Damped random walks and the characteristic polynomial of the weighted Laplacian on a graph

MADHAV P. DESAI* and HARIHARAN NARAYANAN†

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Abstract

For $\lambda > 0$, we define a λ-damped random walk to be a random walk that is started from a random vertex of a graph and stopped at each step with probability $\frac{\lambda}{1+\lambda}$, otherwise continued with probability $\frac{1}{1+\lambda}$. We use the Aldous-Broder algorithm ([1, 2]) of generating a random spanning tree and the Matrix-tree theorem to relate the values of the characteristic polynomial of the Laplacian at $\pm\lambda$ and the stationary measures of the sets of nodes visited by i independent λ-damped random walks for $i \in \mathbb{N}$. As a corollary, we obtain a new characterization of the non-zero eigenvalues of the Weighted Graph Laplacian.

1 Introduction

Consider a weighted graph G in which vertices are labelled $1, \dots, n$. Let the edge between i and j have weight d_{ij} and $\sum_j d_{ij} := d_i$. The laplacian $\mathcal{L}$ is an operator from the space of complex valued functions on the nodes of a graph to itself. For

$$f : [n] \to \mathbb{C},$$

$\mathcal{L}f$ is defined by

$$\forall i \in [n], \mathcal{L}f(i) = \frac{\sum_{j\in[n]} d_{ij}(f(j) - f(i))}{d_i}.$$

Thus, the following matrix equation holds:

$$Df = D'\mathcal{L}f,$$

*Department of Electrical Engineering, Indian Institute of Technology Bombay, 400076, Mumbai, India, madhav@ee.iitb.ac.in

†Department of Computer Science, University of Chicago, 60637, chicago, USA, hari@cs.uchicago.edu

where D is

$$\begin{bmatrix} -d_1 & d_{12} & \cdots & d_{1n} \\ d_{21} & -d_2 & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & -d_n \end{bmatrix},$$

$f = [f(1), \ldots, f(n)]^T$, $\mathcal{L}f = [\mathcal{L}f(1), \ldots, \mathcal{L}f(n)]^T$ and D' is the diagonal matrix whose entry in the i^{th} row and i^{th} column is d_i.

We consider the transition matrix $D'^{-1}\mathcal{L} + I$, which is

$$\begin{bmatrix} 0 & \frac{d_{12}}{d_1} & \cdots & \frac{d_{1n}}{d_1} \\ \frac{d_{21}}{d_2} & 0 & \cdots & \frac{d_{2n}}{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_{n1}}{d_n} & \frac{d_{n2}}{d_n} & \cdots & 0 \end{bmatrix},$$

Whenever we consider random walks on G, the transition from $i \rightarrow j$ will have probability equal to the ij^{th} entry of the above matrix.

The eigenvalues of the Laplacian are the roots of the polynomial

$$C(\lambda) = \det \begin{vmatrix} -d_1(1+\lambda) & d_{12} & \cdots & d_{1n} \\ d_{21} & -d_2(1+\lambda) & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & -d_n(1+\lambda) \end{vmatrix}.$$

Note that λ is an eigenvalue of the laplacian if and only if $1 + \lambda$ is an eigenvalue of the transition matrix $D'^{-1}\mathcal{L} + I$.

Now, let λ take a fixed positive real value. Consider the following procedure of gernerating a random walk:

1. Choose a vertex i with probability

$$\frac{d_i}{\sum_{j \in [n]} d_j}.$$

2. Toss a coin that comes up Heads with probability $\frac{\lambda}{1+\lambda}$.
   If the outcome is Heads **STOP**.
   Else make a random transition to a neighbouring vertex with probability proportional to the weight of the corresponding edge.

3. Goto 2.

Definition 1 *A random walk generated by the above procedure shall be called a "λ-damped random walk".*

Definition 2 *Let $w_1, \ldots, w_i, \ldots$ be a sequence of independent λ-damped walks. We define the weighted fraction of the nodes covered by the first i walks,*

$$f_i := \frac{\sum_{j \in \bigcup_{k=1}^{i} w_k} d_j}{\sum_{j \in [n]} d_j}.$$

Note that this is the measure of $\bigcup_i w_i$ under the stationary distribution of the natural random walk on the graph.

Theorem 1 *For $c > 0$, let $k \geq (c+1) d_{avg} d_{min}^{-1} n \ln n$, where d_{avg} is the arithmetic mean and d_{min} is the minimum of the d_i for $i \in [n]$. Then,*

$$|E[\prod_{j=1}^{k-1}(2f_j - 1)] + \frac{C(-\lambda)}{C(\lambda)}| < \frac{2}{n^c}.$$

Corollary 2 *The non-zero eigenvalues $-\lambda_i$ of the laplacian (which are all negative), are characterized by $E[\prod_{j=1}^{\infty}(2f_j - 1)] = 0$, where f_j are as in definition 2 with λ set to λ_i.*

2 Kirchhoff's Matrix-tree theorem

Definition 3 *Let $\mathbb{T}$ be the set of trees of G. For a tree t of G, let its weight $|t|$ be the product of the weights of its edges.*

Theorem 3 *The sum of the weights of spanning trees of a graph G is equal to any cofactor of the degree matrix of G minus the adjacency matrix of G.*

In particular, Kirchhoff's matrix tree theorem implies that

$$\sum_{t \in \mathbb{T}} w(t) = (-1)^{n-1} \det \begin{vmatrix} -d_2 & d_{23} & \cdots & d_{2n} \\ d_{31} & -d_3 & \cdots & d_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n2} & d_{n3} & \cdots & -d_n \end{vmatrix}.$$

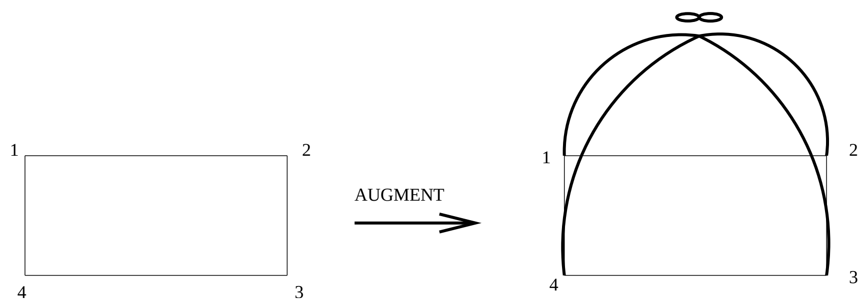

Figure 1: Augmenting G

3 The Aldous-Broder algorithm

Definition 4 *Given a weighted graph G, let $P_{\mathbb{T}}$ be the probability distribution on the trees of G, where the probability assigned to a tree t is $\frac{|t|}{\sum_{t\in\mathbb{T}}|t|}$.*

The Aldous-Broder algorithm [1, 2] described below generates a random spanning tree, from the distribution $P_{\mathbb{T}}$.

1. Simulate the random walk on the graph G starting at an arbitrary vertex s, until every vertex is visited. For each $i \in [n], i \neq s$, collect the edge (j, i) corresponding to the first transition to vertex i.

2. Output t.

Originally this algorithm was proved to give a random spanning tree of a simple graph. However, it has been shown by Mosbah and Saheb [4], that the algorithm works even for a weighted graph provided that the walk makes transitions according to the transition matrix $D'^{-1}\mathcal{L} + I$.

4 Augmenting the graph

We now prove theorem 1.

Definition 5 *Given a graph G as in the Introduction, and $\lambda > 0$, we manufacture an augmented graph G_λ by adding an additional node ∞ and for each $i \in [n]$, a branch joining ∞ and i that has weight $d_i\lambda$.*

Figure 1 illustrates this.

Definition 6 *Let $\mathbb{T}_\lambda$ be the set of trees of G_λ, and $P_{\mathbb{T}_\lambda}$ be the probability distribution that assigns to a tree $t \in \mathbb{T}_\lambda$, a probability proportional to the product of its edge weights.*

The determinant of the laplacian of G is a determinant of a maximal cofacter of G_λ. Thus by the Matrix-Tree theorem, we have

$$C(\lambda) \quad = \quad (-1)^n \sum_{t \in \mathbb{T}_\lambda} \prod_{e \in t} |e|. \tag{1}$$

In the above expression, by $|e|$, we mean the weight of e. For a tree t of G_λ, let us denote by $deg_\infty(t)$ the *unweighted* degree of ∞ in t. This is the number of branches of t that are incident upon ∞. Then,

$$\frac{C(-\lambda)}{C(\lambda)} \quad = \quad \frac{\sum_{t \in \mathbb{T}_\lambda} \prod_{e \in t} (-1)^{deg_\infty(t)} w(e)}{\sum_{t \in \mathbb{T}_\lambda} \prod_{e \in t} w(e)} = E[(-1)^{\deg_\infty(t)}], \tag{2}$$

where the expectation is taken over the distribution $P_{\mathbb{T}_\lambda}$ on $\mathbb{T}_\lambda$.

5 Proof of Theorem 1

We do an infinite random walk w on the graph G_λ, $w := \infty \to i_1 \to i_2 \to \dots$ on G_λ starting at vertex ∞, and generate a random tree t from w using the Aldous-Broder algorithm. Note that no transition that occurs after every vertex has been visited will figure in the tree. We let the walk have infinite length just for notational benefits. Let us partition w into visits to ∞ and the walks on intervals between these visits, which we label $w_1, w_2, \dots$. Thus $\infty \to 3 \to 2 \to 5 \to \infty \to 4 \to 1 \to \infty \to 1 \to 3 \to \dots$ would be partitioned as $\infty, 3 \to 2 \to 5, \infty, 4 \to 1, \infty, 1 \to 3 \dots$, and $w_1 := 3 \to 2 \to 5$, $w_2 := 4 \to 1$, $w_3 := 1 \to 3 \dots$.

Definition 7 *Let x_i be the random variable defined to be 0 if the first vertex of w_i was covered by one of the walks $w_1, \dots, w_{i-1}$, and 1 otherwise.*

Note that if t is the tree obtained by applying the Aldous-Broder algorithm to walk w,

$$\deg_\infty(t) \quad = \quad \sum_{i=1}^{\infty} x_i. \tag{3}$$

Also note that the w_i are independent λ-damped walks on G. Let d_{avg} and d_{min} be the arithmetic mean and the minimum of the weighted degrees d_i. Then, the probability that the first $k := (c+1)\frac{d_{avg}}{d_{min}} n \ln n$ walks do not cover a given node j is less than the probability that j is not the first vertex of one of the walks $\{w_i | i \leq k\}$, which is $(1 - \frac{d_j}{n d_{avg}})^k < \frac{1}{n^{c+1}}$.

Therefore by the union bound, the probability that every node is covered by some w_i, $i \leq k$ is $> 1 - \frac{1}{n^c}$. Thus

$$P[\deg_\infty(t) \;=\; \sum_{i=1}^{k} x_i] > 1 - \frac{1}{n^c}, \tag{4}$$

where $k := (c+1)\frac{d_{avg}}{d_{min}} n \ln n$.

$$\begin{aligned} E[(-1)^{\deg_\infty(t)}] &= E[(-1)^{\sum_{i\leq k} x_i}(-1)^{\deg_\infty(t)-\sum_{i\leq k} x_i}] \\ &= E[(-1)^{\sum_{i\leq k} x_i}] + E[(-1)^{\sum_{i\leq k} x_i}(-1+(-1)^{\deg_\infty(t)-\sum_{i\leq k} n_i})]. \end{aligned} \tag{5}$$

However, from 5,

$$\begin{aligned} |E[(-1)^{\sum_{i\leq k} x_i}(-1+(-1)^{\deg_\infty(t)-\sum_{i\leq k} n_i})]| &= \\ P[\deg_\infty(t) > \sum_{i\leq k} n_i)]|E[(-1)^{\sum_{i\leq k} x_i}(-1+(-1)^{\deg_\infty(t)-\sum_{i\leq k} n_i})|\deg_\infty(t) > \sum_{i\leq k} n_i]| &< \\ \frac{2}{n^c}. \end{aligned} \tag{6}$$

It follows from 2, 4 and 6 that

$$|E[(-1)^{\sum_{i\leq k} x_i}] + \frac{C(-\lambda)}{C(\lambda)}| \;<\; \frac{2}{n^c}. \tag{7}$$

x_i is a function of $w_1, \ldots, w_i$, and is independent of $w_{i+1}, w_{i+2}, \ldots$. So,

$$\begin{aligned} E_{w_1,\ldots,w_i}[(-1)^{\sum_{i\leq k} x_i}] &= E_{w_1}[(-1)^{x_1} E_{w_2,\ldots,w_i}[(-1)^{\sum_{i=2}^k x_i}|w_1] \\ &= (-1)E_{w_1}[E_{w_2,\ldots,w_k}[(-1)^{\sum_{i=2}^k x_i}|w_1] \\ &= (-1)E_{w_1}[(2f_1-1)E_{w_2}[(-1)^{x_2}E_{w_3,\ldots,w_k}[(-1)^{\sum_{i=3}^k x_i}|w_1,w_2]|w_1] \\ &= \ldots \\ &= (-1)E_{w_1,\ldots,w_{k-1}}[\prod_{i=1}^{k-1}(2f_i-1)], \end{aligned} \tag{8}$$

where f_i are as defined in Definition 2. From 2, 7 and 8 Theorem 1 now follows.

6 Conclusion

We used the Aldous-Broder algorithm ([1, 2]) of generating a random spanning tree and the Matrix-tree theorem to relate the values of the characteristic polynomial of the Laplacian at $\pm\lambda$ and the stationary measures of the sets of nodes visited by i independent λ-damped random walks for $i \in \mathbb{N}$. As a corollary, we obtained a new characterization of the non-zero eigenvalues of the Weighted Graph Laplacian. The authors are not aware of any analogue of this result for compact Riemannian manifolds. To find such analogues, appears to be an interesting line of research.

References

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