mu := \frac{\epsilon'^4}{2^{18}n^2}$, $N := \lceil\frac{2^{13}n}{\epsilon'^3}\rceil$.

Step 1. Run the a volume algorithm to obtain an estimate $\hat{V}$ of V that has a relative error ϵ' with probability $>\frac{15}{16}$.

Step 2 Generate a linear transformation T given by a symmetric positive-definite matrix such that TK is 2-isotropic with probability $>\frac{15}{16}$.

Step 3 Compute a lower bound r' to the smallest eigenvalue r_{opt} of $\frac{T^{-1}}{\sqrt{2}}$, that satisfies $\frac{2}{\sqrt{5}}r_{opt}<r'<r_{opt}$. Set $r_{in}:=\max(r, r')$.

Step 4 Set $\sqrt{t} := \frac{\epsilon' r_{in}}{4n}$.

Step 5 Generate N random points $x_2,\ldots,x_N$ from K, such that with probability $15/16$, they are $\frac{\epsilon'^2}{64n}$-uniform and each pair $\{x_i, x_j\}$ for $1\leq i<j\leq N$ is μ-independent.

Step 6 Generate N independent random samples $v_1,\ldots,v_N$ from the spherically symmetric multivariate Gaussian distribution with mean $\vec{0}$ and variance $2nt$.

Step 7 Let $\hat{N} := |\{i|x_i+v_i\notin K\}|$ be the number of times x_i+v_i lands outside of K.

Step 8 Output $\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}\hat{V}$.

3.3 Analysis of the Run-time

Step 1 takes at most

$$O\left(\frac{n^4}{\epsilon^2}\log^9\frac{n}{\epsilon}+n^4\log^8 n\log\frac{R}{r}\right)$$

oracle calls, using the volume algorithm of Lovász and Vempala ([11].) The number of steps in the computation is $O^*(n^6)$.

Step 2 Such a transformation is obtained during the execution of the volume algorithm from [11] for no additional cost.

Step 3 takes $O(n^3)$ steps of computation ([12].)

Step 4 takes $O(1)$ steps.

Step 5 takes

$$O\left(\frac{n^4}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)$$

steps of computation (including oracle calls) once a point x_1 is obtained that is $\frac{\epsilon'^2}{64n}$-uniform, and has an $\mathcal{L}^2$ norm that is bounded above by a constant. Such a point can be obtained from the algorithm in step 1, for no additional cost up to constants. The cost mentioned in this step is incurred because we are required to generate $O(\frac{n}{\epsilon^3})$ random points given the initial random point x_1 and the time per point is $O(n^3 \ln^7 \frac{n}{\epsilon})$. This last fact follows from the complexity per point mentioned on page 4 ([11]), and theorems 7.1 and 7.2 of ([15].)

Step 6 and **Step 7** take $O\left(\frac{n^2}{\epsilon^3}\mathsf{polylog}\frac{n}{\epsilon\delta}\right)$ steps each, assuming that a sample from univariate Gaussian distribution can be obtained upto $O(\mathsf{polylog}(\frac{n}{\epsilon\delta}))$ digits in $O(\mathsf{polylog}(\frac{n}{\epsilon\delta}))$ steps.

Step 8 takes $O(1)$ steps. Finally, to obtain the approximation with a confidence $> 1-\delta$, this algorithm must be run $O\left(\log\left(\frac{1}{\delta}\right)\right)$ times. Therefore the overall cost in terms of oracle calls is

$$O\left(n^4 \log\frac{1}{\delta}\left(\frac{1}{\epsilon^2}\log^9\frac{n}{\epsilon} + \log^8 n \log\frac{R}{r} + \frac{1}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)\right)$$

i. e. $O^*(n^4)$ oracle calls. The number of arithmetic operations is $O^*(n^6)$, on numbers with a polylogarithmic number of digits. This is the same as that for volume computation in [11].

4 Proving correctness of the algorithm

Definition 1 *Let*

$$G_t(x,y) := \frac{\mathrm{e}^{-\|x-y\|^2/4t}}{(4\pi t)^{n/2}}.$$

and

$$F_t := \sqrt{\frac{\pi}{t}}\int_K\int_{\mathbb{R}^n-K} G_t(x,y)dydx.$$

$\sqrt{\frac{t}{\pi}}\frac{F_t}{V}$ is the fraction of heat that would diffuse out of K in time t.

Our proof hinges on two main propositions. The first, Proposition 2, states that F_t is a good approximation for the surface area S. As in the surface area algorithm, let T be a linear transformation such that TK is 2-isotropic. Compute a lower bound r' to the smallest eigenvalue r_{opt} of $\frac{T^{-1}}{\sqrt{2}}$, that satisfies $\frac{2}{\sqrt{5}}r_{opt} < r' < r_{opt}$. Set $r_{in} := \max(r, r')$. Then, the following is true.

Proposition 2 *Let* $\sqrt{t} = \frac{\epsilon' r_{in}}{4n}$ *and* $\epsilon' < 1/2$. *Then,*

$$(1-\epsilon')S < F_t < (1+\epsilon')S.$$

Proposition 3 states that the empirical quantity $\hat{S}$ computed by the surface area algorithm is likely to be an ϵ-approximation of F_t with probability $> 3/4$. Let $x_1, x_2, \ldots, x_N$ from K, be ϵ'-uniform and each pair $\{x_i, x_j\}$ for $1 \le i < j \le N$ be μ-independent with probability $> 15/16$. Let $v_1, \ldots, v_N$ be N independent random samples from the spherically symmetric multivariate Gaussian distribution whose mean is $\vec{0}$ and variance is $2nt$. Let $\hat{N} := |\{i | x_i + v_i \notin K\}|$. Then,

Proposition 3 *Let* $\sqrt{t} = \frac{\epsilon' r_{in}}{4n}$ *and* $\epsilon' < 1/2$. *Then, with probability greater than* $\frac{3}{4}$,

$$(1-\epsilon')(1-2\epsilon')F_t < \frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}\hat{V} < (1+\epsilon')(1+2\epsilon')F_t.$$

These two Propositions together imply that with probability $> 3/4$,

$$(1-\epsilon)S < \frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}\hat{V} < (1+\epsilon)S.$$

The argument for boosting the confidence from $3/4$ to $1-\delta$ is along the lines of ([8],[6].) We devote the rest of this paper to outlining the proofs of Proposition 2 and Proposition 3.

5 Relating surface area S to normalized heat flow F_t

In this section, we prove Proposition 2.

5.1 Notation and Preliminaries

The set of points within a distance δ of a convex body K (including K itself) shall be denoted K_δ. This is called the *outer parallel body* of K and is convex.
The set of points at a distance $\ge \delta$ to $\mathbb{R}^n - K$ shall be denoted $K_{-\delta}$. This is called the *inner parallel body* of K and again is convex. For any body K, we denote by ∂K, its boundary.

Given $x \in K$, let H_x be a closest half-space to x not intersecting $K - \partial K$. For $y \notin K$ define H_y to be the half-space furthest from y containing K.

Observation 1 *If* $x \in \partial K_{-\delta}$ *then the distance between* x *and* H_x *is* δ. *If* $y \in \partial K_\delta$ *then the distance between* y *and* H_y *is* δ.

Definition 2 *Let*

$$e(t,\delta) = \frac{1 - \mathrm{Erf}\left(\frac{\delta}{2\sqrt{t}}\right)}{2}$$

where Erf *is the usual Gauss error function, defined by*

$$\mathrm{Erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt.$$

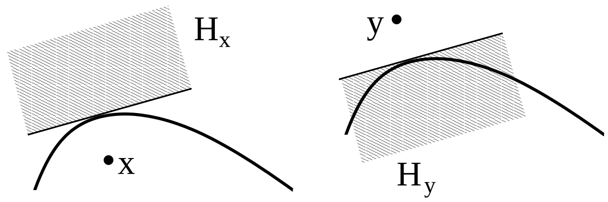

Figure 1. Points x and y and corresponding half-spaces H_x and H_y

Observation 2 *Let $x \in \partial K_{-\delta}$, and $y \in \partial K_\delta$. Then,*

$$\int_{H_y} G_t(z,y)dz = e(t,\delta)$$

and

$$\int_{H_x} G_t(z,x)dz = e(t,\delta)$$

The volume of K_δ is a polynomial in δ, given by the *Steiner formula* (see page 197, [13].)

$$\mathrm{vol}(K_\delta) = a_0 + \cdots + \binom{n}{i} a_i \delta^i + \cdots + a_n \delta^n.$$

The coefficients a_i satisfy the *Alexandrov-Fenchel inequalities* (see page 334, [13],) which state that the coefficients a_i are log-concave; i. e. $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$.

Definition 3 *The surface area* $\mathrm{vol}(\partial K)$ *of an arbitrary convex body K is defined as*

$$\lim_{\delta \to 0} \frac{\mathrm{vol}K_\delta - \mathrm{vol}K}{\delta}.$$

It follows from the *Steiner formula* that this limit exists and is finite. It is a consequence of Lemma 2 that the so defined surface area for an inner parallel body $\mathrm{vol}(\partial K_{-\delta})$ is a continuous function of δ. For an outer parallel body, the *Steiner formula* implies that $\mathrm{vol}(\partial K_\delta)$ is a polynomial in δ.

5.2 Proof of Proposition 2

Lemma 1 is the first step towards proving upper and lower bounds for the normalized heat flow F_t in terms of S. It bounds F_t above by a function of the $\mathrm{vol}(\partial K_\delta)$ and below by a function of $\mathrm{vol}(\partial K_{-\delta})$.

Lemma 1 *1.* $\sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \mathrm{vol}(\partial K_{-\delta}) e(t,\delta) d\delta < F_t$

2. $F_t < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \mathrm{vol}(\partial K_\delta) e(t,\delta) d\delta.$

Proof

Note that for a fixed $x \in \partial K_{-\delta}$

$$\int_{\mathbb{R}^n - K} G_t(x,y)dy > \int_{H_x} G_t(x,y)dy = e(t,\delta).$$

Therefore integrating over shells $\partial K_{-\delta}$, $F_t =$

$$\sqrt{\frac{\pi}{t}} \int_K \int_{\mathbb{R}^n - K} G_t(x,y)dydx > \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \mathrm{vol}(\partial K_{-\delta}) e(t,\delta) d\delta.$$

By the same token for a fixed $y \in \partial K_\delta$

$$\int_K G_t(x,y)dx < \int_{H_y} G_t(x,y)dx = e(t,\delta)$$

and proceeding as before, we have the upper bound

$$F_t < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \mathrm{vol}(\partial K_\delta) e(t,\delta) d\delta.$$

□

The next step is to upper bound $\mathrm{vol}(K_\delta)$ and lower bound $\mathrm{vol}(\partial K_\delta)$, which is done in Lemmas 2 and 3 respectively.

Lemma 2

$$\mathrm{vol}(\partial K_{-\delta}) \geq \left(1 - n\frac{\delta}{r_{in}}\right) \mathrm{vol}(\partial K)$$

Proof:
Let O be the centre of the sphere of radius r_{in} contained inside K. We shall first prove that $K_{-\delta}$ contains $(1 - \frac{\delta}{r_{in}})K$ where this scaling is done from the origin O. Let A be a point on ∂K and let F be the image of A under this scaling. It suffices to prove that $F \in K_{-\delta}$.

Consider Figure 5.2. We construct the smallest cone from A containing the sphere. Let B be a point where the cone touches the sphere. We have $OB = r_{in}$. Now consider the inscribed sphere centered at F. By similarity of triangles, we have

$$\frac{CF}{OB} = \frac{AF}{AO}$$

Noticing that $AF = \frac{\delta}{r_{in}} OA$, we obtain

$$CF = OB\frac{AF}{AO} = \delta$$

We thus see that the radius of the inscribed ball is δ and hence the δ-ball centered in F is contained in K. The fact that $F \in K_{-\delta}$ follows from the definition.

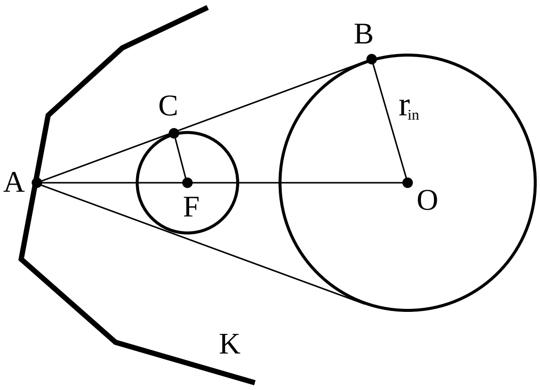

Figure 2. $K_{-\delta}$ **contains** $\left(1 - \frac{\delta}{r_{in}}\right) K$

It is known that the surface area of a convex body is less or equal than the surface area of any convex body that contains it (page 284, [13]). Therefore

$$\mathrm{vol}(\partial K_{-\delta}) \geq \mathrm{vol}\left((1 - \frac{\delta}{r_{in}})\, \partial K\right)$$

Since the volumes of $n-1$-dimensional objects scale as $n - 1^{th}$ powers and observing that for $x < 1$, $\max\{0, (1-x)^{n-1}\} > 1 - nx$, we obtain the final conclusion: $\mathrm{vol}\left((1 - \frac{\delta}{r_{in}})\, \partial K\right) = (1 - \frac{\delta}{r_{in}})^{n-1}\, \mathrm{vol}(\partial K)$ $\geq (1 - \frac{n\delta}{r_{in}})\mathrm{vol}(\partial K)$ □

□

Lemma 3

$$\mathrm{vol}(K_\delta) \leq V\, \exp\left(\delta \frac{S}{V}\right).$$

Proof: The volume of K_δ is a polynomial in δ, given by the *Steiner formula* (see page 197, [13].)

$$\mathrm{vol}(K_\delta) = a_0 + \cdots + \binom{n}{i} a_i \delta^i + \cdots + a_n \delta^n.$$

From the Alexandrov-Fenchel inequalities (see page 334, [13].) the coefficients a_i are log-concave; i. e.

$$a_i^2 \geq a_{i-1} a_{i+1}.$$

As a result

$$\frac{a_i}{a_0} \leq \left(\frac{a_1}{a_0}\right)^i.$$

a_0 is V, the volume of K while na_1 is the surface area S of K. Putting these inequalities together with the fact that $\binom{n}{i} \leq \frac{n^i}{i!}$, the lemma follows.

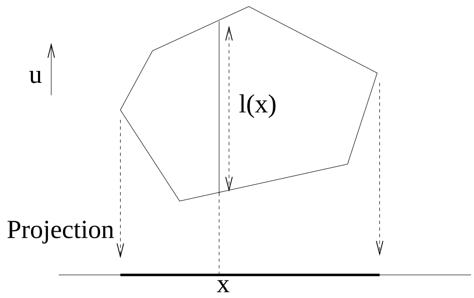

Figure 3. Projecting along a unit vector u **minimising** $\|T^{-1}u\|$

□

Although Lemma 3 is an upper bound on $\mathrm{vol}(K_\delta)$ rather than $\mathrm{vol}(\partial K_\delta)$, it can be applied after transforming the upper bound in Lemma 1 by integrating by parts. Lemmas 2, 3 and 1 together result in the following lemma, which we prove in the Appendix.

Lemma 4 *Let* $\alpha = \left(\frac{S}{V}\right)^2 t$. *Then,*

$$S\left(1 - \frac{n\sqrt{\pi t}}{2r_{in}}\right) < F_t < S\left(\sqrt{\frac{\pi}{\alpha}}\, \frac{\exp(\alpha) - 1}{2} + \exp(\alpha)\right).$$

Finally, we prove the following bounds for the isoperimetric constant $\frac{V}{S}$ of K in terms of r_{in}.

Lemma 5

$$\frac{r_{in}}{n} \leq \frac{V}{S} < 4\, r_{in}$$

Proof: It follows from Lemma 3.4 in [11] that a ball of radius $\frac{1}{\sqrt{2}}$ around the centroid of TK is entirely contained in TK. Therefore

Observation 3 *K contains a ball of radius r_{in}.*

Observation 4 *For any unit vector* u *that minimises* $\frac{\|T^{-1}u\|}{\sqrt{2}}$, *if* x *is chosen uniformly at random from* K, *var*$(\mathrm{u} \cdot \mathrm{x}) \leq 5\mathrm{r}_{\mathrm{in}}^2$.

We are now in a position to present the proof of Lemma 5. The first inequality $\frac{r_{in}}{n} \leq \frac{V}{S}$ is a consequence of Lemma 8 (proved in the Appendix). The only condition on r_{in} there, is that $r_{in}B \subseteq K$, a property that is satisfied by r_{in} by Observation 3.

Fix a unit vector u such that for x chosen uniformly at random from K, $\mathrm{var}(\mathrm{u} \cdot \mathrm{x}) \leq 5\mathrm{r}_{\mathrm{in}}^2$. Observation 4 states that such a vector exists.

Definition 4 *Let π be an orthogonal projection of K onto a hyperplane perpendicular to* u*. Further, for a point $y \in \pi(K)$, let ℓ_y be the length of the preimage $\pi^{-1}(y)$.*

$$\mathrm{var}(\mathrm{u}^{\mathrm{T}}\mathrm{x}) \le 5\mathrm{r}_{\mathrm{in}}^2.$$

The variance of $\mathrm{u} \cdot \mathrm{x}$ under the condition $\pi(x) = y$, is given by $\ell_y^2/12$, since this is the variance of a random variable that takes a value from an interval of length ℓ_y uniformly at random.

$$\mathrm{var}(\mathrm{u} \cdot \mathrm{x}) \ge \frac{\int_{\pi(K)} \mathrm{var}(\mathrm{u} \cdot \mathrm{x} | \pi(\mathrm{x}) = \mathrm{y}) \ell_{\mathrm{y}} \mathrm{dy}}{V} \quad (1)$$

$$= \frac{\int_{\pi(K)} \ell_y^3 dy}{12V}. \quad (2)$$

$$(3)$$

$$\frac{\int_{\pi(K)} \ell_y^3 dy}{\mathrm{vol}(\pi(K))} \ge \left(\frac{\int_{\pi(K)} \ell_y dy}{\mathrm{vol}(\pi(K))}\right)^3 = \left(\frac{V}{\mathrm{vol}(\pi(K))}\right)^3.$$

since for any non-negative random variable X, $E[X^3] \ge E[X]^3$. Therefore,

$$\frac{\int_{\pi(K)} \ell_y^3 dy}{12V} \ge \left(\frac{V^2}{12\mathrm{vol}(\pi(K))^2}\right).$$

Further, $\mathrm{vol}(\pi(K)) \le S/2$. Putting these facts together,

$$5r_{in}^2 \ge \left(\frac{V^2}{12\mathrm{vol}(\pi(K))^2}\right) \ge \left(\frac{V^2}{3S^2}\right),$$

and so $\frac{V}{S} < \sqrt{15}\, r_{in} < 4\, r_{in}$

□

Lemmas 4 and 5 together give the result of Proposition 2, as we show below. The lower bound on F_t is immediate for $\sqrt{t} = \frac{\epsilon r_{in}}{4n}$ using the lower bound in Lemma 4. To prove the upper bound, we observe that $\alpha = \left(\frac{S}{V}\right)^2 t \le \left(\frac{n}{r_{in}}\right)^2 t$ from Lemma 1, which equals $\frac{\epsilon'^2}{16}$. Since $\epsilon < 0.5, \alpha < 1$. Therefore $\mathrm{e}^\alpha < 1 + 2\alpha$. It follows that

$$\begin{aligned} S\left(\sqrt{\frac{\pi}{\alpha}}\frac{\exp(\alpha)-1}{2} + \exp(\alpha)\right) &< S\left(\sqrt{\pi\alpha} + 1 + 2\alpha\right) \\ &< S(1 + 4\sqrt{\alpha}) \\ &< (1+\epsilon)S. \end{aligned}$$

□

6 Proof of Proposition 3

The proof of Proposition 3 is complicated by the large number of parameters involved. Therefore we shall only mention the important steps, and leave the details to the appendix. The result will follow from two lemmas that are proved in the appendix.

Lemma 6 *With probability greater than* $7/8$*,*

$$(1-\epsilon')F_t < E\left[\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V\right] < (1+\epsilon')F_t,$$

and $(1-\epsilon')V < \hat{V} < (1+\epsilon')V$.

Lemma 7 *With probability greater than* $15/16$*,*

$$var\left(\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V\right) < \frac{\epsilon'^2 F_t^2}{16}.$$

Using Chebycheff's inequality and Lemma 7,

$$P\left[\left|\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V - E\left[\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V\right]\right| > \epsilon' F_t\right] < \frac{1}{16}.$$

Putting this together with Lemma 6, we arrive at the desired result. □

Acknowledgment

The authors would like to thank László Babai for helpful discussions and encouragement. We are grateful to Eric Vigoda for pointing out the Dyer-Gritzmann-Hufnagel paper [4] to us. We would also like to thank Ravi Kannan and Santosh Vempala for relevant literature.

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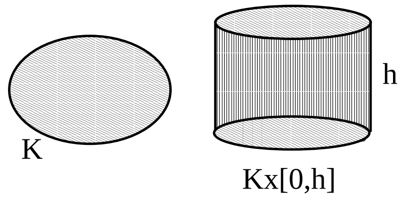

Figure 4. volume of the base and surface area of the cylinder.

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A Relative hardness of approximating surface area and volume

We shall prove Proposition 1 here.

Proof: Given a convex body K, $rB \subseteq K \subseteq RB$ let $C(K,h)$ denote the $n+1$-dimensional cylinder having base K and height h.

We shall show that for $h = \frac{\epsilon r}{n}$,

$$\text{vol } K < \frac{\text{vol } \partial C(K,h)}{2} < (1+\epsilon)\text{vol } K.$$

It is true for any $h > 0$ that

$$\text{vol } K = \frac{\text{vol } \partial C(K,h) - h\text{vol } \partial K}{2}.$$

Together with Lemma 8, this completes the proof. □

B Proofs of lemmas and propositions

Lemma 8 *Let K contain a ball of radius r_{in}. Then,*

$$\frac{r_{in}}{n} \leq \frac{V}{S}.$$

Proof:

Observe that without a loss of generality we can assume that the boundary of K is smooth. Indeed notice that the boundary of K_δ is smooth and from the *Steiner's formula* we have

$$\lim_{\delta \to 0} \text{vol}(K_\delta) = V \qquad \lim_{\delta \to 0} \text{vol}(\partial K_\delta) = S$$

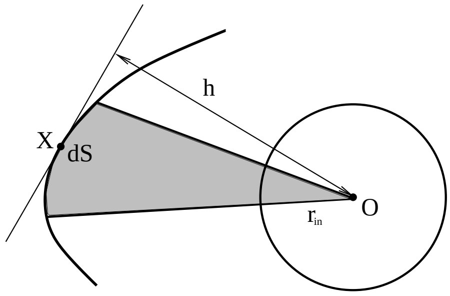

Figure 5. Using infinitesimal cones to show $\frac{r_{in}}{n} \leq \frac{V}{S}$.

By passing to the limit we see that it suffices to prove that for all $\delta > 0$,

$$\frac{r_{in}}{n} \leq \frac{\text{vol}(K_\delta)}{\text{vol}(\partial K_\delta)}$$

and hence K can be replaced by K_δ for some small δ.

We will now show that the inequality holds for infinitesimal cones (see Figure 3). Let $X \in \partial K$ and consider an infinitesimal cone with vertex O and with the base area dS The volume dV of such cone is equal to $\frac{hdS}{n}$, where h is the distance from O to the tangent line to ∂K at X. By observing that $h \geq r_{in}$ we obtain

$$dV = \frac{h}{n}\,dS \geq \frac{r_{in}}{n}\,dS$$

Since K is convex, integrating over dS yields

$$V = \int \frac{h}{n}\,dS \geq \int \frac{r_{in}}{n}\,dS = \frac{r_{in}}{n}\,S$$

□

Proof of Lemma 4
Lemma 2 can be restated in a slightly different form using integration by parts. Let $-\frac{d}{dt}e(t,\delta) = N(t,\delta)$. Then $N(t,\delta)$ is the density of the normal distribution with variance $2t$, evaluated at δ. Let f be any continuous function of δ whose magnitude is bounded above by a polynomial. Then,
$\int_0^\infty f(\delta)e(t,\delta)d\delta = \left[\left(\int_0^\delta f(\delta')d\delta'\right)e(t,\delta)\right]|_0^\infty - \int_0^\infty \left(\int_0^\delta f(\delta')d\delta'\right)N(t,\delta)d\delta =$
$\int_0^\infty \left(\int_0^\delta f(\delta')d\delta'\right)N(t,\delta)d\delta$.
The first term disappears, since $e(t,\delta)$ decays exponentially. Applying this formula, $\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0}\text{vol}(\partial K_\delta)e(t,\delta)d\delta = \sqrt{\frac{\pi}{t}}\int_{\delta\geq 0}(\text{vol}(K_\delta) - \text{vol}(K))N(t,\delta)d\delta$. An application of Lemma 3 now yields that

$$F_t < \sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} V\left(\exp\left(\delta\frac{S}{V}\right) - 1\right)N(t,\delta)d\delta.$$

Following the notation of Lemma 4, we let $\alpha = (S/V)^2 t$.

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} N(t,\delta)d\delta = \sqrt{\frac{\pi}{4t}}.$$

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} V\exp\left(\delta\frac{S}{V}\right)N(t,\delta)d\delta =$$

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} V\,\exp\left(\delta\sqrt{\frac{\alpha}{t}}\right)N(t,\delta)d\delta,$$

which is

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} V\left(\frac{\exp\left(\delta\sqrt{\frac{\alpha}{t}} - \frac{\delta^2}{4t}\right)}{\sqrt{4\pi t}}\right)N(t,\delta)d\delta.$$

Completing the square, this may be rewritten as

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} V\left(\frac{\exp\left(\alpha - \left(\frac{\delta}{2\sqrt{t}} - \sqrt{\alpha}\right)^2\right)}{\sqrt{4\pi t}}\right)N(t,\delta)d\delta.$$

The above expression is

$$\sqrt{\frac{\pi}{t}}V\,e^\alpha\int_{\delta\geq -2\sqrt{t\alpha}} V\,N(t,\delta)d\delta < Se^\alpha + \sqrt{\frac{\pi}{4t}}V\,e^\alpha, \quad (4)$$

since we can bound the integral of $N(t,\delta)$ over $[-2\sqrt{t\alpha}, 0]$ by $\sqrt{\frac{\alpha}{\pi}}$. The upper bound on F_t now follows.

Lemmas 2 and 1 imply that

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} S\left(1 - n\frac{\delta}{r_{in}}\right)e(t,\delta)d\delta < F_t.$$

We transform this integral by parts and obtain

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0} S\left(\delta - n\frac{\delta^2}{2r_{in}}\right)N(t,\delta)d\delta.$$

Applying a change of variables $u := \delta^2$, it is easy to verify that

$$\int_{\delta\geq 0}\delta N(t,\delta)d\delta = \sqrt{\frac{t}{\pi}}.$$

$$\int_{\delta\geq 0} \delta^2 N(t,\delta) d\delta = t,$$

since this is half the variance of the Normal disribution $N(t,\cdot)$. The lower bound on F_t now follows. □

The following lemma is a consequence of (Lemma 7.1, [15]) and summarizes some properties of μ-independence that we shall need.

Lemma 9 *1. Let X and Y be μ-independent, and let f, g be two measurable functions. Then $f(X)$ and $g(Y)$ are also μ-independent.*

2. Let X, Y be μ-independent random variables such that $0 \leq X \leq a$ and $0 \leq Y \leq b$. Then,

$$|E(XY) - E(X)E(Y)| \leq \mu ab.$$

3. Let $x_1, \ldots, x_N$ be a Markov chain and assume that $\forall i > 0$, x_{i+1} is μ-independent from x_i. Then, $\forall i \neq j$, x_i and x_j are μ-independent.

Proof of Lemma 6: Let X_i denote the indicator random variable for the event $x_i + v_i \notin K$. Suppose that x_i is sampled from the probability distribution p. x_i is $\frac{\epsilon'^2}{64n}$-uniform, therefore it has a distribution p for which $\int_K |p(x) - 1/V| dx < \frac{\epsilon'^2}{64n}$. Therefore

$$\begin{aligned}
|\frac{\sqrt{t}F_t}{\sqrt{\pi}V} - E[X_i]| &= |\int_{\mathbb{R}^n - K} \int_K G_t(x,y)(p(x) - \frac{1}{V}) dx dy| \\
&= |\int_K (p(x) - \frac{1}{V}) \int_{\mathbb{R}^n - K} G_t(x,y) dy dx| \\
&< \int_K |(p(x) - \frac{1}{V})| dx \\
&= \frac{\epsilon'^2}{64n}.
\end{aligned}$$

In the above calculations, the inequality is a consequence of the fact that

$$\int_{\mathbb{R}^n - K} G_t(x,y) dy < 1.$$

The calculations above hold for each i, and the Lemma now follows from the linearity of expectation. □

Proof of Lemma 7 Let X_i denote the indicator random variable for the event $x_i + v_i \notin K$ as in the proof of Lemma 6. Then, Let

$$q = \frac{F_t}{V}\sqrt{\frac{t}{\pi}}$$

and $\epsilon'' = \epsilon' q$.

$$\mathrm{var}(\frac{\sum_1^N X_i}{N}) = \frac{\sum_1^N \mathrm{var} X_i + \sum_{i\neq j} \mathrm{cov}(X_i, X_j)}{N^2}.$$

Since we are dealing with $0, 1$ variables,

$$\mathrm{var}(X_i) < E[X_i] < q + \frac{\epsilon'^2}{64n} < q(1+\epsilon').$$

The last statement follows from Lemma 6, and the fact that $\frac{\epsilon'}{64n} < q$, which we show below.

$$\frac{F_t}{V}\sqrt{\frac{t}{\pi}} > \frac{S\sqrt{t}}{2\sqrt{\pi}V} > \frac{S\sqrt{t}}{4V},$$

as a consequence of Proposition 2. Using the values of $\sqrt{t}$ and the inequality $\frac{V}{S} < 4r_{in}$ from Lemma 5, this is $> \frac{\epsilon'}{64n}$.

Continuing the proof, from Lemma 9 we have that,

$$\frac{\sum_{i\neq j} \mathrm{cov}(X_i, X_j)}{N^2} < \mu.$$

$$\frac{\sum_1^N (\frac{F_t}{V}\sqrt{\frac{t}{\pi}} + \epsilon'') + N(N-1)\mu}{N^2} <$$

$$\frac{\frac{F_t}{V}\sqrt{\frac{t}{\pi}}}{N} + \frac{\epsilon''}{N} + \mu < \frac{2q}{N} + \mu.$$

We just showed that $q = \frac{\epsilon''}{\epsilon'} > \frac{\epsilon'}{64n}$. We are required to show that

$$\frac{2q}{N} + \mu < \frac{\epsilon'^2 q^2}{16},$$

The ratio of the left hand side to the right,

$$\frac{32}{N\epsilon'^2 q} + \frac{16\mu}{\epsilon'^2 q^2} < \frac{2^{11} n}{N\epsilon'^3} + \frac{2^{16}\mu n^2}{\epsilon'^4} = 1/2 < 1.$$

This completes the proof. □