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Green's Functions and the Spectrum of the Laplacian on Graphs and Planar Domains

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by

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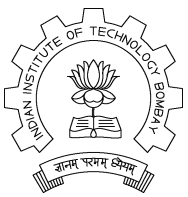

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Abstract

This report explores some relationships between Green's functions and the spectrum of the Laplacian. In Theorem 2.3 we present a characterization of the eigenvalues of a weighted graph using random walks. In Chapter 2, we introduce a "box product", $[a_1, \cdots, a_n]$, for a region R if the boundary ∂R is the clockwise union of arcs $a_1, b_1, \cdots, a_n, b_n$. We show in Theorem 4.2 that $[a_1, \cdots, a_n][b_1, \cdots, b_n] = 1$. Under a certain reasonable hypothesis, we show that scaled eigenvalues of the discrete Laplacian on a triangular grid approximation to R converge to the eigenvalues of the Laplacian on R under Neumann boundary conditions.

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Chapter 1

Introduction

1 Operators and Spectrum

A considerable amount of interest has been shown in the relations between the spectrum of a physical object and its structure. The spectrum is always something that is observable, while intrinsic structure is very often not. Moreover in certain cases, some version of the spectrum is *all* that can be observed and inverse spectral results then become essential to gain any kind of knowledge about the actual object. One is tempted to cite the example of quantum mechanics, where the observables are eigenvalues of certain operators acting upon a system of interest. Any information about the system must necessarily be derived through the spectrum of operators acting upon it. As a concrete example, cracks in mechanical objects are often detected by sending pulses into them and observing their responses. Scans of any kind for humans involve inferences made from a measurement of the body's response to waves of some kind. Many spaces can be well approximated using weighted graphs. Then linear operators on the original spaces get converted to matrices, and the (sufficiently) small eigenvalues of the original operators

are approximated by the spectrum of the corresponding matrix. We survey some existing inverse spectral results on continuous spaces in the following section, and give a brief historical overview.

2 Inverse Spectral Results for manifolds

An isometry of two manifolds M and N is a one-to-one onto map $f : M \to N$, such that the length of a smooth curve in M is the same as the length of its image under f. If such an f exists, M and N are said to be isometric. The Laplacian

$$\triangle = \frac{1}{2} \times \sum_{i=1}^{d} \nabla_i \nabla_i$$

acts upon real valued functions on a compact d-dimensional manifold M, where at the point of consideration, the x_j form an orthonormal system of coordinates. Its eigenvalues form the *spectrum* of M. M and N are said to be *isospectral* if their eigenvalues coincide, counting multiplicity. Isospectral manifolds are not necessarily isometric. In 1964, John Milnor constructed two 16 dimensional flat tori that were isospectral but not isometric . In 1966 Mark Kac posed the question "Can one hear the shape of a drum" (see [1]). This was the question of whether the spectrum of the Laplacian acting on a compact planar domain with a boundary, acting on functions vanishing on the boundary, determined its shape. In 1992, Carolyn Gordon, David Webb and Scott Wolpert answered Kac's question in the negative (see [2]). In a recent paper Steven Zelditch, (see [7]) has proved however that in the special case of analytic surfaces of revolution, the spectrum determines the manifold upto isometry. A result due to Minakshisundaram and Pleijel (see [9]) from 1949 says that the number of eigenvalues of a manifold M less than λ, $N(\lambda)$

has the asymptotic expansion

$$\frac{\beta(d)}{(2\pi)^d} Vol(M)\lambda^{d/2} + o(\lambda^{d/2}).$$

Here $\beta(d)$ and $Vol(M)$ denote the volumes of the d dimensional unit ball and manifold M respectively. The above formula shows that the dimension and volume of a manifold are spectral invariants, i.e are fully determined by the spectrum.

3 An overview

In Chapter 2, three results on the eigenvalues of the discrete Laplacian on a weighted graph have been presented, namely Corollary 2.1, Theorem 2.2 and Theorem 2.3. The latter two are probabilistic in nature. Theorem 2.3 has been verified for a cyclic graph with 4 nodes. Three "dips" seen in Figure 2.5, correspond to the three non-zero eigenvalues of this graph counting multiplicity namely -1, -1 and -2. In Chapter 3 we state the theorems of Kelvin, which relate the minima in power to the solution of a resistive circuit. Raleigh's monotonicity principle states that on increasing the resistance of any branch of a circuit, the equivalent resistance between two points cannot decrease. Dirichlet's principle is a continuous version of one of Kelvin's theorems. As an application of some of these results, we find the rate at which the smallest non-zero eigenvalue of a graph with two components connected by a single branch approaches 0 as the weight of the connecting branch tends to 0. In Chapter 2 we define the "bracket" $[a_1, \cdots, a_n]$, which is shown to be independent of permutation. When $a_1, b_1, \cdots, a_n, b_n$ form the boundary of a simply connected region R in say the clockwise direction, we prove that $[a_1, \cdots, a_n][b_1, \cdots, b_n] = 1$. This result could have practical applications. for example in examining the uniformity of a plate. Suppose we were given a

plate that requires to have constant thickness and conductivity σ. We make contacts on the lateral surface, each along the boundary of the plate, and having straight ends perpendicular to its face. Let these be $\{a_i\}$, and the complementary set of such contacts be $\{b_i\}$. Then $[a_1, \cdots, a_n][b_1, \cdots, b_n]$ should not depend on the positions or sizes of a_i and b_i. Any variation can be used to evaluate the quality of the plate. In the Chapter 5 we consider the eigenvalues $\lambda_{i,\delta}$ of the discrete Laplacian on a graph obtained by triangulating R. We state minimax theorems which express the i^{th} eigenvalue as a minimum of the largest critical value of the Raleigh Quotient over all i-dimensional subspaces of smooth functions or node functions. These are used to prove that $\lim_{\delta \to 0} |\lambda_{i,\delta}|/\delta^2 \leq |\lambda_i/4|$. The reverse inequality has been proved making a reasonable hypothesis concerning the boundary of the approximating graphs. Finally, we have some numerical results on the convergence of the eigenvalues of graphs approximating a torus. Convergence is observed to be quadratic. This indicates that in order to get within ϵ of λ_i, the i^{th} eigenvalue of R, one must use grids having $O(1/\epsilon)$ vertices.

Chapter 2

The Spectrum of the discrete Laplacian on a weighted graph

In this section we present some results on graphs. Except where otherwise specified, the results are believed to be new. In whatever follows, G shall be a undirected weighted graph on nodes labelled $1, \ldots, n$. The edge between i and j shall have weight $g_{ij} = g_{ji}$, a non-negative real number. When there is no edge between i and j, we assume the presence of a branch with weight g_{ij} to be 0. We shall also refer to the branch between i and j as g_{ij}; it will always be clear from the context whether it is being referred to as a branch or branch weight. G shall be connected and will be assumed to contain an odd cycle. (The latter condition is needed for certain convergences we shall encounter.) We shall denote by d_j the weighted degree of node j i.e $\sum_i g_{ij}$. The sum of the degrees of a set of nodes S, will be called d_S, in particular $d_G = \sum_i \sum_j g_{ij}$. (We assume g_{ii} to be 0, for any node i.) When not explicitly mentioned, all summations shall be from 1 to n. For us, the spectrum of G shall signify the roots of the characteristic equation -

$$\det \begin{vmatrix} -\sum g_{1i}(1+\lambda) & g_{12} & \cdots & g_{1n} \\ g_{21} & -\sum g_{2i}(1+\lambda) & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & -\sum g_{ni}(1+\lambda) \end{vmatrix} = 0$$

The L.H.S above shall be called $G(\lambda)$.

$G(\lambda)$ can be opened out and be written as a polynomial in the g_{ij}.

Proposition 2.1 *(Cayley) Let T be the set of trees in G and $\mathbf{t}$, the product of branch weights of the tree $t \in T$. Then,*

$$\det \begin{vmatrix} -\sum g_{2i} & g_{23} & \cdots & g_{2n} \\ g_{32} & -\sum g_{3i} & \cdots & g_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n2} & g_{n3} & \cdots & -\sum g_{ni} \end{vmatrix}$$

is equal to

$$(-1)^{n-1} \sum_{t \in T} \mathbf{t}.$$

Let the matrix above be A, and the e_i for $i = 2, \cdots, n$ denote the column vector having 1 in the $i - 1^{th}$ position and 0 everywhere else. For a column vector v, let v_i denote its $i - 1^{th}$ entry. This shift of one everywhere is necessitated by our having constructed A by deleting the row and column corresponding to node 1. Let $Ae_k = u$. The entries of A^{-1} are rational functions in the g_{ij} given by Cramer's rule. That the determinant of A is a non-zero polynomial can be seen by setting to 1 the weights of branches adjacent to node 1 and setting to 0 all others. G is a connected graph and g_{ij} is taken to be the *conductance* of the branch between nodes i and j, u_k is

the current that would leave node k if a unit voltage were applied between nodes k and 1; in other words the equivalent conductance between 1 and k. If the conductances of branches in a cutset C separating k and 1 tend to 0, the equivalent conductance tends to 0. Therefore every monomial term in the numerator of u_k contains a branch in C. In other words each monomial term in the numerator corresponds to a connected spanning subgraph. Since the numerator of u_k is $|A|$ independent of k, each term in the numerator corresponds to a connected subgraph with $n-1$ branches. This means each monomial term corresponds to a tree. Let two trees be called adjacent if each contains only one branch not in the other. We prove that any two adjacent trees occur with the same coefficient. The union of two adjacent trees would be a connected subgraph r of G having n branches and exactly one cycle. By looking at the expression for the equivalent conductance between two nodes in the cycle after setting to 0 the conductances of branches outside r, one proves that any two trees that are subgraphs of r occur in $|A|$ with the same coefficient. Therefore adjacent trees are present with the same coefficient in $|A|$. Any tree of G can be reached from any other by the successive addition and deletion of branches, i.e through a series of trees, consecutive ones being adjacent. Therefore all trees occur with the same coefficient. The value of this coefficient is $(-1)^{n-1}$, as can be seen be calculating the coefficient of the term

$$\prod_{i=2}^{n} g_{1i},$$

setting the conductances of all other branches not figuring in this term 0. This completes our proof. As a corollary we have the following.

Corollary 2.1 *Let R be the set of all vertex spanning subgraphs of G having exactly n branches (allowing multiplicity) and* $\mathbf{r}$ *be the product of weights*

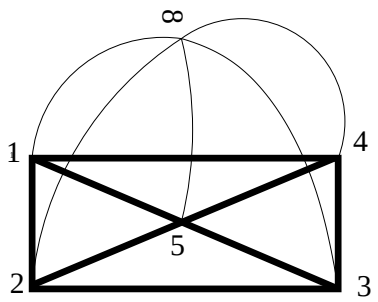

Figure 2.1: G_{aug}.

of the branches in the subgraph $r \in R$. For $i \in 1, \cdots, n$ let $m_i(r)$ be the number of functions whose domain is a set of i nodes in G and range is a set of branches in r, such that any node in the domain is taken to a branch adjacent to it and the range intersects every cycle in r. Then,

$$G(\lambda) = (-1)^n \sum_{i=1}^{n} \sum_{r \in R} \mathbf{r} m_i(r) \lambda^i$$

Definition 2.1 *G_{aug} is the graph obtained from G by adding an additional node ∞ and for each node $i \in G$, a branch between i and ∞ of weight $d_i\lambda$. (Recall that d_i is $\sum_j g_{ij}$.) T_{aug} is set of trees in G_{aug}, just as T is the set of trees in G.*

In the statement of Proposition 2.1, a row and column corresponding to node 1 in G were deleted. Now we consider a matrix in which the row and column corresponding to ∞ in G_{aug} are deleted. Its determinant is $G(\lambda)$, which using proposition 2.1 can be expressed in terms of trees in G_{aug}. The result follows after some collection and rearrangement of terms, writing $\mathbf{t}$ for a tree t_{aug} in G_{aug} as a sum of terms $r\lambda^i$ where i is the degree of ∞ in t_{aug}. Adding up $\mathbf{t_{aug}}$ over trees of T_{aug} *in which* node ∞ has degree i, one obtains

$$\sum_{r \in R} \mathbf{r} m_i(r) \lambda^i.$$

This is because each subgraph r occurs as many times multiplied by λ^i as there are trees t_{aug} in which the degree of node ∞ is equal to i and whose restriction to G is contained in r. This number is precisely $m_i(r)$, which completes our proof.

Theorem 2.1 *The matrix*

$$A = \begin{bmatrix} -\sum g_{1i} & g_{12} & \cdots & g_{1n} \\ g_{21} & -\sum g_{2i} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & -\sum g_{ni} \end{bmatrix}$$

is non - positive definite.

We must prove that for any vector v, $v^t A v \leq 0$. This is straightforward as

$$v^t A v = -\sum_{i \neq j} g_{ij}(v_i - v_j)^2.$$

Theorem 2.2 *Let p be a probability, $0 < p < 1$; $F = \{X_i\}_{i=1}^{i=n}$ be a collection of random variables such that X_j is either a branch of G adjacent to node j, or the null set $\emptyset$, with probabilities given below.*

$$P[X_i = g_{ij}] = \frac{g_{ij}}{d_i} \times p,$$

$$P[X_i = \emptyset] = 1 - p.$$

Let $n(F)$ be the random variable equal to the number of branches in F, and $l(F)$ be the random variable equal to the number of loops in F. Then,

$$E[(-1)^{n(F)} | l(F) = 0] = 0$$

if and only if $1 - 1/p$ is an eigenvalue of G.

As observed in the proof of corollary 2.1, for $\lambda > 0$, $G(\lambda) = (-1)^n \sum_{t \in T_{aug}} \mathbf{t}$. The eigenvalues of G are all real and non-positive, because the matrix involved is symmetric and negative definite (see Theorem 2.1). $-\lambda$ would be a non-zero eigenvalue of G if and only if the magnitudes of terms having an even exponent of λ in $G(\lambda)$ add up to the same as what the magnitudes of the other terms add up to (since all coefficients in $G(\lambda)$ have the same sign). This criterion can be realised by sampling. Indeed, suppose that every node of G_{aug} except ∞ chooses a random branch (of G_{aug}) adjacent to it with probability proportional to weight. If these branches contain no cycle, they form a tree of G_{aug} and for every tree of G_{aug} there is a unique such assignment. Thus it is clear that the probability of getting a particular tree in G_{aug} is proportional to its weight. What is more, the chance that a given node picks the branch joining it to ∞ is $1 - p = \lambda/(1 + \lambda)$. $n - n(F)$ is precisely the number of such branches. The theorem now follows.

1 Spectrum and Random Walks

For a fixed p, $0 < p < 1$, the probability space $(\Omega, \Im, P_p)$ is defined as follows. Ω is the set of all paths on G of finite length. $\Im$ is the power set of Ω. Let w be a walk of length $m - 1$, say $(w_1, \cdots, w_m)$, where the w_j are vertices of G. P_p assigns to w the probability

$$p^{m-1}(1-p) \times \frac{d_{w_1}}{d_G} \prod_{j=1}^{m-1} \frac{g_{w_j w_{j+1}}}{d_{w_j}}.$$

In other words, $m \in \mathbf{N}$ is chosen with a geometric distribution using p, and then a path of length $m - 1$ is chosen among all such, with probability proportional to the product of branch weights. If m is chosen to be 1, a single node comprises the path, and that node is chosen out of G with probability

proportional to the degree.

Theorem 2.3 *Let $\{w_j\}_1^\infty$ be a sequence of random walks chosen independently out of $(\Omega, \Im, P_p)$, S_j the set of nodes covered by $\cup_1^j w_i$, ($S_0 = \emptyset$), and S_j^c the complement of S_j. Then,*

$$E[\prod_{j=0}^{\infty} \frac{d_{S_j} - d_{S_j^c}}{d_G}] = 0$$

if and only if $1 - 1/p$ is a non-zero eigenvalue of G.

Proposition 2.1 applied to G_{aug} gives an expression for characteristic polynomial $G(\lambda)$ involving the trees of G_{aug}, as we have seen in the proof of corollary 2.1. Namely

$$G(\lambda) = (-1)^n \sum_{t \in G_{aug}} \mathbf{t}.$$

Our proof involves constructing random trees of T_{aug} using a standard method (probably introduced for the first time in [3]) and then expressing $G(\lambda)$ in the form of an expectation. Our needs regarding the construction of a random tree are fulfilled by the following lemma. Let $(\hat{\Omega}, \hat{\Im}, P)$ denote the probability space of all random walks w on G, $w : \mathbf{N} \to \{1, \cdots, n\}$ defined as follows. $\hat{\Omega}$ is the space of all walks defined on N, $\hat{\Im}$ is the σ-field generated by the collection of all subsets of $\hat{\Omega}$ that have a common trajectory from time 1 to some finite time $T \geq 1$. It is sufficient to define P on $\Im$. This is done by giving the set of all walks $\{w\}$ that go through the node $\{w_i\}$ at time i (for $i \leq T$) the probability measure

$$\frac{d_{w_1}}{d_G} \prod_{i=1}^{T-1} \frac{g_{w_i w_{i+1}}}{d_i}$$

if $T \geq 2$, and d_{w_1}/d_G if $T = 1$.

Lemma 2.1 *(Anantaram and Tsoucas) Let $\hat{t}(w)$ be the tree obtained from the walk $w \in \hat{\Omega}$ by choosing all those branches g_{ij} that represent the first hop onto a particular node by the walk w. (We note that almost all walks in $\hat{\Omega}$ visit every node of G). In other words $g_{ij} \in \hat{t}(w)$ if and only if the first time i was reached by w was at a time just after settling on j or vice versa. Then*

$$Prob[\hat{t}(w) = t] = \frac{\mathbf{t}}{\sum_{t \in T} \mathbf{t}}.$$

We can define a markov chain M with the state space $\hat{\Omega}$ with transition probability between two walks w and u defined as follows.

$$p[w \rightarrow u] = \begin{cases} \frac{g_{u_1 w_1}}{d_{w_1}} & \text{if} u_{i+1} = w_i \forall i \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since we have assumed right at the beginning that G is connected and has an odd cycle, the basic markov chain whose walks are elements of $\hat{\Omega}$ is aperiodic and irreducible. It can then be shown that P is the unique stationary distribution of M on $\hat{\Im}$. Let A be the set of all rooted trees of G, i.e trees with a special node called the root. All the branches of a rooted tree (also called an arborescence) a are directed *away* from its root $\hat{r}(a)$. We shall define the weight of a, $\mathbf{a}$ to be $\frac{d_{\hat{r}(a)}\mathbf{t}}{\prod_j d_j}$ and call the corresponding tree $\hat{t}(a)$, allowing $\hat{t}$ to act both on random walks and rooted trees.

The markov chain M induces a finite state markov chain A on state space A by the assignment $w \rightarrow a$ where $\hat{t}(a) = \hat{t}(w)$ and $\hat{r}(a) = w_1$. This markov chain can be described using transition probabilities as follows. Let a_1 and a_2 be two rooted trees with roots r_1 and r_2 and the corresponding trees be t_1 and t_2.

$$p[a_1 \rightarrow a_2] = \begin{cases} \frac{g_{r_1 r_2}}{d_{r_1}} & \text{if} t_2 \subset t_1 \cup g_{r_1 r_2} \\ 0 & \text{otherwise.} \end{cases}$$

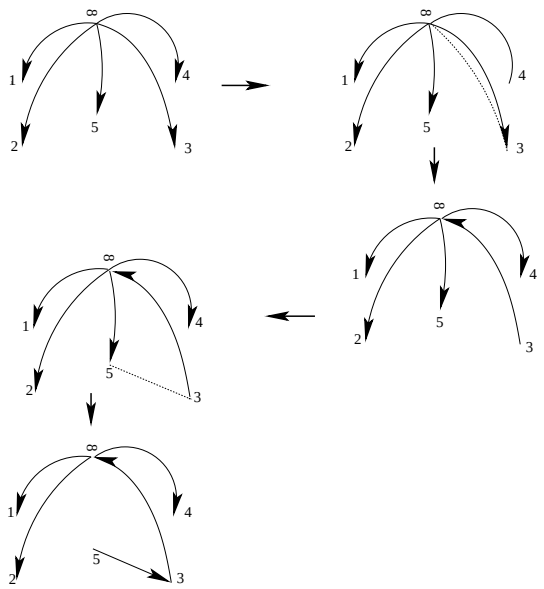

Figure 2.2: The random walk of rooted trees.

We shall now sketch a proof of the fact that a_1 has stationary probability $\mathbf{a_1}/\sum_{a\in A}\mathbf{a}$. Since the evolution of a_1 can be perfectly represented by a random walk starting from its root r_1, we shall talk in terms of evolution of the root. Firstly, any rooted tree a_2 can be reached from a_1. One simply traverses all the branches of $\hat{t}(a_2)$, never using any other branch, ending up at the root r_2 of a_2. Since G has an odd cycle, A is aperiodic, and we have already seen that it is irreducible. Therefore it has a unique stationary distribution, which we only need to verify is the one that gives a_1 the probability $\mathbf{a_1}/\sum_{a\in A}\mathbf{a}$. This is routine - one assumes the claimed stationary distribution at time $T = 1$ and checks that it is unchanged at time $t = 2$. Let a be a rooted tree, with the root i connected to nodes $i_1, \cdots, i_k$. Denote by C_r the set of vertices c for which the unique path from i to c in a passes through i_r. Any arborescence that precedes a has all the branches a has, except one of directed branches $g_{ii_1}^{\rightarrow}, \cdots, g_{ii_k}^{\rightarrow}$. Suppose W.L.O.G it doesn't have $g_{ii_1}^{\rightarrow}$. It

must then have root i_1, and must possess a branch $\vec{g_{c_1 i}}$ for some node c_1 in C_1. The stationary probability measure of the event that, at time $t = 1$ one of *these* arborescences is formed and at time $t = 2$, a is formed is

$$\sum_{c \in C_1} \mathbf{a} \times \frac{d_{i_1}}{g_{i i_1}} \times \frac{g_{ci}}{g_i} \times \frac{g_{i i_1}}{d_{i_1}}.$$

Adding this up for $C_1, \cdots, C_k$ we find the probability of a at time $t = 2$ is

$$\sum_{c \in G} \mathbf{a} \times \frac{g_{ci}}{g_i},$$

which as claimed is simply a. Let us now use Lemma 2.1 to sample the trees of G_{aug}. For a positive λ, the procedure would be to choose a random node of G_{aug} with probability proportional to degree, to perform a random walk until all nodes have been visited, and *to count the number* $n(\lambda)$ of branches adjacent to ∞ that lead to first visits to some node. $-\lambda$ is an eigenvalue if and only if $E[(-1)^{n(\lambda)}] = 0$. We note that if ∞ is reached at time $t = T$, the probability that the next node visited is $i \in G$ is precisely d_i/d_G. In other words, this is exactly the probability of the very first node being i, given that ∞ is not the first node. Restricted to G the result of this procedure is that we have several walks, each terminating when the walker visits ∞. We need to count the number $n(\lambda)$ of these G-walks that were initiated on nodes that were visited then for the first time. At each time, there is a probability $p = \lambda/(\lambda + 1)$ for the walk to reach ∞. Having reached ∞, if the set of nodes visited thus far, S, has total degree d_S, the probability that the next node will be one never visited before is $(d_{S^c})/d_G$, *independent* of the actual G-walks until this event. Let $n_j(\lambda)$ be the random variable that takes value 1 if the j^{th} G-walk begins on a node unvisited by the previous $j-1$ G-walks and 0 otherwise. Then, denoting by d_{S_j}, the total degree of nodes covered

upto and including the j^{th} walk the expected value

$$\begin{aligned}
E[(-1)^{n(\lambda)}] &= E[\prod_{j\geq 1}(-1)^{n_j(\lambda)}] \\
&= E[E[\prod_{j\geq 1}(-1)^{n_j(\lambda)}|w_1]] \\
&= E[(-1)\frac{d_{S_1}-d_{S_1^c}}{d_G}E[\prod_{j\geq 3}(-1)^{n_j(\lambda)}]] \\
&= E[(-1)\frac{d_{S_1}-d_{S_1^c}}{d_G}E[\prod_{j\geq 3}(-1)^{n_j(\lambda)}|w_2]] \\
&= E[-\prod_{j=1}^{2}\frac{d_{S_{j-1}}-d_{S_{j-1}^c}}{d_G}E[\prod_{j\geq 4}(-1)^{n_j(\lambda)}]] \\
&= \cdots \\
&= E[-\prod_{j=1}^{k}\frac{d_{S_{j-1}}-d_{S_{j-1}^c}}{d_G}E[\prod_{j\geq k+1}(-1)^{n_j(\lambda)}]].
\end{aligned}$$

The inner expectation is, in the second and third line over walks $\{w_j\}_2^\infty$, in the fourth and fifth line over $\{w_j\}_3^\infty$, and in the last line over $\{w_j\}_{k+1}^\infty$. The outer expectation in each of these cases is over the walks the inner one does not cover. $\lim_{k\to\infty} Prob[\sum_{j\geq k+1} n_j = 0] = 1$, since the probability that a random walk of length T has not visited all nodes tends to 0 as $T \to \infty$. Therefore,

$$lim_{k\to\infty}E[\prod_{j\geq k+1}(-1)^{n_j(\lambda)}] = 1.$$

Thus we have the final result that $-\lambda = 1 - 1/p$ is a (non-zero) eigenvalue of G if and only if

$$E[\prod_{j=1}^{\infty}\frac{d_{S_j}-d_{S_j^c}}{d_G}] = 0$$

and the theorem is proved. This result has been verified for a cyclic graph with 4 nodes shown in Figure 2.3. The eigenvalues of this graph are 0, -1 and -2. The eigenvalue -1 has multiplicity 2. Figure 2.4 is a plot of

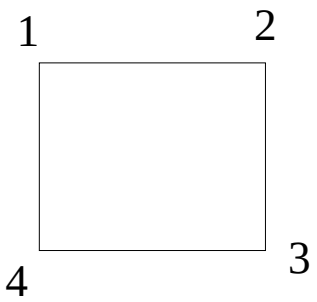

Figure 2.3: The test graph.

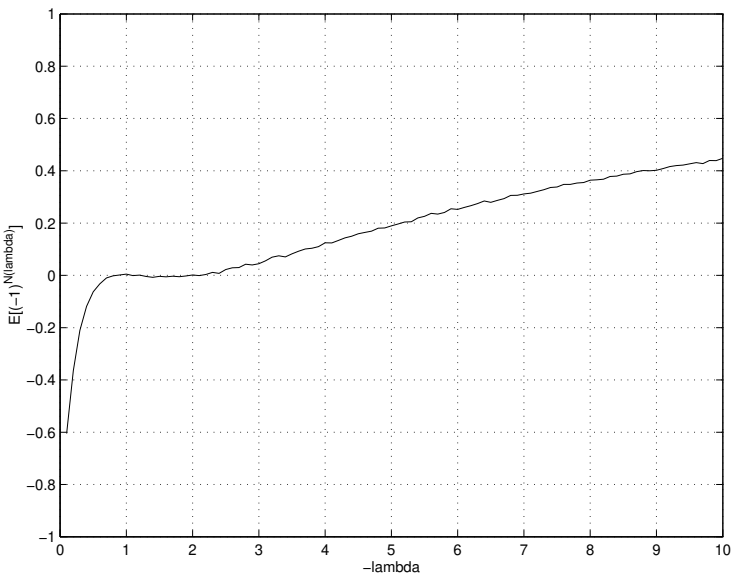

Figure 2.4: Plot of $E[(-1)^{N(\lambda)}]$ versus $-\lambda$.

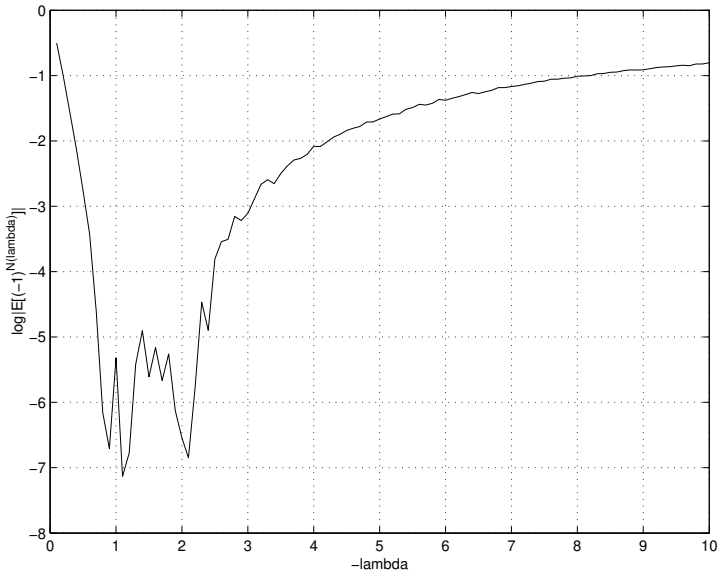

Figure 2.5: Plot of $log|E[(-1)^{N(\lambda)}]|$ versus $-\lambda$.

$E[(-1)^{N(\lambda)}]$ with $-\lambda$, and Figure 2.5 is a plot of $\log|E[(-1)^{N(\lambda)}]|$ with $-\lambda$. The number of samples taken for each λ (that was a multiple of 0.1) was 10^6. In the latter, we see dips corresponding to the two eigenvalues -1 and -2. It is interesting to note that we observe two closely spaced dips around 1, which seems to be a reflection of the fact that -1 is an eigenvalue of multiplicity 2.

2 Spectrum and Kirchchoff's Formula

Definition 2.2 *Let the graph obtained from G by fusing the nodes i and j be denoted by G_{ij}, and the set of its trees by T_{ij}.*

Kirchchoff's formula for the equivalent conductance between the two nodes i and j of G is given by the following proposition.

Proposition 2.2 *(Kirchchoff) The equivalent conductance eq_{ij} between the*

node i and j in G having branch conductances $\{g_{kl}\}$, is given by

$$\frac{\sum_{t\in T}\mathbf{t}}{\sum_{t_{ij}\in T_{ij}}\mathbf{t_{ij}}}.$$

A proof of this result may also be found in Kirchchoff's original paper [4]. Without loss of generality we assume that $i = 1$ and $j = 2$. Let the voltage of node 1 be 1 volt and that of node 2 be 0 volts; then eq_{12} is precisely the current that flows from node 1 to node 2. We shall prove that

$$eq_{12}\sum_{t_{12}\in T_{12}}\mathbf{t_{12}} = \sum_{t\in T}\mathbf{t}.$$

The branches of a tree t_{12} of T_{12} form a forest with two components in G which we shall continue to call t_{12}. one of these components contains 1, the other that contains 2. The total current that flows from the first component to the second is eq_{12}. Let us denote the set of directed branches b having tail in the component of t_{12} containing 1 and head in the other by $c(t_{12})$. Let the current flowing through $b \in c(t_{12})$ from tail to head be $\hat{i}(b)$, the voltage difference be $\hat{v}(b)$ likewise and the conductance of branch b be $\mathbf{b}$.

Then for each tree t_{12} in T_{12},

$$eq_{12} = \sum_{b\in c(t_{12})}\hat{i}(b).$$

Therefore,

$$\begin{aligned}
eq_{12}\sum_{t_{12}\in T_{12}}\mathbf{t_{12}} &= \sum_{t_{12}\in T_{12}}\mathbf{t_{12}}\sum_{b\in c(t_{12})}\hat{i}(b)\\
&= \sum_{t_{12}\in T_{12}}\mathbf{t_{12}}\sum_{b\in c(t_{12})}\mathbf{b}\hat{v}(b)\\
&= \sum_{t\in T}\mathbf{t}\sum_{b\in t, t\setminus b\in T_{12}}\hat{v}(b)\\
&= \sum_{t\in T}\mathbf{t}.
\end{aligned}$$

The last step uses the fact that when one adds the branch voltages over a path from node 1 to node 2, 1 volt is obtained.

Definition 2.3 *Let the graph obtained from G by fusing the set of nodes $S = \{i_1, \cdots, i_k\}$ be denoted by G_S, and the set of its trees by T_S.*

One observes that a tree $t \in G_{aug}$ in which the branches adjacent to ∞ are $g_{i_1\infty}, \cdots, g_{i_k\infty}$ has a weight

$$\mathbf{t} = \mathbf{t_S} \prod_{i_r \in S} d_{i_r} \lambda^k,$$

where t_S is the tree in T_S that contains all the branches of t non-adjacent to ∞.

Definition 2.4 *For a set $S = \{i_1, \cdots, i_k\}$, let $|S|$ be its cardinality and eq_S be*

$$\frac{\sum_{t \in T} \mathbf{t}}{\sum_{t_S \in T_S} \mathbf{t_S}}.$$

If the graph in question is not G, it shall be specified in subscript -

$$[eq_{i_1 i_2}]_{G_{S \setminus i_1}}$$

is the equivalent conductance between i_1 and i_2, which are the equivalence classes of nodes i_1 and i_2 in $G_{S \setminus i_1}$.

The above definition is consistant with our earlier definition of eq_{ij}. We draw attention to the fact that we are forced to define $[eq_{ij}]_{G_{ij}}$ to be ∞. From the definitions, it is clear that

$$eq_S = eq_{i_1 i_2} [eq_{S \setminus i_1}]_{G_{i_1 i_2}}. \tag{2.1}$$

Therefore the set equivalent conductances eq_S can be calculated as the product of the equivalent conductances between pairs of nodes in graphs obtained by successive fusions. Also, we have the formula

$$G(\lambda) = (-1)^n \sum_{t \in T} \mathbf{t} \sum_{\emptyset \neq S \subseteq \{1,\cdots,n\}} \frac{\prod_{i \in S} d_i}{eq_S} \lambda^{|S|}, \tag{2.2}$$

obtained from the expansion of eq_S and the definition of $G(\lambda)$.

Chapter 3

Some Classical Results and an Application

1 Kelvin's theorems, Raleigh's monotonicity principle and Dirichlet's principle

Theorem 3.1 *(Kelvin) In the graph G, if the voltages of nodes $s_1, \cdots, s_k$ in S are respectively set at $v_1, \cdots, v_k$, the remaining nodes $s_{k+1}, \cdots, s_n$ take up voltages $v_{k+1}, \cdots, v_n$, such that the quadratic form*

$$D(w) = 1/2 \times \sum_{i,j} g_{ij}(w_i - w_j)^2$$

takes the minimum possible value at $w = v$.

We first observe that D is a non-negative form as a function of $v_{k+1}, \cdots v_n$ ($v_1, \cdots, v_k$ are assumed to be fixed). Then as $v_s \to \infty$, $D \to \infty$ for any $s \in \{k+1, \cdots, n\}$. Therefore the infimum of D is acheived on a finite vector v. Let v correspond to a global minimum of D. Then, for $j \in k+1, \cdots, n$

$\frac{\partial D(w)}{\partial w_j}|_{w=v} = 0$. In other words,

$$\sum_i g_{ij}(v_i - v_j) = 0.$$

This is precisely Kirchchoff's Current Law. Given the voltages of nodes in S, there is a unique way of assigning voltages to nodes in S^c such that KCL holds in S^c and this assignment gives the voltages corresponding to the solution of the network; also proving that D has a unique local minimum.

Theorem 3.2 *(Kelvin) In the graph G, let the nodes $s_1, \cdots, s_k$ in S have injected currents $j_1, \cdots, j_k$ respectively and the remaining nodes have none. Let I be the space of all current vectors that satisfy KCL corresponding to the condition above. The branch currents i_{kl} in the solution of the network are given by that current vector $i \in I$ for which $E(i) = \sum_{kl} \frac{i_{kl}^2}{g_{kl}}$ is a local minimum within I.*

Let L be a loop consisting of branches $k_1\vec{k}_2, \cdots, k_l\vec{k}_{l+1}$ $(l+1 \equiv 1)$. Let i_L be the current vector in which $i_{k_s k_{s+1}} = 1$ and all other currents are 0. E is a non-negative form and for any k and l, as $i_{kl} \to \infty$, $E \to \infty$. Therefore the infimum of E on I is acheived on a finite vector in I. If $\hat{i}$ is a local minimum of $E(i)$ on I,

$$\lim_{t \to 0} \frac{E(\hat{i} + t i_L) - E(\hat{i})}{t} = 0,$$

which is equivalent to

$$\sum_{k_s \vec{k}_{s+1} \in L} \frac{i_{k_s k_{s+1}}}{g_{k_s k_{s+1}}} = 0.$$

This is precisely Kirchchoff's voltage law. It is known that there is a unique solution to the KCL and KVL equations for a resistive network. Thus $\hat{i}$ is the unique local minimum of E on I and corresponds to the solution of the network.

Theorem 3.3 *(Raleigh's monotonicity principle) Let G be a graph as in the previous two theorems. Let A and B be two vertices. If the resistance of some branch $b_x \in G$ is increased, the equivalent resistance between A and B does not decrease.*

Let us consider two cases, one in which the branch conductance of b_x is g_1, and another in which it is g_2, with $g_1 < g_2$. For any other branch b, we take the conductance of the branch to be $g(b)$ in either case. We inject a current of 1 amp into A and -1 into B. Let i_1 be the current vector that results in the case of g_1 and let i_2 result in the case of g_2. Consider the two quadratic power forms E_1 and E_2 acting on a current vector i given by

$$E_1(i) = \frac{i(b_x)^2}{g_1} + \sum_{b \in G \setminus b_x} \frac{i(b)^2}{g(b)}$$

and

$$E_2(i) = \frac{i(b_x)^2}{g_2} + \sum_{b \in G \setminus b_x} \frac{i(b)^2}{g(b)}.$$

$E_1(i_1)$ is the equivalent resistance in the first case, which is greater than $E_2(i_1)$ with equality if and only if $i_1(b_x) = 0$. $E_2(i_1)$ is greater or equal to $E_2(i_2)$ by Theorem 3.2 with equality if and only if $i_1 = i_2$. Thus we have the stronger result that $E_1(i_1) \geq E_2(i_2)$ with equality if and only if

1. $i_1(b_x) = 0$
2. $i_1 = i_2$

It is clear that if the first of the above conditions holds so does the second, from the uniqueness of the solution to the KCL and KVL equations. Therefore we have the result that $E_1(i_1) \geq E_2(i_2)$ with equality if and only if $i_1(b_x) = 0$, in which case $i_1 = i_2$.

Theorem 3.4 *(Dirichlet's principle) Given a domain G in $\mathbf{R}^2$ whose boundary γ consists of Jordan Curves. Let g be a function continuous in $G+\gamma$, piecewise smooth in G, and with finite Dirichlet integral*

$$D[g] = \int\int_G (g_x^2 + g_y^2) dxdy$$

Consider the class of all functions ϕ continuous in $G+\gamma$, piecewise smooth in G, and having the same boundary values as g.Then the problem of finding a function ϕ for which $D[\phi]$ attains a minimum d has a unique solution $\phi = u$. This function u is the solution of the boundary value problem for $\triangle u = 0$ with the prescribed boundary values $\overline{g}$ on γ.

Theorem 3.5 *(Dirichlet's Principle for free boundary values) Let the boundary γ of a connected domain consist of at least one boundary line α formed by arcs and of connected boundaries $\beta_1, \beta_2, \cdots, \beta_m$. Consider $D[\phi]$ for all piecewise smooth functions ϕ in G and assuming prescribed boundary values on α. Then the solution for the variational problem of minimising $D[\phi]$ is a harmonic function u that assumes the prescribed values of α and satisfies the conditions*

$$\int_{\tilde{\beta}_i} \frac{\partial u}{\partial n} ds = 0.$$

Here $\tilde{\beta}_i$ is an arbitrary polygon bounding, with β_i, a connected subdomain of G, $\frac{\partial}{\partial n}$ denotes the normal derivative, and s arc length on $\tilde{\beta}_i$. Moreover, the conjugate function v takes a constant value on each arc β_i.

The proofs of these theorems may be found in [5], pages 11 and 41 respectively. We note that if an arc β_i is smooth, Dirichlet's theorem for free boundary values implies that $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s} = 0$, as v is constant on β_i.

Corollary 3.1 *Let $P_1 \subset P_2$ and $Q_1 \subset Q_2$ be four subdomains in G, such that there exists a continuous piecewise smooth function of G that is 1 on P_1*

and 0 *on* Q_1 *and another that is* 1 *on* P_2 *and* 0 *on* Q_2. *Consider the boundary value problem* V_1 *on* $G \setminus P_1 \cup Q_1$ *in which* ∂P_1 *takes up potential* 1 *and* ∂Q_2 *takes potential* 0 *and its solution* ϕ_1. *Similarly consider the corresponding boundary value problem* V_2 *on* $G \setminus P_2 \cup Q_2$ *in which* ∂P_2 *takes up potential* 1 *and* ∂Q_2 *takes potential* 0 *and its solution* ϕ_2. *We then have*

$$\int\int_{G \setminus P_2 \cup Q_2} |\nabla \phi_2|^2 dxdy < \int\int_{G \setminus P_1 \cup Q_1} |\nabla \phi_1|^2 dxdy$$

ϕ_2 can be extended to a piecewise smooth continuous function $\tilde{\phi_2}$ that is constant on $P_2 \setminus P_1$ and $Q_2 \setminus Q_1$. Then,

$$\int\int |\nabla \tilde{\phi_2}|^2 dxdy = \int\int |\nabla \phi_2|^2 dxdy.$$

$\tilde{\phi_1}$ satisfies the boundary conditions of problem V_2. Since ϕ_2 corresponds to the minimum of the "power functional" under boundary conditions of V_2, the result follows from Dirichlet's principle.

2 The first non-zero eigenvalue of a graph with two weakly coupled components

We consider the weighted graph G having two connected components G_1 and G_2, each having more than one node, and a single branch b of conductance (weight) w between the two components. We shall prove that the first non-zero eigenvalue $\lambda_1 \to 0$ as $w \to 0$ and give an expression for $\lim_{w \to 0} \frac{\lambda_1}{w}$ in terms of the degrees of nodes in component G_1 and G_2. We shall also prove that the second non-zero eigenvalue λ_2 is bounded away from 0 by a constant as $w \to 0$.

Proposition 3.1 *Let* $A \in G_1$, $B \in G_2$, C *be three distinct vertices of* G, *and* S *be the set* $\{A, B, C\}$. *Then the equivalent conductance between* A *and*

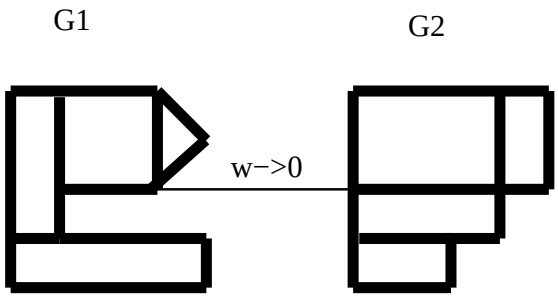

Figure 3.1: The weakly coupled graph G.

B, $eq_{AB} = \frac{C_1 w}{C_1 + w}$ for some non-negative constant C_1 and $eq_S \geq C_2 eq_{AB}$ for a positive constant C_2, where, as mentioned earlier, $eq_S = eq_{AB}[eq_{BC}]_{G_{AB}}$. (Recall that $[eq_{BC}]_{G_{AB}}$ is the equivalent conductance between B and C after A and B are shorted.)

In what follows, $(x||y)$ shall signify $xy/(x+y)$. The branch b of weight w is the only branch across the two subgraphs G_1 and G_2. Let us suppose its end-vertices are s_1 and s_2 in the components G_1 and G_2 respectively. Since any current that flows from A to B must first flow from A to s_1, from s_1 to s_2, and then from s_2 to B, we have

$$eq_{AB} = \{eq_{As_1}||eq_{s_1 s_2}\}||eq_{s_2 B} = eq_{As_1}||eq_{s_2 B}||w.$$

Letting $C_1 = eq_{As_1}||eq_{s_2 B}$, we have $eq_{AB} = \frac{C_1 w}{C_1 + w}$. That $eq_S \geq C_2 eq_{AB}$ for a positive constant C_2, (where $eq_S = eq_{AB}[eq_{BC}]_{G_{AB}}$) is now immediate. Indeed, for $w = 0$ let $[eq_{BC}]^0_{G_{AB}}$ denote the value of $[eq_{BC}]_{G_{AB}}$. From Raleigh's monotonicity principle, $[eq_{BC}]_{G_{AB}} \geq [eq_{BC}]^0_{G_{AB}}$. Letting C_2 to be $\inf_{C \in G}[eq_{BC}]^0_{G_{AB}}$ which is greater than 0 as G_i, $i = 1, 2$ are connected graphs the result now follows.

Theorem 3.6 *Let D_1 and D_2 be the sums of the weighted degrees of the nodes in components G_1 and G_2; we assume that both are greater than 0. Let*

$0 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ *be the* n *eigenvalues of* G*, which are the zeroes of the polynomial*

$$G(\lambda) = \det \begin{vmatrix} -\sum g_{1i}(1+\lambda) & g_{12} & \cdots & g_{1n} \\ g_{21} & -\sum g_{2i}(1+\lambda) & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & -\sum g_{ni}(1+\lambda) \end{vmatrix}.$$

Then,

1. $$\lim_{w \to 0} \frac{\lambda_1}{w} = -\frac{D_1 + D_2}{D_1 D_2}.$$

2. $$\limsup_{w \to 0} \lambda_2 < 0.$$

Equation 2.2 gives us

$$G(\lambda) = \sum_{t \in T} \mathbf{t} \sum_{\emptyset \neq S \subseteq \{1, \cdots, n\}} \frac{\prod_{i \in S} d_i}{eq_S} \lambda^{|S|}. \tag{3.1}$$

Therefore from the relations that exist between the roots of a polynomial and its coefficients and the fact that $\lambda_0 = 0$,

$$\sum_{i=1}^{n-1} 1/\lambda_i = -\frac{\sum_{|S|=2}(1/eq_S)\prod_{i \in S} d_i}{\sum_{|S|=1}(1/eq_S)\prod_{i \in S} d_i}.$$

By Theorem 2.1 all the λ_i are less or equal to 0. So

$$\sum_{i=1}^{n-1} 1/|\lambda_i| = \frac{\sum_{|S|=2}(1/eq_S)\prod_{i \in S} d_i}{\sum_{|S|=1}(1/eq_S)\prod_{i \in S} d_i}.$$

If S contains two nodes, one in G_1 and the other in G_2, $\beta + 1/w \geq 1/eq_S \geq 1/w$ for some positive β from proposition 3.1. If both the nodes belong either to G_1 or to G_2, $1/eq_S$ is bounded above by a constant as in the proof

of proposition 3.1. Thus considering in the right hand side, only these pairs, we have (for some constant γ),

$$\gamma + \frac{D_1 D_2}{w} \geq \sum_{|S|=2} (1/eq_S) \prod_{i \in S} d_i \geq \frac{D_1 D_2}{w}.$$

From this it follows that

$$\frac{\gamma}{D_1 + D_2} + \frac{D_1 D_2}{wD_1 + wD_2} \geq \sum_{i=1}^{n-1} 1/|\lambda_i| \geq \frac{D_1 D_2}{wD_1 + wD_2}.$$

We similarly have,

$$\sum_{1 \leq i < j \leq n-1} \frac{1}{\lambda_i \lambda_j} = \frac{\sum_{|S|=3} (1/eq_S) \prod_{i \in S} d_i}{\sum_{|S|=1} (1/eq_S) \prod_{i \in S} d_i}.$$

Proposition 3.2 *Let $|S| = 3$. If S is contained entirely in G_1 or in G_2, $1/eq_S$ is independent of w. If S is not entirely contained in either, there exists $\hat{C}$ independent of S such that*

$$1/eq_S \geq \frac{\hat{C}}{w}.$$

If $S = \{A, B, C\}$ and S is not contained in G_1 or G_2, from proposition 3.1 $1/eq_S \geq \frac{C_2(C_1+w)}{C_1 w}$ for some C_1 non-negative and C_2 positive. Letting $\hat{C}$ to be the minimum of C_2 over the finitely many possibilities of S (not entirely contained in G_1 or G_2) we have

$$1/eq_S \geq \frac{\hat{C}}{w}.$$

Now assume (W.L.O.G) that $\{A, B, C\} = S \subseteq G_1$.

$$\frac{1}{eq_S} = \frac{1}{eq_{AB}} \times \frac{1}{[eq_{BC}]_{G_{AB}}}$$

$\frac{1}{eq_{AB}}$ is independent of w because if current flows from A to B, that through branch b is the net current entering G_2, which is 0. Therefore from theorem 3.2 $\frac{1}{eq_{AB}}$ is unchanged when branch b is deleted, (i.e w is set to 0). Thus

$\frac{1}{eq_S}$ is independent of w. Applying the same argument to graph G_{AB}, which we recall is the graph obtained by fusing A and B, we see that

$$\frac{1}{[eq_{BC}]_{G_{AB}}}$$

too is independent of w. Therefore $1/eq_S$ is independent of w. From the above proposition

$$\sum_{|S|=3} (1/eq_S) \prod_{i \in S} d_i \leq \tilde{C} + \frac{\hat{C}}{w}$$

for some positive $\hat{C}$ and non-negative $\tilde{C}$. So, we have

$$\sum_{1 \leq i < j \leq n-1} \frac{1}{\lambda_i \lambda_j} \leq \frac{\tilde{C} + \frac{\hat{C}}{w}}{D_1 + D_2}.$$

We absorb $D_1 + D_2$ into the constants to obtain

$$\sum_{1 \leq i < j \leq n-1} \frac{1}{\lambda_i \lambda_j} \leq \tilde{C} + \frac{\hat{C}}{w}.$$

We have already shown

$$\sum_{i=1}^{n-1} 1/|\lambda_i| \geq \frac{D_1 D_2}{wD_1 + wD_2}.$$

Let

$$\sum_{i=1}^{n-1} 1/|\lambda_i| = \theta$$

and

$$\frac{D_1 D_2}{D_1 + D_2} = K.$$

$$\sum_{1 \leq i < j \leq n-1} \frac{1}{\lambda_i \lambda_j} > (\theta - \frac{1}{|\lambda_1|}) \frac{1}{|\lambda_1|}.$$

We therefore have for positive constants $K_1 = \frac{\gamma}{D_1 + D_2}$ and $K = \frac{D_1 D_2}{D_1 + D_2}$,

1.

$$K_1 + \frac{K}{w} \geq \theta \geq \frac{K}{w}$$

2.

$$(\theta - \frac{1}{|\lambda_1|})\frac{1}{|\lambda_1|} < \tilde{C} + \frac{\hat{C}}{w}.$$

Let $\mu = \frac{1}{|\lambda_1|}$. Then the second of the above two conditions translates to $\mu(\theta - \mu) < a/w + b$ (replacing $\tilde{C}$ by b and $\hat{C}$ by a). Therefore,

$$\frac{\mu}{\theta}(1 - \frac{\mu}{\theta}) < \frac{aw}{K^2} + \frac{bw^2}{K^2}.$$

Because $|\lambda_1| < \cdots |\lambda_{n-1}|$, we have

$$\frac{\mu}{\theta} > \frac{1}{n-1}.$$

Therefore,

$$1 - \frac{\mu}{\theta} < (n-1)(\frac{aw}{K^2} + \frac{bw^2}{K^2}) < \alpha w$$

for constant α when w is sufficiently small. (All statements in the rest of this proof require the qualification "when w is sufficiently small", which we refrain from stating for the sake of readability.) Therefore,

$$\frac{\mu}{\theta} > 1 - \alpha w.$$

This means

$$\mu > \frac{k}{w},$$

for some positive k, from 1 above. Therefore,

$$\theta - \mu < \frac{a/w + b}{k/w} < c,$$

for some fixed c when w is small. Therefore,

$$\mu \in [-c + \frac{K}{w}, K_1 - c + \frac{K}{w}].$$

And so replacing μ by $1/|\lambda_1|$, and K by $(D_1 D_2)/(D_1 + D_2)$ we have

$$\lim_{w \to 0} \frac{|\lambda_1|}{w} = \frac{D_1 + D_2}{D_1 D_2}.$$

Next, $\theta - \mu < c$, implies that $1/|\lambda_2| < c$ and therefore that

$$\liminf_{w \to 0} |\lambda_2| > 0.$$

This in conjunction with Theorem 2.1 which says that $\lambda_2 \leq 0$, completes the proof.

Chapter 4

Green's functions on a planar domain

In this section two theorems about Green's functions are presented. The first, Theorem 4.1 holds for arbitrary Riemannian manifolds, and its (second) proof contains nothing specifically related to planar domains. Theorem 4.2 on the other hand holds only for a simply connected planar domain. The result might be useful in the following situation. Suppose one has a flat plate/membrane that is required to have constant electrical conductivity σ, taken as a two dimensional object. Then

$$[A_1, \cdots, A_n][B_1, \cdots, B_n] = \sigma^{2n-2},$$

where $[A_1, \cdots, A_n]$ and $[B_1, \cdots, B_n]$ are experimentally observable quantities. If their product significantly differs from this quantity, one can conclude that the plate does not have constant electric conductivity any more. The technique followed for the proof of Theorem 4.1 has been to first prove it for positively weighted graphs, then to approximate a continuous region by finer and finer grids and finally show the convergence of relevant quantities. A

more direct proof has also been obtained through examination of the "power quadratic form" which for given input voltages takes the value of the resultant power.

1 $[a_1, \cdots, a_n]$ and its invariance under permutation

Let R be a connected region in $\mathbf{R^2}$, with a smooth boundary ∂R. We define the Neumann spectrum of R as the set of all λ for which there exist non-trivial $f_\lambda : R \to \mathbf{R}$ such that $\frac{\partial f}{\partial n} = 0$ on ∂R, and $\nabla^2 f_\lambda = \lambda f_\lambda$ on R. The multiplicity of λ is the dimension of the solution space of these equations.

Definition 4.1 *For $\epsilon > 0$ and point $P \in R$, we define $B(\epsilon, P)$ to be the intersection of the open ball of radius ϵ and center P with R.*

Definition 4.2 *We define $[B(\epsilon, P_0), \cup_{i=1}^n B(\epsilon, P_i)]$ to be the equivalent resistance between $B(\epsilon, P_0)$ and $\cup_{i=1}^n B(\epsilon, P_i)$, which is calculated as follows. If $\overline{B(\epsilon, P_0)} \cap \overline{\cup_{i=1}^n B(\epsilon, P_i)} \neq \phi$,*

$$[B(\epsilon, P_0), \cup_{i=1}^n B(\epsilon, P_i)] = 0.$$

Else consider the unique function $f : R \setminus \cup_{i=0}^n B(\epsilon, P_i) \to \mathbf{R}$ such that $P \in \partial B(\epsilon, P_0) \Rightarrow f(P) = 1$, $P \in \partial B(\epsilon, P_i), i > 0 \Rightarrow f(P) = 0$, $\frac{\partial f}{\partial n} = 0$ on ∂R and $\nabla^2 f = 0$ in the interior of $R \setminus \cup_{i=0}^n B(\epsilon, P_i)$.

$$[B(\epsilon, P_0), \cup_{i=1}^n B(\epsilon, P_i)] = \{2 \int_{R \setminus \cup_{i=0}^n B(\epsilon, P_i)} \|\nabla f\|^2 dxdy\}^{-1}.$$

It is clear that this definition makes sense if instead of $B(\epsilon, P_i)$ we had an open subset with smooth boundary B_i, or even if this were an arc belonging to ∂R.

In the above definition of equivalent resistance, the dissipated power has been used rather than the total current.

Definition 4.3 *We define* $[B_1, \cdots, B_n]$ *as*

$$\prod_{i=0}^{n-1} [B_i, \cup_{j=i+1}^{n} B_j]$$

Let $\sigma : \{1, \cdots, n\} \mapsto \{1, \cdots, n\}$ be a permutation.

Theorem 4.1

$$[B_1, \cdots, B_n] = [B_{\sigma(1)}, \cdots, B_{\sigma(n)}].$$

A discrete analogue of this theorem has already been proved namely equation 2.1. There we had considered a weighted graph G (the weights were taken to be conductances in a resistive network), and given a subset $S = \{1, \cdots k\}$ of its vertices, had defined eq_S. eq_S was shown to be equal to $eq_{S\setminus 1} \times [eq_{12}]_{G_{S\setminus 1}}$. We note that allowing S_i to be B_i,

$$[eq_{12}]_{G_{S\setminus 1}} = [B_1, \cup_{j=2}^{n} B_j]^{-1}.$$

And then, the fact that eq_S was defined without respect to the order of the nodes proves the theorem.

We now proceed to derive the continuous case from the discrete one through a process of examining finer and finer grids. R is a region in $\mathbf{R^2}$ whose boundary is a piecewise differentiable closed curve. We consider all those points of R whose x and y coordinates are integer multiples of δ. We consider the "grid graph" G_δ formed by these points, with the natural adjacency pattern (according to which interior vertices have four neighbours but boundary vertices have less). We assign a weight of 1 to each branch. Let P and Q be two regions in R with disjoint closures satisfying certain regularity

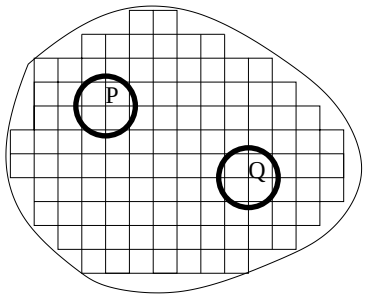

Figure 4.1: The region R and the graph G_δ.

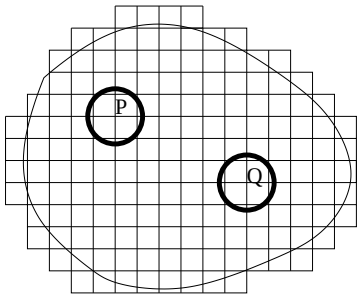

Figure 4.2: The region R and the graph $\tilde{G}_\delta$.

conditions (that will be seen in the course of the proof). Let S_p and S_q be the nodes of G that respectively lie in the regions P and Q. We do not require that P or Q be connected.

Definition 4.4 *Let S_p and S_q be two disjoint sets of nodes $P_1, \cdots, P_{i_p}$ and $Q_1, \cdots, Q_{i_q}$ of graph G_δ. We denote by G_{S_p,S_q}, the graph obtained from G by fusing together elements of S_p and then fusing together nodes of S_q. We define*

$$eq_{S_p,S_q} = [eq_{P_1Q_1}]_{G_{S_p,S_q}}.$$

Lemma 4.1

$$\lim_{\delta\to 0}[eq_{S_p,S_q}]_{G_\delta} = [P,Q]^{-1}.$$

We assume R is such that there exists a function g on $R \setminus P^o \cup Q^o$ (P^o = interior of P) that is 1 on ∂P, 0 on ∂Q and has continuous partial derivatives in the interior of this region. These are the conditions necessary for the applicability of Theorem 3.5. Let $f : R \to \mathbf{R}$ then be the unique function that is 1 on $\overline{P}$ and 0 on $\overline{Q}$, is harmonic on R and $\frac{\partial f}{\partial n} = 0$ on ∂R (which exists by Theorem 3.5). We assume that the partial derivatives of f are uniformly continuous on $R \setminus P \cup Q$. Let R_δ be the closed subset of R consisting of those points (x, y) such that the points $(\delta\lfloor x/\delta \rfloor, \delta\lfloor y/\delta \rfloor)$, $(\delta\lfloor x/\delta \rfloor, \delta\lceil y/\delta \rceil)$, $(\delta\lceil x/\delta \rceil, \delta\lfloor y/\delta \rfloor)$, $(\delta\lceil x/\delta \rceil, \delta\lceil y/\delta \rceil)$, are vertices of G_δ. From uniform continuity of the partial derivatives of f, it follows that $|\partial f/\partial x|$ and $|\partial f/\partial y|$ are bounded on $R \setminus P \cup Q$ say M. The area of $R \setminus R_\delta$ tends to 0 as $\delta \to 0$, which can be seen as follows. We divide ∂R into arcs, so that each arc, represents a monotonic function of x or a monotonic function of y. Let the number of such arcs be c. The total length of of edges in G_δ that are adjacent to only one square is bounded above by $2|\partial R| + 2c\delta$ as can be seen by projecting each arc of ∂R first vertically onto the boundary of R_δ, so that it falls on the horizontal edges, and then horizontally so that it falls on the vertical edges. The area of $R \setminus R_\delta$ can be covered by squares adjacent to these boundary edges of R_δ. Therefore,

$$\int\int_{R \setminus R_\delta} ({f_x}^2 + {f_y}^2) dx dy \to 0$$

as $\delta \to 0$. f, when restricted to the nodes of G_δ gives a set of node potentials v_δ, or a voltage vector on the graph $G_{S_p S_q}$. Let the power corresponding to a voltage vector v in $G_{S_p S_q}$ be denoted $D_\delta(v)$ as in Theorem 3.1. Then $P(\delta) \doteq D_\delta(v_\delta) \geq eq_{S_p S_q}$. We now proceed to prove that $\lim_{\delta \to 0} P(\delta) = [P, Q]^{-1}$.

$$[P, Q]^{-1} = \{2 \int\int_R |\nabla f|^2 dx dy\}^{-1}.$$

$$\{2 \lim_{\delta \to 0} \int_{R_\delta} |\nabla f|^2 dx dy\}^{-1}.$$

The region R_δ can be broken up into squares of side δ, each of which has corners that are nodes of G_δ. Given $\epsilon > 0$, we choose δ such that if $|x_0 - x_1| + |y_0 - y_1| < \delta$, $|f_x(x_0, y_0) - f_x(x_1, y_1)| < \epsilon$ and $|f_y(x_0, y_0) - f_y(x_1, y_1)| < \epsilon$. Now, for any "interior" square $\Box$, with end-vertices $(x, y), (x + \delta, y), (x, y + \delta), (x+\delta, y+\delta)$ in which each edge neighbours another square too, we compare $\int_\Box (f_x^2 + f_y^2) dxdy$ with

$$1/2 \times (((f(x, y) - f(x + \delta, y))^2 + (f(x, y) - f(x + \delta, y))^2 +$$

$$(f(x, y + \delta) - f(x + \delta, y + \delta))^2 + (f(x + \delta, y) - f(x + \delta, y + \delta))^2).$$

$$|\int_\Box (f_x^2 + f_y^2) dxdy - \delta^2 (f_x(x + \delta/2, y + \delta/2)^2 +$$

$$f_y(x + \delta/2, y + \delta/2))^2| < \delta^2 (4M\epsilon).$$

Similarly,

$$|1/2 \times (((f(x, y) - f(x + \delta, y))^2 + (f(x, y) - f(x + \delta, y))^2 +$$

$$(f(x, y + \delta) - f(x + \delta, y + \delta))^2 + (f(x + \delta, y) - f(x + \delta, y + \delta))^2))$$

$$- \int_\Box (f_x^2 + f_y^2) dxdy| < \delta^2 (4M\epsilon).$$

The total length of of edges in G_δ that are adjacent to only 1 square is bounded above by $2|\partial R| + 2c\delta$ So the number of edges is bounded above by $2|\partial R|/\delta + 2c$. The contribution of each to $P(\delta)$ is bounded above by $\delta^2 M\epsilon$, and so the squares adjacent to such branches can be neglected as $\epsilon \to 0$. Adding over all interior squares we obtain $\lim_{\delta \to 0} P(\delta) = [P, Q]^{-1}$. Therefore by Kelvin's theorem $[P, Q]^{-1} \geq lim_{\delta \to 0} eq_{S_p S_q}$. We now work with a slightly larger graph $\tilde{G_\delta}$, which contains G_δ as a subgraph and has as vertices all points $(l\delta, m\delta)$ such that a point among $((l \pm 1)\delta, (m \pm 1)\delta)$ or $((l \pm 1)\delta, (m \mp 1)\delta)$ is a vertex of G_δ. The branches of $\tilde{G_\delta}$ are given by the natural adjacency. It is clear by Raleigh's monotonicity principle (Theorem 3.3) that

the equivalent conductance between the set of nodes S_p and S_q in $\tilde{G_\delta}$ is less or equal to $eq_{S_pS_q}$, i.e that

$$eq_{\tilde{S_p}S_q} \geq eq_{S_pS_q}.$$

It is therefore sufficient to prove that $[P,Q]^{-1} \leq lim_{\delta \to 0} eq_{\tilde{S_p}S_q}$. The strategy is the same as that used earlier - Given the distribution of potentials $\tilde{v_\delta}$ on $\tilde{G_\delta}$ corresponding to the solution of the KCL and KVL equations, we find a function ϕ_δ on R that is continuous and piecewise smooth, satisfies required boundary conditions, and has $\tilde{D_\delta}(\tilde{v_\delta}) \geq D(\phi_\delta)$. Here $\tilde{D_\delta}(v)$ for a voltage vector on $\tilde{G_\delta}$ is the associated power on the graph $\tilde{G_\delta}$. From Theorem 3.5, $D(\phi_\delta) \geq [P,Q]^{-1}$. This implies $[P,Q]^{-1} \leq lim_{\delta \to 0} eq_{S_pS_q}$. To construct ϕ_δ, we consider a square $\Box$, this time in $\tilde{G_\delta}$, with end-vertices $(x,y),(x+\delta,y),(x,y+\delta),(x+\delta,y+\delta)$. Let $\tilde{v_\delta}$ take values a,b,c and d respectively on these points. For a point $p = (x+\alpha\delta, y+\beta\delta)$ inside it, we define

$$\phi_\delta(p) \doteq \alpha\beta d + \alpha(1-\beta)b + (1-\alpha)\beta c + (1-\alpha)(1-\beta)a.$$

It is clear that $\phi_\delta(p)$ is piecewise smooth. It is equal to 1 in a closed set contained in P, namely all those points of P contained in squares of $\tilde{G_\delta}$ whose end-vertices are in P. It is equal to 0 in a closed set contained in Q, namely all those points of Q contained in squares of $\tilde{G_\delta}$ whose end-vertices are in Q. By the corollary to Theorem 3.5, $2D(\phi_\delta) \geq [P,Q]^{-1}$.

Lemma 4.2 $\tilde{D_\delta}(\tilde{v_\delta}) \geq D(\phi_\delta)$.

It is sufficient to prove that

$$\int\int_{\Box} |\nabla \phi_\delta|^2 dxdy \leq 1/2((a-b)^2 + (b-d)^2 + (d-c)^2 + (c-a)^2)$$

The L.H.S above is

$$\int_{\alpha=0}^{1}\int_{\beta=0}^{1}\{(\beta d+(1-\beta)b-\beta c-(1-\beta)a)^2+(\alpha d+(1-\alpha)b-\alpha c-(1-\alpha)a)^2\}d\alpha d\beta.$$

Letting $a-b, b-d, d-c, c-a$ respectively be t_1, t_2, t_3, t_4, the above becomes

$$t_1^2/3 + t_2^2/3 + t_1t_2/3 + t_3^2/3 + t_4^2/3 + t_3t_4/3.$$

Thus L.H.S-R.H.S $= -1/6((t_1 - t_2)^2 + (t_3 - t_4)^2) \leq 0$. This proves that

$$\lim_{\delta \to 0}[eq_{S_p,S_q}]_{G_\delta} = [P,Q]^{-1}.$$

This completes our proof of the theorem.

Aliter:(see [8])

When $[B_1, \cdots, B_n]$ have currents $J = (j_1, \cdots, j_n)^t$ emanating out of them, let their voltages be $V = (v_1, \cdots, v_n)^t$, where $v_1 + \cdots + v_n = 0$.Let e_i be the vector with 0s at all positions except the i^{th}, where there is a 1. Consider the matrix R, such that

1. $RJ = V$.
2. $R\sum_i e_i = 0$.

Since J can be any vector whose entries add up to 0 and no others, there is a unique matrix satisfying the above 2 conditions. As the sums and columns of R both add up to 0, all principal cofactors, i.e those corresponding to diagonal elements have the determinants of the same magnitude - indeed one obtains cofactor matrix C_{ii} from cofactor matrix C_{jj}, $i < j$ by deleting the i^{th} row, inserting at the $j-1^{th}$ row the negative of the sum of all rows of C_{ii}, deleting the i^{th} column and inserting at the $j-1^{th}$ column the negative of the sum of all the columns of the resulting matrix. Now consider the matrix R_n, obtained from R by subtracting from each row of R, its n^{th} row. The last row of R_n is 0. Given a current vector J, $R_nJ = V$ gives a voltage vector

in which B_n is the ground. Let

$$R_n = \begin{bmatrix} r_{11} & \cdots & r_{1n-1} & r_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ r_{n-11} & \cdots & r_{n-1n-1} & r_{n-1n} \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

It can be seen that

$$\det \begin{vmatrix} r_{11} & \cdots & r_{1n-1} \\ \vdots & \ddots & \vdots \\ r_{n-11} & \cdots & r_{n-1n-1} \end{vmatrix} = n \det C_{ii}.$$

And the matrix

$$r_n = \begin{bmatrix} r_{11} - r_{1n} & \cdots & r_{1n-1} - r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n-11} - r_{n-1n} & \cdots & r_{n-1n-1} - r_{n-1n} \end{bmatrix}$$

has determinant $n^2 \det C_{ii}$. We observe that $r_n j = v$, $j = (j_1, \cdots, j_{n-1})$ and $v = (v_1, \cdots, v_{n-1})$, where $v_n = 0$, i.e the potential of $B(\epsilon, P_n)$ is taken as ground. The statement

$$[B_1, \cdots, B_n] = [B_{\sigma(1)}, \cdots, B_{\sigma(n)}].$$

would be proved true if we could show that

$$[B_1, \cdots, B_n] = \det \begin{vmatrix} r_{11} - r_{1n} & \cdots & r_{1n-1} - r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n-11} - r_{n-1n} & \cdots & r_{n-1n-1} - r_{n-1n} \end{vmatrix}$$

Let the above matrix be written as

$$r_n = \left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{21} & k_{22} \end{array}\right]$$

where k_{11} is an entry and k_{22} is an $n-2 \times n-2$ matrix. For a non-zero vector j, $j^t r_n j > 0$, since this is twice the dissipated power. Therefore for any "principal" submatrix M, $\det M \neq 0$.

$$\det r_n = \det \left| \begin{array}{c|c} x & 0 \\ \hline k_{21} & k_{22} \end{array} \right|$$

where $x = k_{11} - k_{12} k_{22}^{-1} k_{21}$. Thus $\det r_n = x \det k_{22}$. Consider the equation

$$\left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{21} & k_{22} \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \alpha \end{array} \right] = \left[\begin{array}{c} y \\ \hline 0 \end{array} \right]$$

Clearly y equals the equivalent resistance

$$[B_1, \cup_{i=2}^n B_i].$$

But by row operations we then obtain

$$\left[\begin{array}{c|c} x & 0 \\ \hline k_{21} & k_{22} \end{array} \right] \left[\begin{array}{c} 1 \\ \hline \alpha \end{array} \right] = \left[\begin{array}{c} y \\ \hline 0 \end{array} \right].$$

which implies that $x = y$. In other words

$$\det r_n = [B_1, \cup_{i=2}^n B_i] \times \det k_{22}.$$

Proceeding in this way, we obtain

$$\det r_n = \prod_{j=1}^{n-1} [B_j, \cup_{i=j+1}^n B_i].$$

Since $|\det r_n|$ is independent of the order in which $1, \cdots, n-1$ are taken and also is $n^2 |\det C_{ii}|$ independent of i, i.e since a similarly constructed r_i has the same determinant,

$$[B_1, \cdots, B_n] = [B_{\sigma(1)}, \cdots, B_{\sigma(n)}].$$

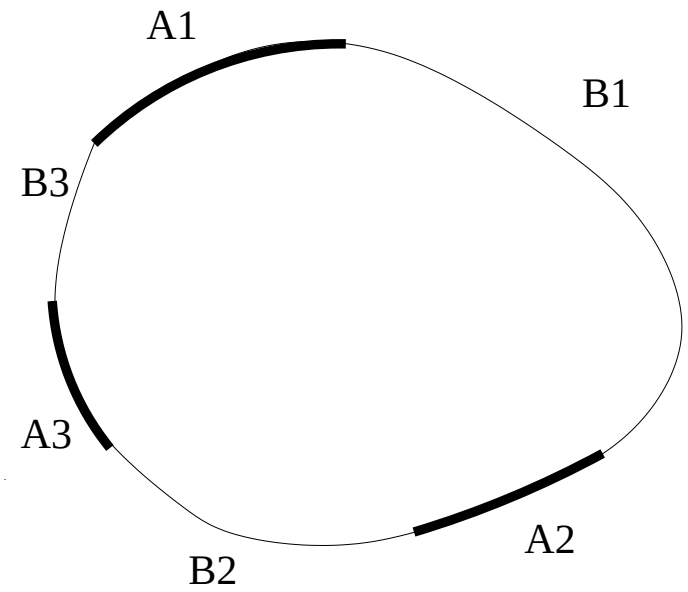

Figure 4.3: The region R and the arcs A_i, B_i.

2 Planar duality

Theorem 4.2 *Let R be a simply connected region with a smooth boundary ∂R, which consists of the union of arcs $A_1, B_1, \cdots, A_n, B_n$ in clockwise direction. Then*

$$[A_1, \cdots, A_n][B_1, \cdots, B_n] = 1.$$

In the case where the arcs A_i are given potentials $v^A = (v_1^A, \cdots v_n^A)$, let $j^A = (j_1^A, \cdots, j_n^A)$ be the currents flowing out of the arcs $A_1, \cdots, A_n$. The boundary conditions in this case are

1. $u|_{A_i} = v_i^A$.

2. $$\frac{\partial u}{\partial n}|_{B_i} = 0.$$

If r_n^A is the unique matrix such that

$$r_n^A \begin{bmatrix} j_1^A \\ j_2^A \\ \vdots \\ j_{n-1}^A \end{bmatrix} = \begin{bmatrix} v_1^A - v_n^A \\ v_2^A - v_n^A \\ \vdots \\ v_{n-1}^A - v_n^A \end{bmatrix}.$$

We have seen in the second proof of Theorem 4.1 that $|\det r_n^A| = [A_1, \cdots, A_n]$. Let r^A be the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} r_n^A$$

Then,

$$r^A \begin{bmatrix} j_1^A \\ j_2^A \\ \vdots \\ j_{n-1}^A \end{bmatrix} = \begin{bmatrix} v_1^A - v_n^A \\ v_2^A - v_1^A \\ \vdots \\ v_{n-1}^A - v_{n-2}^A \end{bmatrix}.$$

Clearly $\det r^A = \det r_n^A$. Let u be the potential on R corresponding to the vector v^A. Since u is harmonic, it has a conjugate $\tilde{u}$, unique up to an additive constant such that $u_x = \tilde{u}_y$ and $u_y = -\tilde{u}_x$. Let ds denote a line element in the clockwise direction around ∂R and let dn denote a normal element directed towards the interior. The boundary conditions satisfied by $\tilde{u}$ are

1.

$$u|_{A_i} = v_i^A \Rightarrow \frac{\partial \tilde{u}}{\partial n} = 0.$$

2.

$$\frac{\partial u}{\partial n}|_{B_i} = 0 \Rightarrow \frac{\partial \tilde{u}}{\partial s}|_{B_i} = 0.$$

Let $\tilde{u}|_{B_i} = v_i^B$. Further, we see that $j_i^A = v_i^B - v_{i-1}^B$, and $j_i^B = v_i^A - v_{i+1}^A$, writing the j_i and v_i as line integrals. Thus

$$r^A \begin{bmatrix} j_1^A \\ j_2^A \\ \vdots \\ j_{n-1}^A \end{bmatrix} = \begin{bmatrix} -j_n^B \\ -j_1^B \\ \vdots \\ -j_{n-2}^B \end{bmatrix}.$$

Let r^B be the unique matrix so that

$$r^B \begin{bmatrix} j_1^B \\ j_2^B \\ \vdots \\ j_{n-1}^B \end{bmatrix} = \begin{bmatrix} v_1^B - v_n^B \\ v_2^B - v_1^B \\ \vdots \\ v_{n-1}^B - v_{n-2}^B \end{bmatrix}.$$

As before, $\det r^B = [B_1, \cdots, B_n]$.

$$r^B \begin{bmatrix} j_1^B \\ j_2^B \\ \vdots \\ j_{n-1}^B \end{bmatrix} = \begin{bmatrix} j_1^A \\ j_2^A \\ \vdots \\ j_{n-1}^A \end{bmatrix}.$$

Therefore

$$r^A r^B \begin{bmatrix} j_1^B \\ j_2^B \\ \vdots \\ j_{n-1}^B \end{bmatrix} = \begin{bmatrix} -j_n^B \\ -j_1^B \\ \vdots \\ -j_{n-2}^B \end{bmatrix}.$$

Therefore,

$$r^A r^B = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Therefore, $\det r^A \det r^B = 1$. Hence proved.

Chapter 5

Eigenvalues of the Laplacian, discrete and continuous

1 Convergence of eigenvalues

We shall now concern ourselves with the Neumann eigenvalues of the Laplacian. Our aim in this section is to prove that the Neumann eigenvalues of a region R with a smooth boundary can be obtained by taking limits of the (scaled) eigenvalues of certain graphs. f is called a Neumann eigenfunction of R if $\partial f/\partial n$ is 0 on the boundary of R and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \lambda f.$$

In this case λ is called a Neumann eigenvalue of R. The graphs in question are obtained by taking all points which are integer linear combinations of the two vectors $(0, \delta)$ and $(\delta/2, \sqrt{3}\delta/2)$, and have a neighbour (with the natural adjacency pattern) within R. Following the notation of lemma 4.1, we call this graph $\tilde{G}_\delta$, although there the grid consisted of squares and here they consist of equilateral triangles. The technique of using minimax theorems to

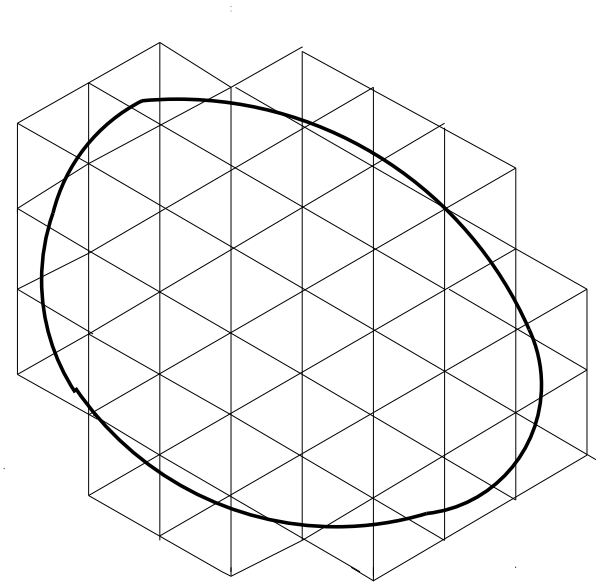

Figure 5.1: An "equilateral grid" graph approximating R.

prove convergences has been used before, notably in [10], where simplicial approximations to manifolds *without* boundary are considered. This section is about a planar domain R which has a boundary, which is not covered in the paper mentioned.

λ_δ is an eigenvalue of the graph $\tilde{G}_\delta$ if there exists a complex-valued function on the nodes of $\tilde{G}_\delta$, v such that

$$\lambda_\delta v_a = \frac{\sum_{b \leftrightarrow a}(v_a - v_b)}{d_a},$$

where $b \leftrightarrow a$ means that b is connected to a. All vertices inside R have degree 6.

Definition 5.1 *1. The Raleigh quotient of a C^∞ function f on R is the ratio*

$$Q(f) = \frac{\int_R |\nabla f|^2 dxdy}{\int_R f^2 dxdy} = \frac{E(f)}{\|f\|^2}.$$

2. *The Raleigh quotient of a discrete node function f_δ on $\tilde{G}_\delta$ is*

$$Q(f_\delta) = \frac{\sum_{a \leftrightarrow b}(f_\delta(a) - f\delta(b))^2}{\sum_a d_a f_\delta(a)^2} = \frac{E(f_\delta)}{\|f_\delta\|^2}.$$

Here d_a is the degree of node a.

Theorem 5.1 *(The minimax principles):*

1. *For any finite dimensional subspace E of functions on R, which are in the closure of the set of all smooth functions under the norm $\|\nabla f\|^2$. set*

$$m(E) = \max\{Q(f), f \in E\}.$$

Then, the k^{th} Neumann eigenvalue of R

$$\lambda_k = -\inf\{m(E), \dim E = k\}.$$

2. *For any finite dimensional subspace E_δ of discrete node functions on $\tilde{G}_\delta$, set*

$$m(E) = \max\{Q(f), f \in E\}.$$

Then the k^{th} eigenvalue of $\tilde{G}_\delta$

$$\lambda_{k,\delta} = -\inf\{m(E_\delta), \dim E = k\}.$$

The continuous case appears in [6], page 178, Theorem 4.62. Both are based on linear algebra and we do not prove them here.

Theorem 5.2

$$\lim_{\delta \to 0} \frac{-\lambda_{k,\delta}}{\delta^2} \leq \frac{-\lambda_k}{4}.$$

We assume the (normalised) eigenfunctions of R to have bounded second derivatives. Let $\phi_1, \cdots, \phi_n, \cdots$ be orthonormal eigenfunctions of R, corresponding to eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq \cdots$ respectively. Consider the vectorspace E generated by the first k eigenfunctions $\phi_1, \cdots, \phi_k$.

Definition 5.2 *Given a function f on R, we define its projection $\rho_\delta(f)$, a function on the nodes of $\tilde{G_\delta}$, by the formula $\rho_\delta(f)(a) = f(x,y)$ if a has coordinates (x,y). If a lies outside R, we draw a normal from a to ∂R at say (x_1,y_1), and $\rho_\delta(f)(a) = f(x_1,y_1)$. If the normal is not unique, we take the one among those of minimum length that has the maximum slope.*

For sufficiently small δ, the map ρ_δ is injective. Indeed, consider the unit sphere $S(E)$ of all elements in E of norm 1, The magnitudes of the second derivatives for any function $f \in S(E)$ are bounded above by some positive real M, since each is a finite linear combination of $\phi_1, \cdots, \phi_k$ with coefficients less or equal to i. $\max_{(x,y)\in R} |f(x,y)| \geq 1/\sqrt{|R|}$. The partial derivatives of f at this point would be 0. This holds even if (x,y) belongs to ∂R, as then the normal derivative is 0 due to Neumann boundary conditions. Thus it is clear that one can obtain a ball centered around (x,y) with radius in terms of M and $|R|$ (that is the area of R) in which f is non-zero. Therefore as δ becomes small enough, we would have a vertex of $\tilde{G_\delta}$ in it, which implies that $\rho_\delta(f) \neq 0$. Therefore ρ_δ is injective. Further, since the second partial derivatives are bounded, it can be seen that

$$\lim_{\delta\to 0} \frac{4Q(\rho_\delta(f))}{\delta^2} = Q(f).$$

for any f in E. $\dim \rho_\delta(E) = k$ for δ sufficiently small. Theorem 5.1 now implies that

$$\lim_{\delta\to 0} \frac{-4\lambda_{k,\delta}}{\delta^2} \leq -\lambda_k.$$

Let $\phi_{1,\delta}, \cdots, \phi_{n,\delta}, \cdots$ be orthonormal eigenfunctions of R, corresponding to eigenvalues $\lambda_{1,\delta} \geq \cdots \geq \lambda_{n,\delta} \geq \cdots$ respectively. By orthonormal we mean

1. $\sum_a d_a \phi_{i,\delta}(a)^2 = 1$.
2. If $i \neq j$, $\sum_a d_a \phi_{i,\delta}(a)\phi_{j,\delta}(a) = 0$.

Consider the vectorspace E_δ generated by the first k eigenfunctions $\phi_{1,\delta}, \cdots, \phi_{k,\delta}$.

Definition 5.3 *Given a function f_δ on G_δ, we define its projection $\tilde{\rho}_\delta(f_\delta)$, a function on R, by the formula $\tilde{\rho}_\delta(f)(p) = \alpha f_\delta(a) + \beta f_\delta(b) + \gamma f_\delta(c)$ where the points a, b and c are nearest to p and $p = \alpha a + \beta b + \gamma c$, $\alpha + \beta + \gamma = 1$.*

Our objective is to show that

$$\lim_{\delta \to 0} \sup_{f_\delta \in E_\delta} \frac{4Q(f_\delta)}{\delta^2} - Q(\tilde{\rho}_\delta(f_\delta)) \geq 0.$$

It can be seen that

$$Q(\sum_i \alpha_i \phi_{i,\delta}) = -\sum_i \alpha_i^2 \lambda_{i,\delta}.$$

Therefore it is sufficient to prove that

$$\lim_{\delta \to 0} \frac{4Q(\phi_{i,\delta})}{\delta^2} - Q(\tilde{\rho}_\delta(\phi_{i,\delta})) \geq 0; i \in \{1, \cdots, k\}.$$

Consider a single equilateral triangle $\triangleright$ of side δ, a subgraph of $\tilde{G}_\delta$. Let the values that $\phi_{i,\delta}$ takes on its corners be a, b and c.

$$Q(\tilde{\rho}_\delta(\phi_{i,\delta})|_\triangleright) = \frac{\sqrt{3}/3(a^2 + b^2 + c^2 - ab - bc - ca)}{\delta^2 \sqrt{3}/24(a^2 + b^2 + c^2 + ab + bc + ca)}.$$

$$Q(\phi_{i,\delta})) = \frac{\sum_{a \leftrightarrow b} (\phi_{i,\delta}(a) - \phi_{i,\delta}(b))^2}{\sum_a d_a \phi_{i,\delta}(a)^2}.$$

We now make an assumption about the sum of the squares of the values of $\phi_{\delta,i}$ on the boundary of $\tilde{G}_\delta$, namely that it tends to 0 as δ tends to 0. We indicate why this is reasonable in Theorem 5.3.

Lemma 5.1

$$\lim_{\delta \to 0} \frac{\sum_{a \leftrightarrow b \in \tilde{G}_\delta} \phi_{i,\delta}(a) \phi_{i,\delta}(b)}{\sum_{a \in \tilde{G}_\delta} d_a \phi_{i,\delta}(a)^2} = 1/2.$$

$\sum_{b,b\leftrightarrow a}(\phi_{i,\delta}(a)-\phi_{i,\delta}(b))=-d_a\lambda_{i,\delta}(a)\phi_{i,\delta}(a)$. Therefore,

$$\sum_{a\leftrightarrow b\in\tilde{G}_\delta} 2\phi_{i,\delta}(a)\phi_{i,\delta}(b)-\sum_{a\in\tilde{G}_\delta} d_a\phi_{i,\delta}(a)^2=d_a\lambda_{i,\delta}(a)\phi_{i,\delta}(a)^2.$$

However,

$$\lim_{\delta\to 0}\frac{-4\lambda_{i,\delta}}{\delta^2}\leq-\lambda_i.$$

So,

$$\lim_{\delta\to 0}\lambda_{i,\delta}=0,$$

which from equation 1 proves the lemma. The above lemma together with the assumption of "insignificance of boundary values" in the limit imply that

$$\lim_{\delta\to 0}\frac{4Q(\phi_{i,\delta})}{\delta^2}-Q(\tilde{\rho}_\delta(\phi_{i,\delta}))\geq 0; i\in\{1,\cdots,k\}.$$

It can be seen that the piecewise interpolated function $\tilde{\rho}_\delta(\phi_{i,\delta})$ lies in the closure mentioned in the minimax theorems by an argument involving "partitions of unity" of the domain R.. Therefore by the minimax principles,

$$\lim_{\delta\to 0}\frac{-4\lambda_{k,\delta}}{\delta^2}\geq-\lambda_k.$$

As the reverse inequality has already been proved, we have

$$\lim_{\delta\to 0}\frac{4\lambda_{k,\delta}}{\delta^2}=\lambda_k.$$

Theorem 5.3 *Let $\phi_{i,\delta}$ be the i^{th} eigenfunction of $\tilde{G}_\delta$. Let C_δ be the "contour of boundary points", i.e those with degree less than* 6. *Let $\mu>|\lambda_{i,\delta}|$. Then*

$$\lim_{\delta\to 0}\frac{\sum_{a\in C_\delta} d_a\phi_{i,\delta}^2}{\sum_{a\in\tilde{G}_\delta} d_a\phi_{i,\delta}^2}=0,$$

provided the following is true.

$$\lim_{\mu\to 0}\lim_{\delta\to 0} P_{\delta,\mu}=0$$

where $P_{\delta,\mu}$ is the probability that a random walk of length N on $\tilde{G}_\delta$, starting at any point ends on the boundary, N being a random variable having the geometric distribution $P(N = n) = \mu(1-\mu)^n$.

We shall show that

$$\phi_{i,\delta}(a) < 2 \times \sum_{b \in \tilde{G}_\delta} \alpha_{a,\delta}(b)\phi_{i,\delta}(b),$$

for a certain probability measure $\alpha_{a,\delta}$.

$$\frac{\phi_{i,\delta}(a)^2}{4} < (\sum_{b \in \tilde{G}_\delta} \alpha_{a,\delta}(b)\phi_{i,\delta}(b))^2 \leq \sum_{b \in \tilde{G}_\delta} \alpha_{a,\delta}(b)\phi_{i,\delta}(b)^2,$$

as $E[\phi_{i,\delta}(b)^2] \geq E[\phi_{i,\delta}]^2$ when b is chosen according to the probability distribution $\alpha_{a,\delta}(b)$. Adding up over points in the boundary, we would have

$$\sum_{a \in C_\delta} d_a \phi_{i,\delta}(a)^2 < 4 \sum_{b \in \tilde{G}_\delta} (\sum_{a \in C_\delta} \alpha_{a,\delta}(b))\phi_{i,\delta}(b)^2,$$

We show that given arbitrary $\epsilon > 0$, it is possible to choose δ small enough so that for any $k < \delta$

$$\max_{b \in \tilde{G}_k} (\sum_{a \in C_k} \alpha_{a,k}(b)) < \epsilon.$$

It clearly follows that

$$\frac{\sum_{a \in C_k} d_a \phi_{i,k}^2}{\sum_{a \in \tilde{G}_k} d_a \phi_{i,k}^2} < 4\epsilon,$$

which would complete the proof. In order to define $\alpha_{a,\delta}(b)$, consider a random walk that starts at a on $\tilde{G}_\delta$ that is terminated after the n^{th} step with probability $(1-\mu)^{n-1}\mu$, for some $\mu > -\lambda_{i,\delta}$. Let $\alpha_{a,\delta}(b)$ be defined to be the probability that this walk ends at b.

Lemma 5.2

$$\phi_{i,\delta}(a) < 2 \sum_{b \in \tilde{G}_\delta} \alpha_{a,\delta}(b)\phi_{i,\delta}(b).$$

Indeed, consider the random variables X_i and X to be the values of $\phi_{i,\delta}$ on the i^{th} vertex of a random walk starting on a, and the final vertex of the walk respectively. We then have $E[X_i|X_{i-1}] = X_{i-1}/(1-\lambda_{i,\delta})$. Taking expected values on either side repeatedly, we have

$$E[X_i] = \frac{E[X_0]}{(1-\lambda_{i,\delta})^i}.$$

$E[X] = \sum_{i\geq 0} \mu(1-\mu)^{i-1}E[X_i]$, which yeilds

$$E[X] = \frac{E[X_0](1-\lambda_{i,\delta})}{1-\lambda_{i,\delta}/\mu},$$

$$E[X] > \frac{E[X_0](1-\lambda_{i,\delta})}{2} > \frac{E[X_0]}{2}.$$

$E[X_0] = X_0$, which is $\phi_{i,\delta}(a)$. This completes our proof as $E[X] = \sum_{b\in\tilde{G}_\delta} \alpha_{a,\delta}(b)\phi_{i,\delta}(b)$.

We now only need to prove the following - given $\epsilon > 0$, it is possible to find δ such that $k < \delta$ implies that $\sum_{a\in C_k} \alpha_{a,k}(b) < \epsilon$ for all $b \in \tilde{G}_k$. Observe, that since $\alpha_{a,k}(b)$ is the probability that a "μ-damped" random walk starting at a terminates at b,

$$\alpha_{a,k}(b) = \frac{\sum_{\tilde{b}\leftrightarrow b} \alpha_{a,k}(\tilde{b})(1-\mu)}{d_b} + \delta_{ab}\mu,$$

where δ_{ab} is the Kronecker delta function, equal to 1 if $a = b$ and 0 otherwise. From the above equation it is clear that if neither b nor any of its neighbours is on the boundary C_k,

$$\sum_{a\in C_k} \alpha_{a,k}(b) = (1-\mu)\sum_{\tilde{b}\leftrightarrow b} \frac{\sum_{a\in C_k} \alpha_{a,k}(\tilde{b})}{6}.$$

Thus, $\max_{b\in\tilde{G}_k}(\sum_{a\in C_k} \alpha_{a,k}(b))$ occurs at a point b on the boundary C_k or adjacent to it. It is clear from the reversible nature of the random walk that

$d_a \alpha_{a,k}(b) = d_b \alpha_{b,k}(a)$. Let b be the point at which the above maximum is achieved. Then

$$\sum_{a \in C_k} \alpha_{a,k}(b) = \sum_{a \in C_k} \frac{d_b}{d_a} \alpha_{b,k}(a)),$$

which is less than $6 \sum_{a \in C_k} \alpha_{b,k}(a)$. It is sufficient to prove that there exists a δ so that $k < \delta$ implies $\sum_{a \in C_k} \alpha_{b,k}(a) < \epsilon/6$, which proves the theorem.

2 Experimental results

We present some plots that show the eigenvalues of graphs obtained by taking successively finer grids on the 2 dimensional torus $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$. It is observed that in each case, $\lambda_{i,\delta}/\delta^2$ is less than λ_i, thus the estimated eigenvalues undershoot the actual ones. In Figure 5.3, which is a plot of

$$(4\lambda_{i,\delta}/\delta^2 - \lambda_i)/\delta^2.$$

against i, we see that as $\delta \to 0$, the curves increase monotonically to a fixed curve. Thus convergence is seen to be quadratic, in the sense that the error is $O(\delta^2)$.

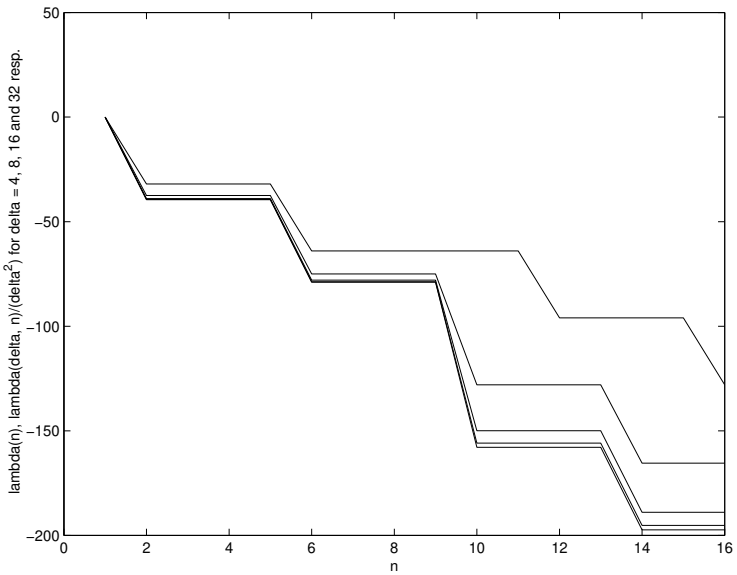

Figure 5.2: The scaled eigenvalues of a square grid approximating the torus $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$.

From top to bottom, the following graphs have been displayed in figure 5.2.

1. $4\lambda_{\delta,n}/\delta^2$ vs n for $\delta = 1/4$.
2. $4\lambda_{\delta,n}/\delta^2$ vs n for $\delta = 1/8$.
3. $4\lambda_{\delta,n}/\delta^2$ vs n for $\delta = 1/16$.
4. $4\lambda_{\delta,n}/\delta^2$ vs n for $\delta = 1/32$.
5. λ_n vs n.

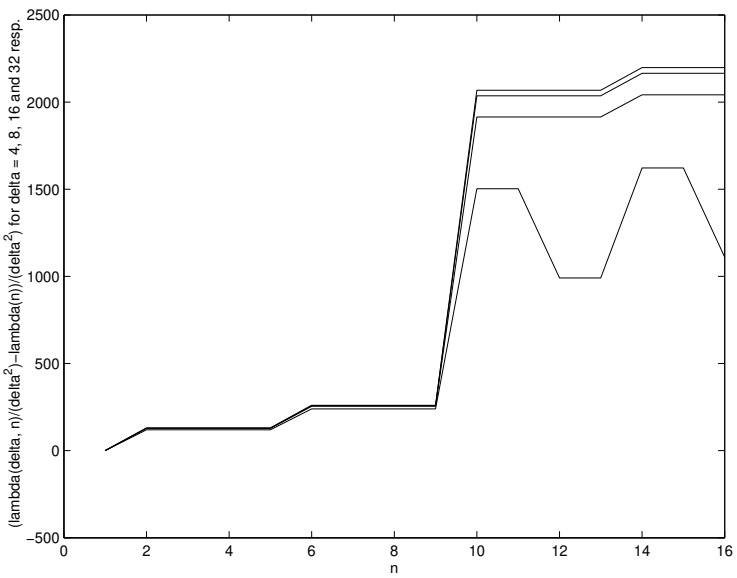

Figure 5.3: The scaled errors in the eigenvalues of the grid approximation.

From bottom to top, the following graphs have been displayed in Figure 5.3.

1. $(4\lambda_{\delta,n}/\delta^2 - \lambda_n)/\delta^2$ vs n for $\delta = 1/4$.
2. $(4\lambda_{\delta,n}/\delta^2 - \lambda_n)/\delta^2$ vs n for $\delta = 1/8$.
3. $(4\lambda_{\delta,n}/\delta^2 - \lambda_n)/\delta^2$ vs n for $\delta = 1/16$.
4. $(4\lambda_{\delta,n}/\delta^2 - \lambda_n)/\delta^2$ vs n for $\delta = 1/32$.

Chapter 6

Conjectures on the Spectrum of a Riemannian Manifold

1 Two conjectures

In chapter 2 we have derived some results relating the spectrum of a combinatorial graph and certain parameters related to equivalent conductance as well as random walks (see Theorem 2.3). In this section we conjecture analogues of them for compact Riemannian manifolds.

Let P_i, $i \in \mathbf{N}$ be independent uniformly distributed random points on R, (each having probability distribution dictated by the Lebegue measure on R). Let

$$a_{n+1}(\epsilon) = E[B(\epsilon, P_0), B(\epsilon, P_1), \cdots, B(\epsilon, P_n)].$$

We define $\lambda_n(\epsilon)$ to be the n^{th} zero by magnitude of the power series

$$A(\epsilon, \lambda) = \sum_{i \geq 1} \frac{a_i(\epsilon)\lambda^i}{i!}. \tag{6.1}$$

As a matter of fact, for each $\epsilon > 0$, the above power series is a polynomial, since all coefficients of large enough exponent of λ turn out to be 0. If n is larger than the degree of $A(\epsilon, \lambda)$, we define $\lambda_n(\epsilon)$ to be $-\infty$.

Conjecture 6.1 $\lim_{\epsilon \to 0} \lambda_n(\epsilon) = \lambda_n$, *where* λ_n *is the* n^{th} *Neumann eigenvalue of* R.

In the discrete domain, Theorem 2.3 relates the spectrum of the discrete Laplacian on a graph to some quantities involving random walks. Conjecture 6.2 is an analogue of this for continuous spaces.

Let M be a compact d-dimensional Riemannian Manifold with volume V. For a set of points b on M, let $Vol_\epsilon(b)$ denote the volume of the ϵ-neighbourhood of b. (which is the volume of the set of all points at geodesic distance less or equal to ϵ from some point in b). Let $t_1, \cdots, t_n, \cdots$ be an infinite sequence of independent exponentially distributed random variables, each with expected value $-1/\lambda$ for some negative real λ. Let $w_1, \cdots, w_n, \cdots$ be the image sets of independent brownian motions, w_i having starting point chosen with the uniform distribution on M and lasting time t_i. Let b_k be the union of $w_1, \cdots, w_k$. For a small ϵ consider all λ for which,

$$E[\prod_i^\infty (2Vol_\epsilon(b_k)/V - 1)] = 0.$$

Define $\tilde{\lambda}_i(\epsilon)$ to be the i^{th} such λ.

Conjecture 6.2

$$lim_{\epsilon \to 0} \tilde{\lambda}_i(\epsilon) = \lambda_i.$$

2 The spectrum of S^1

We verify conjecture 6.1 in the case of the circle S^1. For us S^1 shall mean a circle with perimeter 1. For a point $P \in S^1$ and $\epsilon < 1/2$, $B(\epsilon, P)$ is the open

interval of length 2ϵ symmetric about P. Consider

$$a_{n+1}(\epsilon) = E[B(\epsilon, P_0), B(\epsilon, P_1), \cdots, B(\epsilon, P_n)].$$

In the 1-dimensional case, $\lim_{\epsilon \to 0} a_n(\epsilon) = a_n(0)$ is a finite real number, unlike in dimensions ≥ 2.

Proposition 6.1 *Let $P_0, \cdots, P_n$ be $n+1$ points on S^1, taken in clockwise order. Then,*

$$[P_0, \cdots, P_n] = \prod_{i=0}^{n} |P_i P_{i+1}|,$$

where P_{n+1} is interpreted as P_0.

We proceed by induction. For $n = 2$ this is $P_0P_1 \times P_1P_0/(P_0P_1 + P_1P_0)$ where each arc is taken clockwise. This is $P_0P_1 \times P_1P_0$ as we assume the circumference to be 1. (For $n = 1$ we take the value to be the volume of S^1 for reasons of consistancy, which is 1.) Assume the proposition for $n = m-1$.

$$[P_0, \cdots, P_n] = [P_0, \cup_{i=1}^{n} P_i][P_1, \cdots, P_n].$$

$$[P_0, \cup_{i=1}^{n} P_i] = \frac{|P_0P_1||P_nP_0|}{|P_nP_1|},$$

from the formula for the equivalent resistance of two parallel resistors. Substituting this, we obtain the desired result.

Thus

$$a_{n+1}(0) = E[\prod_{i=0}^{n} |P_i P_{i+1}|].$$

Proposition 6.2

$$a_n(0) = \frac{(n-1)!}{(2n-1)!}.$$

We map S^1 onto the unit interval $[0,1)$, preserving length, so that P_0 is mapped onto 0. We calculate $a_n(0)$ by integrating over the position of P_1.

$$\begin{aligned} a_n(0) &= \int_0^1 (n-1)(1-x)^{n-2}x(1-x)^{n-1} \times a_{n-1}(0)dx \\ &= (n-1)a_{n-1}(0)\int_0^1 (1-x)^{2n-3}xdx \\ &= \frac{n-1}{(2n-2)(2n-1)} \end{aligned}$$

$a_1(0) = 1$, therefore $a_n(0) = 2(n!)/2n!$.

Proposition 6.3 *If* $n\epsilon \geq 1$, $a_n(\epsilon) = 0$. *Otherwise*

$$a_n(\epsilon) > (1 - \prod_{i=1}^{n-1}(1-4\epsilon i))(1-2\epsilon n)^n a_n(0).$$

Let $P_0, \cdots, P_n$ be n points on S^1 chosen randomly. With probability greater than $\prod_{i=1}^{n}(1-4\epsilon i)$, the n neighbourhoods $B(P_i,\epsilon)$ are mutually disjoint, as can be seen by successive addition. Therefore

$$E[B(P_0,\epsilon),\cdots,B(P_n,\epsilon)|B(P_i,\epsilon)\text{disjoint}] > (1-2\epsilon(n))^n a_n(0).$$

Since given that the intervals $B(P_i,\epsilon)$ are disjoint, the expected value of $[B(P_0,\epsilon),\cdots,B(P_{n-1},\epsilon)]$ is $(1-2n\epsilon)^n E[B(P_0,0),\cdots,B(P_{n-1},0)] = (1-2n\epsilon)^n a_n(0)$.

$\lim_{\epsilon\to 0} a_n(\epsilon) = a_n(0)$

$\forall\lambda \in \mathbf{C}$,

$$\sum_{n=1}^{\infty} |\frac{a_n(0)\lambda^n}{n!}| = 2\cosh(|\sqrt{\lambda}|) - 2.$$

is convergent. Therefore it follows that

$$\lim_{\epsilon\to 0} A(\epsilon,\lambda) = A(0,\lambda),$$

where

$$A(\epsilon, \lambda) = \sum_{n=1}^{\infty} \frac{a_n(\epsilon)\lambda^n}{n!}.$$

$$A(0, \lambda) = \sum_{i \geq 1} \frac{2\lambda^n}{2n!} = 2(\cosh(\sqrt{\lambda}) - 1).$$

Thus, the zeroes of the above power series are given by $\sqrt{\lambda} = 2n\Pi\imath$.

$$\lambda = -4n^2\Pi^2,$$

$\forall n \in \mathbf{N} \cup \{0\}$ For each $n \in \mathbf{N}$, this is a double zero, while for $n = 0$ this is a simple zero. We know that an orthonormal set of solutions to

$$\nabla^2 f = \lambda f$$

on $\mathbf{R}/\mathbf{Z}$ is given by $\sqrt{2}\sin 2n\Pi x$, $\sqrt{2}\cos 2n\Pi x$ correspond to $\lambda = -4n^2\Pi^2$ for $n \in \mathbf{N}$, and the constant function 1 corresponding to $\lambda = 0$. Thus the statement of conjecture 6.1 has been verified for S^1.

Chapter 7

Conclusions

In this report we explored some connections between Green's functions and the spectrum of the Laplacian. In the discrete domain, Theorem 2.3 relates the spectrum of the discrete Laplacian on a graph to some quantities involving random walks. Conjecture 6.2 is an analogue of this for continuous spaces. In Chapter 2 a new bracket $[a_1, \cdots, a_n]$ was introduced where a_i are boundary arcs of a region R in $\mathbf{R^2}$. This was shown to be permutation invariant in Theorem 4.1. A duality result $[a_1, \cdots, a_n][b_1, \cdots, b_n] = 1$ was proved in the case that R is simply connected. In the above formula, the arcs $a_1, b_1, \cdots, a_n, b_n$ form the boundary ∂R taken counter-clockwise. Conjecture 6.1 is about the polynomial $\sum_i E[a_1, \cdots, a_i]\lambda^i/i!$, where the a_i are the boundaries of random ϵ-balls cut out of R. It states that as ϵ tends to 0, the roots of this polynomial tend to the eigenvalues of the Laplacian on R under Neumann conditions. In Chapter 5 we attempted to prove the convergence of (scaled) eigenvalues of the discrete Laplacian of approximating graphs to the Neumann spectrum of R. This was acheived under certain assumptions on the structure near the boundary of the approximating graphs (see Theorem 5.3). With proper results on convergence of eigenvalues from discrete to

continuous cases, many results for continuous spaces can be proved by first proving them for their discrete counterparts. It seems as though both conjectures 6.1 and 6.2 could be proved if appropriate theorems on convergence were available. From experimental results presented in Chapter 5, Section 2, it appears that on making a grid have "mesh" δ, the error in calculated eigenvalues is $O(\delta^2)$. Conjecture 6.1 has been verified for the circle S^1 in the final section of this report.

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