Performance Analysis of Hybrid Estimation Algorithms *

Inseok Hwang, Hamsa Balakrishnan, and Claire Tomlin
Hybrid Systems Laboratory
Department of Aeronautics and Astronautics
Stanford University, Stanford, CA 94305
ishwang, hamsa, tomlin@stanford.edu

Abstract

In this paper, we analyze the performance of estimation algorithms for discrete-time stochastic linear hybrid systems. The problem of being able to estimate both the discrete and continuous states of a hybrid system given only the continuous output sequence is a difficult one, and while algorithms [1, 2] exist for this purpose, little has been proved on the limitations of these algorithms, or even the dependence of their performance on system parameters. We find conditions that must be satisfied to guarantee the convergence of these hybrid estimation algorithms. We also derive expressions to determine bounds on the discrete mode detection delay. These conditions also provide a means to predict *a priori* which transitions in a hybrid system are relatively easy to detect, as a function of the system parameters. Finally, we validate our conditions and predictions using a simple yet illustrative 1-D example, and a more complex aircraft tracking example.

KEYWORDS: hybrid systems, estimation, mode transitions.

1 Introduction

Many complex systems can be modeled by hybrid systems with a number of discrete modes having different continuous dynamics and discrete transition relations between the modes. There has been considerable interest in hybrid estimation among researchers in estimation theory. The objective of hybrid estimation is to estimate both the mode and the continuous state of a hybrid system at any given time. Hybrid estimators usually consist of the combination of a bank of continuous state estimators (usually Kalman filters), designed for the different discrete modes, and a hypothesis testing (mode selecting) algorithm. The form of the hypothesis testing algorithm depends on the type of output data available. The class of hybrid estimators analyzed in this paper addresses a challenging problem - that of mode detection and state estimation given only the continuous output data of the system. In such cases, during hypothesis testing, it is necessary to use the differences in

*This research is supported by ONR under MURI contract N00014-02-1-0720 and by an NSF Career Award (ECS-9985072). H. Balakrishnan is supported by a Stanford Graduate Fellowship.

statistical properties (such as mean and covariance) of the outputs from the different Kalman filters to choose the most likely mode.

Hybrid estimation algorithms have been developed for discrete-time stochastic linear hybrid systems in which the discrete transitions are governed by finite-state Markov chains. The Multiple Model Adaptive Estimation Algorithm (MMAE) [1] is an algorithm in which, during hypothesis testing, the residuals of the different Kalman filters are used to form likelihood functions for the different modes, which in turn are used as *adaptive weights* to find the most probable mode. A refinement of this algorithm, the Interacting Multiple Model Algorithm (IMM) [2], mixes the initial conditions of each of the Kalman filters at each time-step. The original reason behind this refinement was to enable keeping information from N^2 discrete time histories (where N is the number of discrete modes) in just N Kalman filters, thus reducing the complexity. [1, 2, 3, 4] and references therein give various other hybrid estimation algorithms and their applications.

The performance of hybrid estimators has been studied for several decades. Magill [5] provides sufficient conditions for the convergence of the adaptive weights described above in the hybrid estimation of a specific class of systems when a constant parameter vector is unknown and there is a single output from all the modes. Lainiotis et al. [6, 7, 8] extend the results in [5] to multiple outputs and derive the recursive form of the optimal adaptive estimator as well as its exact error covariance. Hawkes et al. [4] examine the asymptotic behavior of the adaptive weights in hybrid estimation algorithms, which in turn determines the performance of the hybrid estimators. The authors show, using the Kullback information function [9], that the weight corresponding to the true model converges almost surely to unity and that the other weights converge to zero. Many other approaches to the performance analysis of adaptive estimation (hybrid estimation in which the hypothesis testing is performed using adaptive weights) can be found in the references cited in [4]. Baram et al. [10] provide conditions under which, for a set of systems driven by stationary white Gaussian inputs and no discrete transitions, the mode probability of the true model converges to unity, i.e,. the probability that the estimated model is the true model converges asymptotically to unity. Baram [11] shows that for the hybrid system in [10], the uniqueness of the prediction error covariance matrix is a sufficient condition for the true model to be estimated asymptotically. However, the system in [10, 11] is a set of stochastic, stationary Gaussian models which do not interact with each other. Thus, the conditions in [10, 11] are more relevant to the observability of stochastic linear hybrid systems [12]. Caputi [13] derives a necessary condition for the performance of hybrid estimation algorithms through the analysis of steady state residuals and shows that the performance of the hybrid estimator depends on the DC gain of the linear systems. This condition is only valid for a specific class of hybrid systems, in which the continuous dynamics for all the modes is the same, but the inputs are distinct and consist of a constant bias vector and zero-mean white Gaussian random noise.

The research summarized above analyzes hybrid estimation in several special classes of systems, yet general analytical analysis techniques for the performance of hybrid estimation algorithms have not been investigated in detail. Maybeck [1] gives qualitative reasons for the performance of hybrid estimators but adds that no rigorous general proofs are available for the (asymptotic convergence) properties of the hypothesis conditional probabilities. In this paper, we analyze the properties of hybrid estimation algorithms and derive conditions under which the computed hybrid estimates converge exponentially to the exact hybrid states. The results of this analysis give some insight into which mode transitions are more detectable than others and also into how to improve the performance of hybrid estimators. (We say that a mode transition is *more detectable* than another if the time taken for the mode estimate to converge to the true mode is less for the former transition than for the latter). We then compare the performance of the Multiple Model Adaptive Estimation (MMAE) algorithm with that of the Interacting Multiple Model (IMM) algorithm which has been widely (and successfully) used in the area of multiple target tracking. We show analytically why the IMM algorithm has better performance than the MMAE.

This paper is organized as follows: Section 2 presents the structure of a stochastic hybrid system and the probabilistic analysis of its sojourn time, as motivation for this research. In Section 3, we describe a generic hybrid estimation algorithm, which could be the MMAE, the IMM, or any other refinement of the basic hybrid estimation algorithm. In Section 4, we analyze the performance of this hybrid estimation algorithm and derive upper bounds for the mode estimation delay, as well as conditions that guarantee instant mode estimation and exponential convergence of hybrid estimation algorithms. We begin with a steady-state analysis in Section 4.1 to derive necessary conditions for mode detection. We then proceed to perform a transient analysis in Sections 4.2, 4.3 and 4.4 to find bounds on the mode detection delay, as well as sufficient conditions for instantaneous mode detection. Section 5 presents a comparison of the performance of different hybrid estimation algorithms. Illustrative examples and conclusions are presented in Sections 6 and 7 respectively.

2 Discrete-time stochastic linear hybrid systems

We consider a discrete-time stochastic linear hybrid system [14]

$$H : \left\{ \begin{array}{rcl} x(k+1) & = & A_i x(k) + B_i u(k) + w_i(k) \\ z(k) & = & C_i x(k) + v_i(k) \end{array} \right. , \ k \in \mathbb{N} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^l$ and $z \in \mathbb{R}^p$ are the continuous state, control input, and output variables respectively. The index $i \in \{1, 2, \cdots, N\}$ requires the discrete state whose evolution is governed by the finite state Markov chain

$$\mu(k+1) = \Pi \mu(k) \tag{2}$$

where $\Pi = \{\pi_{ij}\} \in \mathbb{R}^{N \times N}$ is the *mode transition matrix* and $\mu(k) \in \mathbb{R}^N$ is the *mode probability* at time k. The system matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times l}$, and $C_i \in \mathbb{R}^{p \times n}$ for $i \in \{1, 2, \cdots, N\}$ are assumed known. We denote the covariance of the initial state $x(k_0)$ as $\pi_0 \in \mathbb{R}^N$, and assume that the process noise $w_i(k)$ and the measurement noise $v_i(k)$ are uncorrelated, zero-mean white Gaussian sequences with the covariance matrices $E[w_i(k)w_i(k)^T] = Q_i \in \mathbb{R}^{n \times n}$ and $E[v_i(k)v_i(k)^T] = R_i \in \mathbb{R}^{p \times p}$ respectively. $E[\cdot]$ and $(\cdot)^T$ denote expectation and matrix transpose respectively. It is assumed that $w_i(k)$ and $v_i(k)$ are both uncorrelated with the initial state, i.e., $E[x(k_0)w_i(k)^T] = E[x(k_0)v_i(k)^T] = 0$. We define $Z(k) = \{z(0), \cdots, z(k)\}$ as the measurement sequence up to time k. Since the state evolution of a hybrid system has continuous trajectories as well as discrete jumps, we define a hybrid time trajectory:

Definition 1 *(Hybrid time trajectory) A hybrid time trajectory is a sequence of intervals*

$$[k_0, k_1 - 1][k_1, k_2 - 1] \cdots [k_i, k_{i+1} - 1] \cdots$$

where $k_i (i \geq 1)$ is the time at which the i-th discrete state transition occurs.

In this paper, by 'exponential convergence of a hybrid estimator' we mean:

Definition 2 *(Exponential convergence of a hybrid estimator) Given a hybrid system H with N discrete modes, we say that a hybrid estimator is exponentially convergent if its discrete state estimate exhibits correct identification of the discrete-state transition sequence of the original system after a finite number of steps; and at any given time, the continuous state estimate is unique, with an estimation error mean which converges exponentially to zero.*

2.1 Sojourn time analysis for stochastic linear hybrid systems

In [12], we derived a condition under which hybrid estimation algorithms converge exponentially to the true hybrid state given the maximum mode estimation delay and the minimum sojourn time. If the minimum sojourn time ($\tilde{\Delta} = \min_i \tilde{\Delta}_i$ where $\tilde{\Delta}_i$ is the sojourn time at mode i) satisfies the following condition, the hybrid estimation algorithms converge exponentially [12]:

$$\tilde{\Delta} > \beta + \delta \tag{3}$$

where β is the time between mode changes required for the convergence of the Kalman filters and δ is the maximum mode estimation delay, i.e. $\delta = \max_i \delta_i$ where δ_i is the mode estimation delay in correctly detecting a transition into mode i. Intuitively, until the correct mode is detected after a transition, the estimator is mismatched and its error diverges (for a time δ in the worst case) and therefore it needs enough time (β) in the correct mode for the error to decrease before the next transition occurs. Therefore, $\tilde{\Delta} > \beta + \delta$ is the minimum sojourn time of the system in any mode in order to guarantee exponential convergence of the hybrid estimator. Since the sojourn time of the stochastic linear hybrid system is governed by a finite Markov chain, we cannot compute the minimum sojourn time exactly, rather with some probability. The probability of mode j at time $k + \Delta_i$ given mode i at time k (i.e. no mode transition in $(k, k + \Delta_i)$) is:

$$P(m_j(k + \Delta_i)|m_i(k)) = \pi_{ii}^{\Delta_i - 1}\pi_{ji} \tag{4}$$

where $m_i(k)$ denotes the event that the mode at time k is mode i. Thus, the probability that the sojourn time is greater than or equal to $\tilde{\Delta}_i$ is

$$P(\Delta_i \geq \tilde{\Delta}_i) = \sum_{\Delta_i = \tilde{\Delta}_i}^{\infty} \pi_{ii}^{\tilde{\Delta}_i - 1}(1 - \pi_{ii}) = \pi_{ii}^{\tilde{\Delta}_i - 1} \tag{5}$$

Proposition 1 *Let $\tilde{\Delta} = \min_i \tilde{\Delta}_i$ $(i \in \{1, \cdots, N\})$ be the minimum sojourn time for the stochastic linear hybrid system (1)-(2) and $j = \arg\min_i \tilde{\Delta}_i$. If $\tilde{\Delta} \geq \beta + \delta$, then hybrid estimation algorithms for the stochastic linear hybrid system (1)-(2) can be guaranteed to converge exponentially to the true hybrid state with probability $\pi_{jj}^{\tilde{\Delta}-1}$.*

Proof: The proof follows from (5) and Theorem 3 in [12]. ■

However, the probability in (5) becomes small rapidly as the required minimum sojourn time $\tilde{\Delta}$ becomes large. In this case, Proposition 1 holds with very small probability and we can obtain very little useful information about the performance of hybrid estimation algorithms through sojourn time analysis alone. This observation motivates the present work.

3 Hybrid estimation algorithm

In this section, we consider a generic hybrid estimation algorithm [1] for the discrete-time stochastic linear hybrid system (1)-(2). Following the Bayesian estimation derivation in [1], the state estimate of hybrid estimation is the conditional mean:

$$\hat{x}(k+1) = E[x(k+1)|Z(k+1)] = \int_{-\infty}^{\infty} x(k+1)p(x(k+1)|Z(k+1))dx(k+1) \tag{6}$$

where $p(\cdot|\cdot)$ is the conditional probability density function, given by:

$$p(x(k+1)|Z(k+1)) = \frac{p(x(k+1), Z(k+1))}{p(Z(k+1))} = \frac{\sum_{i=1}^{N} p(x(k+1), Z(k+1), m_i(k+1))}{p(Z(k+1))} \tag{7}$$

Thus, the state estimate (6) is

$$\begin{aligned} \hat{x}(k+1) &= \int_{-\infty}^{\infty} x(k+1) \sum_{i=1}^{N} p(x(k+1)|Z(k+1), m_i(k+1))p(m_i(k+1)|Z(k+1))dx(k+1) \\ &= \sum_{i=1}^{N} \hat{x}_i(k+1)p(m_i(k+1)|Z(k+1)) \end{aligned} \tag{8}$$

where $\hat{x}_i(k+1) = \int_{\infty}^{\infty} x(k+1)p(x(k+1)|Z(k+1), m_i(k+1))dx(k+1)$ is the mode-conditioned state estimate of $x(k+1)$ given $m_i(k+1)$. $\hat{x}_i(k+1)$ is computed by the state estimator matched to mode i. Therefore, the state estimate (8) is a weighted sum of N mode-conditioned state estimates produced by each Kalman filter with the weight $p(m_i(k+1)|Z(k+1))$. The weight can be expressed by

$$\begin{aligned} p(m_i(k+1)|Z(k+1)) &= \tfrac{p(z(k+1)|m_i(k+1), Z(k))p(m_i(k+1)|Z(k))}{p(z(k+1)|Z(k))} \\ &= \tfrac{\Lambda_i(k+1)p(m_i(k+1)|Z(k))}{p(z(k+1)|Z(k))} \end{aligned} \tag{9}$$

where $\Lambda_i(k+1) := \mathcal{N}(r_i(k+1); 0, S_i(k+1))$ is the likelihood function of mode i, $r_i(k+1) = z(k+1) - C_i\hat{x}_i(k+1)$ is the residual produced by the Kalman filter i, $S_i(k+1) \in \mathbb{R}^{p\times p}$ is the corresponding residual covariance, and $\mathcal{N}(a; b, c)$ is the probability at a of a normal distribution with mean b and covariance c. $p(m_i(k+1)|Z(k))$ is the mode probability estimate at time $k+1$. If the mode transitions are governed by a finite Markov chain, the mode probability estimate is

$$\begin{aligned} p(m_i(k+1)|Z(k)) &= \sum_{l=1}^{N} p(m_i(k+1)|m_l(k))p(m_l(k)|Z(k)) \\ &= \sum_{l=1}^{N} \pi_{il} p(m_l(k)|Z(k)) \end{aligned} \tag{10}$$

Thus, the weight (mode probability) (9) is

$$p(m_i(k+1)|Z(k+1)) = \frac{1}{c(k+1)} \Lambda_i(k+1) \sum_{l=1}^{N} \pi_{il} p(m_l(k)|Z(k)) =: \mu_i(k+1) \tag{11}$$

where $c(k+1)$ is a normalization constant. The mode estimate at time k is chosen to be the mode which has the maximum mode probability at that time. The mode probability depends not only on the finite Markov chain but also on the likelihood produced by each Kalman filter. The sojourn time analysis based on a finite Markov chain in Section 2.1, takes into account only the discrete dynamics of the hybrid system, and suggests that the typical sojourn time in any mode is very small. However, we know that the assumption that a finite Markov chain models the discrete dynamics well is not very realistic, since most physical hybrid dynamical systems have longer sojourn times. We therefore need to incorporate knowledge from the continuous dynamics (obtained through the likelihood functions of the Kalman filters) while computing the mode probability. Thus, the accuracy of the mode probability is affected greatly by the likelihood function. The state estimate (8) is

$$\hat{x}(k+1) = \sum_{i=1}^{N} \hat{x}_i(k+1)\big[\frac{1}{c(k+1)}\Lambda_i(k+1)\sum_{l=1}^{N}\pi_{il}p(m_l(k)|Z(k))\big] \tag{12}$$

(11)-(12) is referred to as the Multiple Model Adaptive Estimation (MMAE) algorithm [1]. In the MMAE, all individual Kalman filters run independently at every time step (which is different from the Interacting Multiple Model (IMM) that will be described next). Equation (12) shows that the state estimate depends on the likelihood function, the performance of the hybrid estimator thus greatly depends on the behavior of the likelihood function.

We now describe the general structure of the IMM algorithm [2]. The IMM has the same structure as the

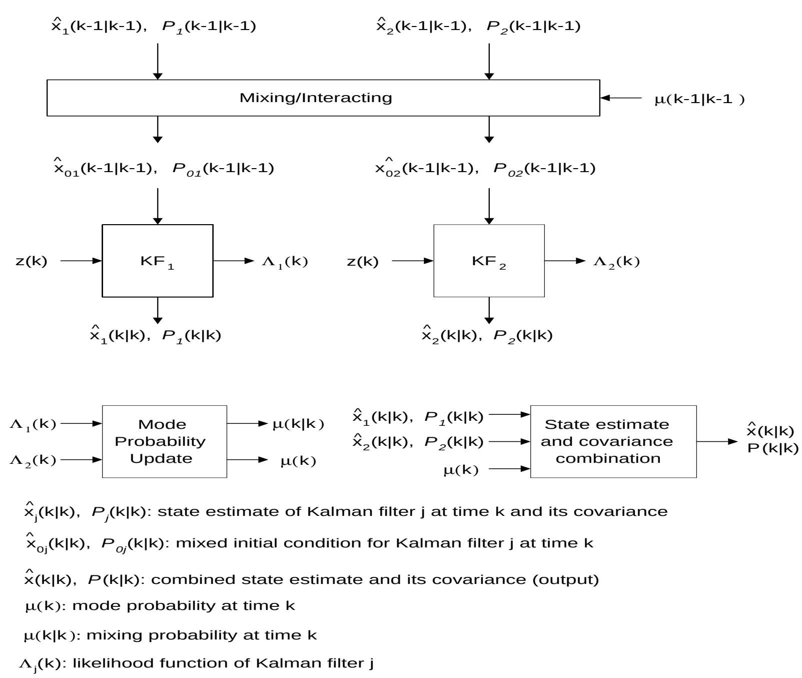


Figure 1: Structure of the IMM algorithm (for two modes) from [15]

MMAE except that it has the *Mixing/Interacting* step at the beginning of the estimation process so that the initial conditions for each Kalman filter are adjusted at the beginning of every time step, using a weighted sum of the state estimates from the previous time step, as shown in Figure 1. The optimal hybrid estimator which minimizes the mean-square estimation error has to keep track of all the mode histories up to the current time, and the number of such histories grows exponentially with time. The optimal estimator is therefore impossible to implement in practice. The IMM is a suboptimal algorithm which, at each time step, keeps information from N^2 mode histories (where N is the number of modes) in just N mode histories by mixing the N mode histories of the previous time step into inputs to the estimators that produce the N mode histories of the current time step. Its complexity involves just N discrete histories, but its performance is close to that of more complex algorithms that keep N^2 mode histories [2].

4 Performance analysis for hybrid estimation

In this section, we first analyze the performance of the hybrid estimation algorithm (either the MMAE or the IMM)) by analyzing the steady-state mean residuals. Since the steady-state analysis gives only necessary conditions on the performance of hybrid estimation, we then analyze the transient behavior of mode probabilities, which are functions of the likelihoods and therefore the residuals.

4.1 Analysis for the steady-state mean residuals

Motivated by Caputi [13], we derive the steady-state mean residual for each mode i for the hybrid system (1)-(2). Using the notations in [16], we define the following quantities:

$$\begin{array}{rcl} \Delta A_i & := & A_T - A_i \\ \Delta B_i & := & B_T - B_i \\ \Delta C_i & := & C_T - C_i \\ \Delta u_{i_{ss}} & := & u_{T_{ss}} - u_{i_{ss}} := \lim_{k\to\infty} u_T(k) - \lim_{k\to\infty} u_i(k) \\ \hat{x}_{i_{ss}} & := & \lim_{k\to\infty} \hat{x}_i(k) \\ \bar{e}_{i_{ss}} & := & \lim_{k\to\infty} E[e_i(k)] = \lim_{k\to\infty} E[(x(k) - \hat{x}_i(k))] \end{array} \tag{13}$$

where the subscript $T \in \{1, \cdots, N\}$ represents the true mode. Following a procedure similar to the one adopted by Hanlon et al. [16], the steady-state mean residual for mode i is

$$\begin{array}{rcl} \bar{r}_{i_{ss}} & = & C_T A_T \bar{e}_{i_{ss}} + (C_T \Delta A_i + \Delta C_i A_T - \Delta C_i \Delta A_i)\hat{x}_{i_{ss}} + C_T B_T \Delta u_{i_{ss}} + (C_T \Delta B_i + \Delta C_i B_T - \Delta C_i \Delta B_i) u_{i_{ss}} \\ & = & \{C_T A_T [I - (I - K_i C_T) A_T]^{-1} [(I - K_i C_T)\Delta A_i - K_i \Delta C_i A_i] + (C_T \Delta A_i + \Delta C_i A_T - \Delta C_i \Delta A_i)\}\hat{x}_{i_{ss}} \\ & & +\{C_T A_T [I - (I - K_i C_T) A_T]^{-1} (I - K_i C_T) B_T + C_T B_T\} \Delta u_{i_{ss}} \\ & & +\{C_T A_T [I - (I - K_i C_T) A_T]^{-1} [(I - K_i C_T)\Delta B_i - K_i \Delta C_i B_i] + (C_T \Delta B_i + \Delta C_i B_T - \Delta C_i \Delta B_i)\} u_{i_{ss}} \end{array} \tag{14}$$

where K_i is the steady-state Kalman filter gain for mode i. We assume that the matrix inverse in (14) exists. If mode i is the correct mode ($i = T$), then $\bar{r}_{i_{ss}} = 0$. If $\bar{r}_{j_{ss}} \neq 0$ ($\forall j \neq i$), then the correct mode can be detected. However, even if mode i is not the correct mode ($i \neq T$), the steady-state mean residual for mode i is zero if the following is true: $(I - K_i C_T)\Delta A_i - K_i \Delta C_i A_i = 0 \bigwedge (C_T \Delta A_i + \Delta C_i A_T - \Delta C_i \Delta A_i) = 0 \bigwedge (I - K_i C_T)\Delta B_i - K_i \Delta C_i B_i = 0 \bigwedge (C_T \Delta B_i + \Delta C_i B_T - \Delta C_i \Delta B_i) = 0 \bigwedge \Delta u_{i_{ss}} = 0$. This means that if at least two models are identical and the corresponding control inputs are the same, then the steady-state residuals of both the corresponding modes are zero. In this case, the hybrid estimation algorithm will not work. In other words, the performance of the hybrid estimation algorithm depends on the differences between the residuals which in turn arise from model differences and input differences. In the above condition, the first four equalities come from model differences and the last equality comes from input differences. From this condition, we know that the mode differences and/or the input differences should be large enough for hybrid estimation to work; this supports Maybeck's heuristic observation that the performance of the MMAE depends on a significant difference between the residual characteristics [1]. We now study the transient characteristics of the mode probability (a function of the residuals) in detail to estimate the mode estimation delay, which is a good measure of the performance of hybrid estimation.

4.2 Transient analysis for mode probability

In this paper, we consider the mean behavior of a hybrid estimator, and analyze its performance in the sense of exponential convergence as defined in Definition 2. A steady-state Kalman filter is assumed to be used as the state estimator for each mode. For the sake of notational simplicity, we define:

$$\mu_i^-(k) \quad := \quad \sum_{l=1}^{N} \pi_{il}\mu_l(k-1) \tag{15}$$

Using $\mu_i(k) := p(m_i(k)|Z(k))$ in (11), the condition for correct mode detection at time k is: $\forall i \neq T$

$$\begin{aligned} & \mu_T(k) > \mu_i(k) \\ \iff \quad & \tfrac{1}{c(k)}\Lambda_T(k)\mu_T^-(k) > \tfrac{1}{c(k)}\Lambda_i(k)\mu_i^-(k) \\ \iff \quad & \Lambda_T(k) > \Lambda_i(k)\tfrac{\mu_i^-(k)}{\mu_T^-(k)} \end{aligned} \tag{16}$$

Since $\Lambda_i(k) = (2\pi)^{-n/2}|S_i|^{-1}\exp[-\frac{1}{2}\bar{r}_i(k)^T S_i^{-1}\bar{r}_i(k)]$ ($S_i = S_i^T > 0$) where $\bar{r}$ is the mean residual, (16) becomes

$$0 \leq \bar{r}_T(k)^T S_T^{-1}\bar{r}_T(k) < \bar{r}_i(k)^T S_i^{-1}\bar{r}_i(k) + 2\ln\left(\frac{|S_i|}{|S_T|}\right) + 2\ln\left(\frac{\mu_T^-(k)}{\mu_i^-(k)}\right) \tag{17}$$

To detect the correct mode exactly for any $k \in \mathbb{N}$, (17) must hold for all $k \in \mathbb{N}$ ($\forall i \neq T$). If there is a time delay (δ_T) for correct mode detection when a mode transition into mode T occurs at time k_l ($l \in \mathbb{N}^+$), (17) holds for $k \in [k_l + \delta_T, k_{l+1})$. For the existence of a $\bar{r}_T(k)$ satisfying (17), the right-hand-side in (17) must be greater than or equal to zero. The following condition holds for a positive definite matrix S_i^{-1} [17]:

$$\lambda_{min}(S_i^{-1})\bar{r}_i(k)^T\bar{r}_i(k) \leq \bar{r}_i(k)^T S_i^{-1}\bar{r}_i(k) \leq \lambda_{max}(S_i^{-1})\bar{r}_i(k)^T\bar{r}_i(k) \tag{18}$$

where $\lambda_{min}(S_i^{-1})$ and $\lambda_{max}(S_i^{-1})$ are the minimum and the maximum eigenvalues of S_i^{-1} respectively. Thus, we have the following condition:

Proposition 2 *The correct mode can be detected in δ_T time steps after a mode transition at time k_l if there exists $\delta_T \in \mathbb{N}^+$ such that for $k \in [k_l + \delta_T, k_{l+1})$ ($l \in \mathbb{N}^+$, $\forall i \neq T$), Condition 1 holds and either Condition 2 or Condition 3 is true.*

1. $\bar{r}_i(k)^T S_i^{-1}\bar{r}_i(k) + 2\ln\left(\frac{|S_i|}{|S_T|}\right) + 2\ln\left(\frac{\mu_T^-(k)}{\mu_i^-(k)}\right) > 0.$

2. $\bar{r}_T(k)^T S_T^{-1} \bar{r}_T(k) < \bar{r}_i(k)^T S_i^{-1} \bar{r}_i(k) + 2\ln\left(\frac{|S_i|}{|S_T|}\right) + 2\ln\left(\frac{\mu_T^-(k)}{\mu_i^-(k)}\right)$.

3. $\|\bar{r}_T(k)\|^2 < \frac{\lambda_{min}(S_i^{-1})}{\lambda_{max}(S_T^{-1})}\|\bar{r}_i(k)\|^2 + \frac{2}{\lambda_{max}(S_T^{-1})}\left[\ln\left(\frac{|S_i|}{|S_T|}\right) + \ln\left(\frac{\mu_T^-(k)}{\mu_i^-(k)}\right)\right]$.

Condition 3 in Proposition 2 is a sufficient condition on, and might be a very conservative test for, the correct mode detection. However, Condition 3 gives valuable insight into the performance of hybrid estimation. Fast mode detection is dependent not only on the magnitudes of the residuals produced by each Kalman filter but also on the residual covariances. If $\frac{\lambda_{min}(S_i^{-1})}{\lambda_{max}(S_T^{-1})}$ is small and/or $\frac{|S_i|}{|S_T|}$ is small, it is difficult for Condition 3 to hold and thus to detect the correct mode. Therefore, by checking the eigenvalues of S_i^{-1} and the determinant of its inverse, we can tell which mode transitions are more detectable than the others. This is similar to the idea of the observability grammian as a measure of which states are more observable than others [18]. If we consider the steady-state mean of the residual, Condition 3 becomes

$$\|\bar{r}_{T_{ss}}\|^2 < \frac{\lambda_{min}(S_i^{-1})}{\lambda_{max}(S_T^{-1})}\|\bar{r}_{i_{ss}}\|^2 + \frac{2}{\lambda_{max}(S_T^{-1})}\left[\ln\left(\frac{|S_i|}{|S_T|}\right) + \ln\left(\frac{\mu_{T_{ss}}^-}{\mu_{i_{ss}}^-}\right)\right], \ \forall i \neq T \tag{19}$$

Therefore, if the asymptotic behavior of the residuals satisfies (19) and the minimum sojourn time is long enough for the residual to converge to its steady-state value, then the MMAE is guaranteed to estimate hybrid states correctly.

4.3 Mode estimation delay

In this section, we derive the mode estimation delay δ_i using Condition 3 in Proposition 2. The mean residual of the correct filter at time $k_l + \delta_i$ ($l \in \mathbb{N}^+$), when $i = T$, are:

$$\bar{r}_T(k_l + \delta_T) \quad = \quad C_T A_T[(I - K_T C_T)A_T]^{\delta_T - 1}\bar{e}_T(k_l) \tag{20}$$

The mean residual and the mean estimation error of the incorrect filter i ($i \neq T$) at time k are

$$\begin{aligned}
\bar{r}_i(k) &= C_T A_T \bar{e}_i(k-1) + [C_T \Delta A_i + \Delta C_i A_T - \Delta C_i \Delta A_i]\hat{x}_i(k-1) + [C_T \Delta B_i + \Delta C_i B_T - \Delta C_i \Delta B_i]u(k-1) \\
&= \begin{bmatrix} C_T A_T & C_T \Delta A_i + \Delta C_i A_T - \Delta C_i \Delta A_i & C_T \Delta B_i + \Delta C_i B_T - \Delta C_i \Delta B_i \end{bmatrix} \begin{bmatrix} \bar{e}_i(k-1) \\ \hat{x}_i(k-1) \\ u(k-1) \end{bmatrix} \\
\bar{e}_i(k) &= (I - K_i C_T)A_T \bar{e}_i(k-1) + [(I - K_i C_T)\Delta A_i - K_i \Delta C_i A_i]\hat{x}_i(k-1) + [(I - K_i C_T)\Delta B_i - K_i \Delta C_i B_i]u(k-1) \\
&= \begin{bmatrix} (I - K_i C_T)A_T & (I - K_i C_T)\Delta A_i - K_i \Delta C_i A_i & (I - K_i C_T)\Delta B_i - K_i \Delta C_i B_i \end{bmatrix} \begin{bmatrix} \bar{e}_i(k-1) \\ \hat{x}_i(k-1) \\ u(k-1) \end{bmatrix}
\end{aligned} \tag{21}$$

For the sake of notational simplicity, we define

$$\begin{array}{rcl} F_T & := & (I - K_T C_T) A_T \\ F_i & := & (I - K_i C_T) A_T \\ H_i^x & := & C_T \Delta A_i + \Delta C_i A_T - \Delta C_i \Delta A_i \\ H_i^u & := & C_T \Delta B_i + \Delta C_i B_T - \Delta C_i \Delta B_i \\ G_i^x & := & (I - K_i C_T) \Delta A_i - K_i \Delta C_i A_i \\ G_i^u & := & (I - K_i C_T) \Delta B_i - K_i \Delta C_i B_i \\ L_i & := & \left[\begin{array}{ccc} C_T A_T & H_i^x & H_i^u \end{array} \right] \end{array} \tag{22}$$

The norm of the mean residual of the correct filter at time $k_l + \delta_T$ is

$$\|\bar{r}_T(k_l + \delta_T)\| = \|C_T A_T F_T^{\delta_T - 1} \bar{e}_T(k_l)\| \leq \bar{\sigma}(C_T A_T) \bar{\sigma}(F_T)^{\delta_T - 1} \|\bar{e}_T(k_l)\| \tag{23}$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. Similarly, from (21), the norm of the mean residual of the incorrect filter at time $k_l + \delta_T$ is

$$\|\bar{r}_i(k_l + \delta_T)\| \geq \underline{\sigma}(L_i) \|\bar{e}_i(k_l + \delta_T - 1)\| \geq \underline{\sigma}(L_i) \underline{\sigma}\left(\left[\begin{array}{ccc} F_i & G_i^x & G_i^u \end{array} \right] \right)^{\delta_T - 1} \|\bar{e}_i(k_l)\| \tag{24}$$

where $\underline{\sigma}(\cdot)$ denotes the minimum singular value. We define

$$\begin{array}{rcl} \alpha & := & \frac{\lambda_{min}(S_i^{-1})}{\lambda_{max}(S_T^{-1})} \\ \beta(k) & := & \frac{2}{\lambda_{max}(S_T^{-1})} \left[\ln\left(\frac{|S_i|}{|S_T|} \right) + \ln\left(\frac{\mu_T^-(k)}{\mu_i^-(k)} \right) \right] \end{array} \tag{25}$$

Using (23), (24), and (25) in Condition 3 of Proposition 2, we obtain the following condition:

$$\bar{\sigma}(C_T A_T)^2 \bar{\sigma}(F_T)^{2(\delta_T - 1)} \|\bar{e}_T(k_l)\|^2 \quad < \quad \alpha \underline{\sigma}(L_i)^2 \underline{\sigma}\left(\left[\begin{array}{ccc} F_i & G_i^x & G_i^u \end{array} \right] \right)^{2(\delta_T - 1)} \|\bar{e}_i(k_l)\|^2 + \beta(k + \delta_T) \tag{26}$$

Even though we cannot find the mode estimation delay δ_T explicitly from (26), (26) will be used to derive a condition for instantaneous mode estimation in the next section. To find δ_T explicitly, we try a different approach. From (21), the mean residual of the incorrect filter at time $k_l + \delta_T$ can be written as:

$$\begin{array}{rcl} \bar{r}_i(k_l + \delta_T) & = & C_T A_T F_i^{\delta_T - 1} \bar{e}_i(k_l) + H_i^x \hat{x}_i(k_l + \delta_T - 1) + C_T A_T [F_i^{\delta_T - 2} G_i^x \hat{x}_i(k_l) + \cdots + G_i^x \hat{x}_i(k_l + \delta_T - 2)] \\ & & + H_i^u u(k_l + \delta_T - 1) + C_T A_T [F_i^{\delta_T - 2} G_i^u u(k_l) + \cdots + G_i^u u(k_l + \delta_T - 2)] \end{array} \tag{27}$$

We define

$$\begin{array}{rcl} b_i(k_l + \delta_T - 1) & := & H_i^x \hat{x}_i(k_l + \delta_T - 1) + C_T A_T [F_i^{\delta_T - 2} G_i^x \hat{x}_i(k_l) + \cdots + G_i^x \hat{x}_i(k_l + \delta_T - 2)] \\ & & + H_i^u u(k_l + \delta_T - 1) + C_T A_T [F_i^{\delta_T - 2} G_i^u u(k_l) + \cdots + G_i^u u(k_l + \delta_T - 2)] \end{array} \tag{28}$$

Then, Condition 3 of Proposition 2 becomes

$$\bar{\sigma}(C_T A_T)^2 \bar{\sigma}(F_T)^{2(\delta_T - 1)} \|\bar{e}_T(k_l)\|^2 \quad < \quad \alpha \underline{\sigma}(C_T A_T)^2 \underline{\sigma}(F_i)^{2(\delta_i - 1)} \|\bar{e}_i(k_l)\|^2 + J_i(k_l + \delta_T) \tag{29}$$

where $J_i(k_l + \delta_T) := \alpha \|b_i(k_l + \delta_T - 2)\|^2 + \beta(k_l + \delta_T)$. $\beta(k_l + \delta_T) = \frac{2}{\lambda_{max}(S_T^{-1})} \left[\ln\left(\frac{|S_i|}{|S_T|} \right) + \ln\left(\frac{\mu_T^-(k_l + \delta_T)}{\mu_i^-(k_l + \delta_T)} \right) \right]$ might be negative, yet its magnitude is usually not big because it is in a logarithmic scale. Thus, $J_i(k_l + \delta_T) \geq 0$ is easily satisfied. If $J_i(k_l + \delta_T) \geq 0$, we can have the following condition:

$$\delta_T > 1 + \left\{ 2 \ln\left[\frac{\underline{\sigma}(F_i)}{\bar{\sigma}(F_T)} \right] \right\}^{-1} \left\{ -\ln\alpha + 2 \ln\left[\frac{\bar{\sigma}(C_T A_T)}{\underline{\sigma}(C_T A_T)} \right] + 2 \ln\left[\frac{\|\bar{e}_T(k_l)\|}{\|\bar{e}_i(k_l)\|} \right] \right\} \tag{30}$$

Proposition 3 *The correct mode can be detected δ_T time steps after a mode transition if Condition 1 of Proposition 2 holds and there exists $\delta_T \in \mathbb{N}^+$, $\delta_T < k_{l+1} - k_l$, $l \in \mathbb{N}^+$, $\forall i \neq T$, such that either of the following conditions is true.*

1. $\bar{\sigma}(C_T A_T)^2 \bar{\sigma}(F_T)^{2(\delta_T - 1)} \|\bar{e}_T(k_l)\|^2 < \alpha \underline{\sigma}(L_i)^2 \underline{\sigma}\left(\begin{bmatrix} F_i & G_i^x & G_i^u \end{bmatrix}\right)^{2(\delta_T-1)} \|\bar{e}_i(k_l)\|^2 + \beta(k + \delta_T)$
2. $\delta_T > 1 + \left\{2 \ln \left[\frac{\underline{\sigma}(F_i)}{\bar{\sigma}(F_T)}\right]\right\}^{-1} \left\{-\ln \alpha + 2 \ln \left[\frac{\bar{\sigma}(C_T A_T)}{\underline{\sigma}(C_T A_T)}\right] + 2 \ln \left[\frac{\|\bar{e}_T(k_l)\|}{\|\bar{e}_i(k_l)\|}\right]\right\}$, *when* $J_i(k_l + \delta_T) \geq 0$.

For a system with only two discrete modes, we can further simplify (29) and also relax the condition that $J_i(k_l + \delta_T) \geq 0$. To do this, we first consider (16). For a system with two modes, for any time $k \in [k_l, k_{l+1})$, if T is the true mode in the interval $[k_l, k_{l+1})$,

$$\frac{\mu_i(k+1)}{\mu_T(k+1)} = \frac{\Lambda_i(k+1)}{\Lambda_T(k+1)} \left(\frac{\pi_{ii}\mu_i(k) + (1 - \pi_{TT})\mu_T(k)}{(1 - \pi_{ii})\mu_i(k) + \pi_{TT}\mu_T(k)}\right) \tag{31}$$

Let $\Omega(k) = \frac{\mu_i(k)}{\mu_T(k)}$. Then, for a diagonally dominant transition matrix Π,

$$\begin{aligned} \Omega(k+1) &= \frac{\Lambda_i(k+1)}{\Lambda_T(k+1)} \left(\frac{\pi_{ii}\Omega(k) + (1 - \pi_{TT})}{(1 - \pi_{ii})\Omega(k) + \pi_{TT}}\right) & (32) \\ &\approx \frac{\Lambda_i(k+1)}{\Lambda_T(k+1)} \left(\frac{\pi_{ii}\Omega(k)}{(1 - \pi_{ii})\Omega(k) + \pi_{TT}}\right) & (33) \end{aligned}$$

We assume that the estimator converges before the transition takes place, which gives us

$$\begin{aligned} \mu_i(k_l - 1) &\approx 1 & (34) \\ \mu_T(k_l - 1) &\approx 0 & (35) \\ \Omega(k_l) &= \frac{\Lambda_i(k_l)}{\Lambda_T(k_l)} \left(\frac{\pi_{ii}}{1 - \pi_{ii}}\right) & (36) \end{aligned}$$

After the transition at time k_l and before detection,

$$\begin{aligned} 1 &\leq \Omega(k) \leq \infty \\ \implies \frac{\Lambda_i(k+1)}{\Lambda_T(k+1)} \left(\frac{\pi_{ii}}{(1-\pi_{ii}) + \pi_{TT}}\right) &\leq \Omega(k+1) \leq \frac{\Lambda_i(k+1)}{\Lambda_T(k+1)} \left(\frac{\pi_{ii}}{1-\pi_{ii}}\right) \end{aligned} \tag{37}$$

Detection occurs at the smallest δ_T when $\Omega(k_l + \delta_T) < 1$. Thus we try to find the smallest δ_T such that

$$\frac{\Lambda_i(k_l + \delta_T)}{\Lambda_T(k_l + \delta_T)} \left(\frac{\pi_{ii}}{1 - \pi_{ii}}\right) < 1 \tag{38}$$

$$\implies \bar{r}_T(k_l + \delta_T)^T S_T^{-1} \bar{r}_T(k_l + \delta_T) < \bar{r}_i(k_l + \delta_T)^T S_i^{-1} \bar{r}_i(k_l + \delta_T) + 2 \ln \left(\frac{|S_i|}{|S_T|}\right) - 2 \ln \left(\frac{\pi_{ii}}{1 - \pi_{ii}}\right) \tag{39}$$

Using (23), (24), and (25) in (40), we obtain the following condition:

Proposition 4 *For a hybrid system with two discrete modes, the correct mode can be detected δ_T time steps after a mode transition if there exists $\delta_T \in \mathbb{N}^+$, $\delta_T < k_{l+1} - k_l$, $l \in \mathbb{N}^+$, $i \neq T$, such that*

$$\begin{aligned} \bar{\sigma}(C_T A_T)^2 \bar{\sigma}(F_T)^{2(\delta_T - 1)} \|\bar{e}_T(k_l)\|^2 \; < \; & \alpha \underline{\sigma}(L_i)^2 \underline{\sigma}\left(\begin{bmatrix} F_i & G_i^x & G_i^u \end{bmatrix}\right)^{2(\delta_T-1)} \|\bar{e}_i(k_l)\|^2 \\ & + \frac{2}{\lambda_{max}(S_T^{-1})} \left[\ln\left(\frac{|S_i|}{|S_T|}\right) - \ln\left(\frac{\pi_{ii}}{1-\pi_{ii}}\right)\right] \end{aligned} \tag{40}$$

Proposition 4 implies that if (40) is satisfied, then the mode probability of the correct mode is definitely greater than those of the other modes after δ_T time steps after a mode transition at time k_l. Thus, the correct mode is detected δ_T time steps after a mode transition.

4.4 Instantaneous Mode Estimation

In this section, we derive the conditions under which the mode change detection is instantaneous. Consider the case of a system with two modes. Assuming that the time between discrete transitions is sufficient to allow the Kalman filters to converge, we can assume that the mode probabilities before the transition have converged. Following a procedure similar to (31)-(40) in Section 4.3, we obtain

$$\bar{\sigma}(C_T A_T)^2 \|\bar{e}_T(k_l-1)\|^2 \quad < \quad \alpha \underline{\sigma}(L_i)^2 \underline{\sigma}\left(\begin{bmatrix} F_i & G_i^x & G_i^u \end{bmatrix}\right)^2 \|\bar{e}_i(k_l-1)\|^2 \tag{41}$$

$$+\frac{2}{\lambda_{max}(S_T^{-1})}\left[\ln\left(\frac{|S_i|}{|S_T|}\right) - \ln\left(\frac{\pi_{ii}}{1-\pi_{ii}}\right)\right] \tag{42}$$

We can extend this to the instantaneous detection of a transition in a system with n modes.

Proposition 5 *The correct mode is detected instantaneously if the following condition holds:*

$$\begin{aligned}\bar{\sigma}(C_T A_T)^2 \|\bar{e}_T(k_l-1)\|^2 \quad < \quad & \alpha \underline{\sigma}(L_i)^2 \underline{\sigma}\left(\begin{bmatrix} F_i & G_i^x & G_i^u \end{bmatrix}\right)^2 \|\bar{e}_i(k_l-1)\|^2 \\ & +\tfrac{2}{\lambda_{max}(S_T^{-1})}\left[\ln\left(\tfrac{|S_i|}{|S_T|}\right) + \ln\left(\tfrac{\pi_{jT}}{\pi_{jj}}\right)_{min}\right]\end{aligned} \tag{43}$$

where $\left(\frac{\pi_{jT}}{\pi_{jj}}\right)_{min}$ *is the smallest ratio of off-diagonal to diagonal elements in any row of the transition matrix.*

4.5 Exponential convergence of hybrid estimation algorithms

Finally, to present a complete picture, we present a result from the authors' previous work [12], of the conditions to guarantee exponential convergence of the hybrid estimator once the correct mode sequence has been detected.

Theorem 1 *(Theorem 3 from [12]): Consider a given stochastic linear hybrid system, an error convergence set* M_0 *and rate of convergence* ζ, $|\zeta| < 1$, $|\alpha(A_i - K_i C_i)| \leq |\zeta|$ *for all* $i = 1 \ldots N$, *where* $\alpha(A)$ *is the maximal absolute value of the eigenvalues of* A. *Let* $\kappa(A) = \|Q\|\|Q^{-1}\|$, *the condition number of* A *under the inverse, where* $Q^{-1}AQ = \mathcal{J}$, *the Jordan canonical form. Then if the following seven conditions are satisfied:*

1. *The system is observable in the sense of a hybrid system [12]*
2. $\{A_i, C_i\}$ *couples are observable for all* $i = 1 \ldots N$
3. $\{A_i, Q_i^{1/2}\}$ *couples are controllable for all* $i = 1 \ldots N$
4. $(A_i - K_i C_i)$ *is stable for all* $i = 1 \ldots N$ *with all distinct eigenvalues*
5. *There exists* $X > 0$ *such that* $\|x(k)\|_\infty \leq X$, $i, j = 1...N$, $k = 1, 2, ...$ *such that*

$$\|[(A_i - A_j) - K_i(C_i - C_j)]\, \bar{x}(k)\|_\infty \leq U = \max \|(A_i - A_j) - K_i(C_i - C_j)\|_1 X \tag{44}$$

6. *The discrete decision time,* δ *satisfies the relation*

$$\delta \leq \frac{M_0}{\sqrt{n} U \max[\kappa(A_i - K_i C_i)]} \tag{45}$$

7. The time between switching events (sojourn time), $\tilde{\Delta}$ satisfies the conditions

$$\tilde{\Delta} > \beta_{min} + \delta, \text{ where} \tag{46}$$

$$\beta_{min} > \max[\frac{1}{|\log\zeta|}\log\left|\left(1 - \frac{\sqrt{n}U\delta\kappa(A_i - K_iC_i)}{M_0}\right)\right|, \max\frac{\log[\kappa(A_i - K_iC_i)]}{|\log[\alpha(A_i - K_iC_i)]|}] \tag{47}$$

we can design a hybrid estimator that converges to the set M_0 with a rate of convergence greater than or equal to ζ.

The discrete decision time assumed in this theorem is in fact the mode detection delay δ that we have now derived. This theorem presents the conditions under which if there is a mode detection delay δ, the sojourn time is long enough for the error convergence during the period of correct detection $(\tilde{\Delta} - \delta)$ to balance the divergence of the error during the mode mismatch. From the present results, we now have a way of determining δ. Therefore, combining the two results, we can evaluate the performance of a given hybrid estimator and also find the minimum sojourn time required in each mode to guarantee exponential convergence of the mean square error.

5 Performance comparison between hybrid estimation algorithms

In this section, we discuss the performance of hybrid estimation and also compare the performance of the MMAE and the IMM algorithms. We focus on the mode estimation delay since usually, the smaller the mode estimation delay, the smaller the estimation error. By investigating (30), we can explain the performance of hybrid estimation algorithms qualitatively. For the mode estimation delay to be small, the following must be small if $J_i(k_l + \delta_T) \geq 0$:

$$\left\{\log\left(\frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}\right) + 2\log\left[\frac{\bar{\sigma}(C_TA_T)}{\underline{\sigma}(C_TA_T)}\right] + 2\log\left[\frac{\|\bar{e}_T(k_l)\|}{\|\bar{e}_i(k_l)\|}\right]\right\}, \ (\forall T \in \{1, \cdots, N\}, \ \forall i \neq T) \tag{48}$$

where mode T is the correct mode after the mode transition at time k_l ($l \in \mathbb{N}^+$). Firstly, $\frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}$ must be small. Here, the pre-computed residual covariance $S_i = C_iP_i^{ss}C_i + R_i$, and P_i^{ss} (the steady-state error covariance matrix computed by Kalman filter i) satisfies the algebraic Riccati equation. Therefore, $\frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}$ depends only on the system parameters A_i, C_i, Q_i, R_i and A_T, C_T, Q_T, R_T. Thus, by checking the residual covariance matrices for each Kalman filter (which can be done without any measurements), we can tell which mode transition is more detectable than the others. In addition, since Q_i and R_i are design parameters for the Kalman filter i, and Q_T and R_T are design parameters for the Kalman filter T, we can make $\frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}$ small by adjusting these parameters (also known as Kalman filter tuning) and thus reduce the mode estimation delay. Secondly, if the condition number of C_TA_T is close to 1, the second term becomes small. Thus, we also say which mode is more easily estimated than the others by checking the condition number of C_TA_T for all T. Thirdly, $\frac{\|\bar{e}_T(k_l)\|}{\|\bar{e}_i(k_l)\|}$ must be small, i.e. the mean state estimation errors produced by mode-mismatched Kalman filters should be small (and close to the error produced by the correct Kalman filter).

The mixing step was originally devised to reduce the complexity of the algorithm, yet it also keeps the estimation errors produced by mismatched Kalman filters small. The IMM readjusts, through the mixing step at each time instant, the initial conditions for each Kalman filter, and shifts them closer to the (correct) estimate computed by the IMM at the previous time step. Therefore the means of the state estimation errors produced by the incorrect Kalman filters are close to that of the correct Kalman filter. Thus, the mode estimation delay of the IMM is smaller than that of the MMAE (which does not have this mixing mechanism). The smaller time delay translates to better estimation performance of the IMM compared to the MMAE. Maybeck [1] proposes two ad hoc methods to improve adaptability of the MMAE: enforcing a lower bound on the mode probabilities and adding pseudonoise to the the Kalman filter models. The IMM does both inherently. We illustrate this through examples in the next section.

6 Examples

We first consider mode detection in a simple, one-dimensional system such as the one in [14]. The dynamics is of the form

$$x(k) = a_i x(k-1) + b_i u(k) + w_i(k) \tag{49}$$

$$y(k) = c_i x(k) + v_i(k) \tag{50}$$

$$u(k) = 5cos(\frac{2\pi t}{100}) \tag{51}$$

where the state variables and model parameters are scalar, there are 2 discrete modes, and the input is deterministic and sinusoidal. We estimate the hybrid state sequence from the output sequence using both the MMAE and the IMM. We first check for instantaneous mode detection at a switch using (42). We then compute the maximum mode detection delay (or the minimum sojourn time needed to guarantee correct mode detection) using (30) and (40). We perform this experiment for various values of the model parameters and compare our predictions with the simulations.

MMAE and IMM Performance vs. Parameters (Monte Carlo Simulation, 100 trials)													
Case	Algorithm	Mode Parameters ($b_1 = b_2 = 1$)				Instant Detection Condition		Mode detection delay				$\frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}$	
								Predicted		Observed			
		a1	a2	c1	c2	$2 \rightarrow 1$	$1 \rightarrow 2$	δ_1^*	δ_2^*	δ_1	δ_2	T=1	T=2
1	MMAE	0.95	0.25	1	0.8	$\surd$	$\times$	0	2	0	2	0.48	2.09
	IMM					$\surd$	$\times$	0	2	0	2		
2	MMAE	0.85	0.85	0.8	0.2	$\times$	$\times$	6	2	4	0	1.29	0.77
	IMM					$\times$	$\surd$	2	0	0	0		
3	MMAE	0.95	0.85	1	0.4	$\times$	$\times$	10	2	5	1	1.13	0.89
	IMM					$\surd$	$\surd$	0	0	0	0		

Table 1: Two mode example

Clearly, the IMM performs better than the MMAE, especially in cases 2 and 3. Also, since we only compute

MMAE and IMM Performance vs. Parameters (Monte Carlo Simulation, 100 trials)																
Case	Algorithm	Mode Parameters ($b_1 = b_2 = b3 = 1$)						Mode detection delay						$\max \frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}$		
								Predicted			Observed					
		a1	a2	a3	c1	c2	c3	δ_1^*	δ_2^*	δ_3^*	δ_1	δ_2	δ_3	T=1	T=2	T=3
1	MMAE	1.2	0.25	0.95	0.8	1.0	0.8	2	0	7	1	0	5	1.2	0.80	1.26
	IMM							1	0	5	1	0	4			

Table 2: Three mode example

a conservative estimate of the mode detection delay, it is quite possible that the observed delay is less than the computed bound (as in cases 2 and 3). Figure 2 shows the mode probabilities and estimates for case 2. The reason for the difference in the performance of the MMAE and IMM algorithms is clear when we consider the estimation errors in Figure 3. At the mode transition times, the errors of the matched and mismatched filters of the IMM are almost equal, thus keeping (48), and therefore the mode detection delay, small. In our example, (48) simply reduces to $\frac{\lambda_{max}(S_T^{-1})}{\lambda_{min}(S_i^{-1})}$ in the case of IMM estimation. Comparing these values for the different transitions with the values of the delay in Table 1, it is seen that as predicted, the smaller the value of (48), the smaller the mode detection delay. The biggest advantage of this result is that given a system and its error bounds, this gives us a way to determine *a priori* transitions to which modes are the most detectable. We also try a three-mode example for the same system. The results are as expected, and are shown in Table 2.

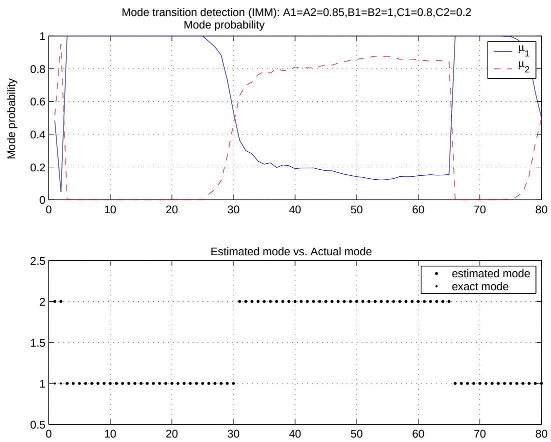
We now consider an aircraft tracking example, which has two discrete states, the constant velocity (CV) mode and the coordinated turn (CT) mode. This represents flight trajectories composed of straight lines and circular arcs. For brevity, we only include the two mode example in this paper but the performance analysis conditions can be applied to multiple-mode cases. The dynamics of both modes is given by

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} T^2/2 & 0 \\ T & 0 \\ 0 & T^2/2 \\ 0 & T \end{bmatrix} u_i(k) + w_i(k) && (52) \\
y(k) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) + v_i(k), \qquad (i \in \{CV, CT\}) && (53)
\end{aligned}
$$

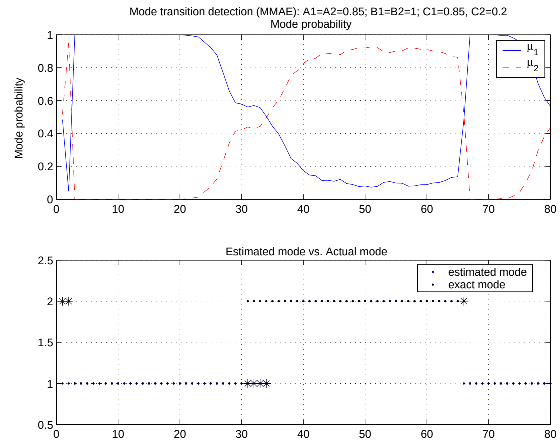
where $x = [\ x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2\ ]^T$ where x_1 and x_2 are the position coordinates, $u = [\ u_1 \quad u_2\ ]^T$ where u_1 and u_2 are the acceleration components. The control input has a different constant value for each mode:

$$
u_{CV} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for CV mode, } \; u_{CT} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \text{ for CT mode} \tag{54}
$$

T is the sampling interval, w_i is the process noise, and v_i is the sensor noise. We choose an operating velocity of 150 knots. We design a flight trajectory such that the mode change from CV mode to CT mode occurs at time=45 seconds and the mode change CT mode to CV mode occurs at time=56 seconds. Using Proposition 3 for the IMM, we find that the mode estimation delay for the mode switching from mode CV to mode CT is $\delta_{ct} = 1$ and the mode estimation delay for the mode switching from mode CT to mode CV is $\delta_{cv} = 2$. Thus,

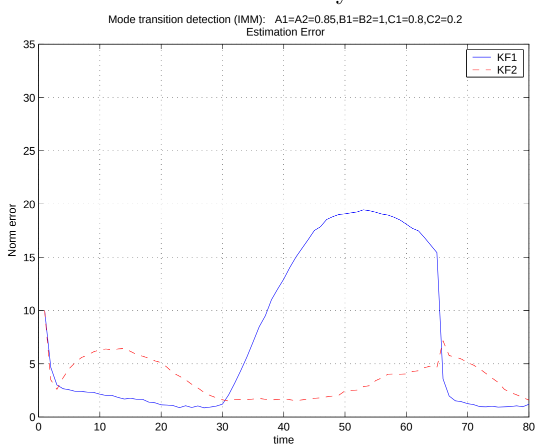


(a)

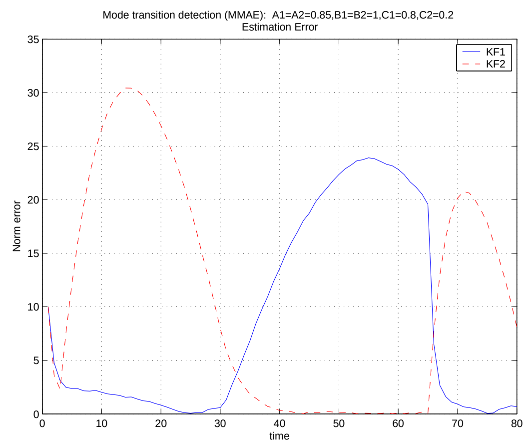


(b)

Figure 2: Mode estimates by (a) IMM and (b) MMAE: 100 trial Monte Carlo simulation results. The asterisks denote mode estimation delay.



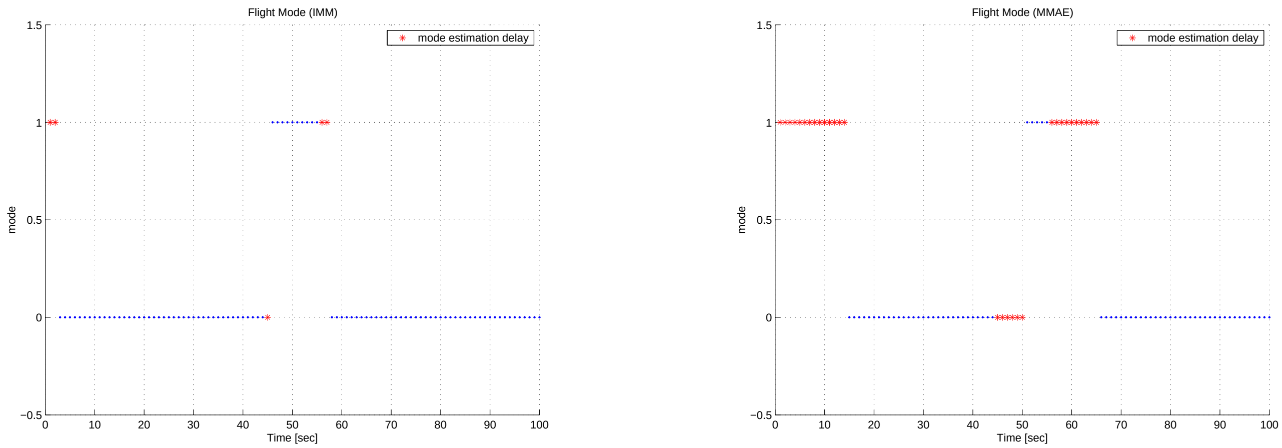
(a)



(b)

Figure 3: Estimation error by (a) IMM and (b) MMAE: 100 trial Monte Carlo simulation results.

we expect the mode switching from mode CV to CT is more detectable than the mode switching from mode CT to CV. Similarly, we obtain $\delta_{ct} = 7$ and $\delta_{cv} = 12$ for the MMAE. Figure 4-(a) shows that the IMM detects the mode change from mode CV to mode CT after a 1 time step delay and the mode change from mode CT to mode CV after a 2 time step delay. Similarly, Figure 1-(b) shows that the MMAE detects the mode change from mode CV to mode CT after a 6 time step delay and the mode change from mode CV to mode CT after a 10 time step delay. The performance analysis predicts that the IMM performs better than the MMAE and the simulation results support this.



(a) (b)

Figure 4: Mode estimates by (a) IMM and (b) MMAE: 100 trial Monte Carlo simulation results (mode CV = 0, mode CT = 1).

7 Conclusions

Although several algorithms have existed for many years to estimate stochastic linear hybrid systems, the issues of their performance and limitations have not been addressed in much detail. In this paper, we have performed a detailed steady-state and transient analysis of the behavior of these algorithms and derived necessary conditions for correct mode detection, bounds on their performance in terms of the mode detection delay and the minimum sojourn time, and also proposed a way to predict *a priori* which mode transitions are the easiest to detect. We have also validated our results using simulated experiments. Most importantly, our results give a mathematical yet intuitive explanation for why the IMM algorithm achieves its high levels of performance in the estimation of stochastic linear hybrid systems.

References

[1] P.S. Maybeck. *Stochastic Models, Estimation, and Control*, volume 2. New York: Academic Press, 1982.

[2] Y. Bar-Shalom, X.R. Li, and T. Kirubarajan. *Estimation with Applications to Tracking and Navigation*. John Wiley & Sons, 2001.

[3] D.D. Sworder and J.E. Boyd. *Estimation problems in Hybrid Systems*. Cambridge University Press, 1999.

[4] R.M. Hawkes and J.B. Moore. Performance bounds for adaptive estimation. *Proceedings of the IEEE*, 64(8):1143–1150, August 1976.

[5] D.T. Magill. Optimal adaptive estimation of sampled stochastic processes. *IEEE Transactions on Automatic Control*, 10(4):434–439, October 1965.

[6] D.G. Lainiotis and F.L. Sims. Performance measures for adaptive Kalman estimators. *IEEE Transactions on Automatic Control*, 15(2):249–250, April 1970.

[7] F.L. Sims, D.G. Lainiotis, and D.T. Magill. Recursive algorithm for the calculation of the adaptive Kalman filter weighting coefficients. *IEEE Transactions on Automatic Control*, 14(2):215–218, April 1969.

[8] D.G. Lainiotis. Optimal adaptive estimation: Structure and parameter adaptation. *IEEE Transactions on Automatic Control*, 16(2):160–170, April 1971.

[9] T.M. Cover and J.A. Thomas. *Elements of Information Theory*. John Wiley & Sons, 1991.

[10] Y. Baram and N.R. Sandell. Consistent estimation on finite parameter sets with application to linear system identification. *IEEE Transactions on Automatic Control*, 23(3):451–454, June 1978.

[11] Y. Baram. A sufficient condition for consistent discrimination between stationary Gaussian models. *IEEE Transactions on Automatic Control*, 23(5):958–960, October 1978.

[12] I. Hwang, H. Balakrishnan, and C. Tomlin. Observability criteria and estimator design for stochastic linear hybrid systems. In *Proceedings of IEE European Control Conference*, Cambridge, UK, September 2003. Submitted.

[13] M.J. Caputi. A necessary condition for effective performace of the multiple model adaptive estimator. *IEEE Transactions on Aerospace and Electronic Systems*, 31(3):1132–1138, July 1995.

[14] H.A.P. Blom and Y. Bar-Shalom. The interacting multiple model algorithm for systems with Markvian switching coefficients. *IEEE Transactions on Automatic Control*, 33(8):780–783, 1988.

[15] X.R. Li and Y. Bar-Shalom. Design of an interacting multiple model algorithm for air traffic control tracking. *IEEE Transactions on Control Systems Technology*, 1(3):186–194, September 1993.

[16] P.D. Hanlon and P.S. Maybeck. Characterization of Kalman filter residuals in the presence of mismodeling. In *Proceedings of the* 37^{th} *IEEE Conference on Decision and Control*, pages 1254–1259, Tampa, FL, December 1998.

[17] R.A. Horn. *Matrix Analysis*. Cambridge University Press, 1985.

[18] T. Kailath. *Linear Systems*. Prentice Hall, 1980.