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Using Fluid Models to Prove Stability of Adversarial Queueing Networks

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***Abstract*—A digital communication network can be modeled as an adversarial queueing network. An adversarial queueing network is defined to be stable if the number of packets stays bounded over time. A central question is to determine which adversarial queueing networks are stable under every work-conserving packet routing policy. Our main result is that stability of an adversarial queueing network is implied by stability of an associated fluid queueing network.**

***Index Terms*—Packet routing, queueing networks, stability.**

I. INTRODUCTION

In this paper we investigate the behavior of communication networks in which packets are generated dynamically and are routed from sources to destinations. Such a communication network is usually modeled as a graph, in which each arriving packet has its own prescribed simple path that it will follow from its source to its destination.

A surge of research activity in the area of packet routing in communication networks has been motivated by digital communication technology. In particular, requirements for providing specific level of performance quality can be translated into questions of stability and performance of packet routing schedules in communication networks. A packet routing schedule is defined to be stable if the number of packets stays bounded as the system runs over a long time period. It is known [1], [19], [21] that the natural load condition that the packet arrival rate is not bigger than the processing rate, is not sufficient for stability, and in general, deciding the stability of a given network under every policy or under a given policy is an unresolved problem.

One direction of stability research has been conducted within the framework of adversarial queueing networks. In such networks an adversary injects packets for processing. There are no probabilistic assumptions on the arrival of packets. This model was introduced by Borodin *et al.* [3]. It was shown by Andrews *et al.* that certain policies like Furthest-To-Go (FTG), Shortest-In-System (NIS), and Longest-In-System (LIS) are stable in all networks, but First-In-First-Out (FIFO) and Nearest-To-Go (NTG) policies are unstable in some networks [1]. Acyclic graphs and unidirectional rings were proven to be stable for all work-conserving policies [1], [3]. Also a complete characterization of graphs which are stable under any work-conserving policy and for any arrival rate smaller than one was constructed by Goel [17] and Gamarnik [15]. In general, however, it is not known which networks are stable under all work-conserving policies for a given specific arrival rate.

A parallel research activity has been conducted for stochastic queueing networks [5], [19], [21]. Here, stability is usually defined as finiteness of the expected number of customers (packets) in steady state. Stability results similar to the ones for adversarial queues have been established: acyclic networks and unidirectional rings are stable for all work-conserving policies [11], [12], [22]. Most of these stability results can be established through analysis of an associated deterministic fluid model. It was proven by Dai [8] that stability of fluid queueing network implies the stability of the underlying stochastic queueing network. Partial converse results were proven by Dai [9] and Meyn [20].

The two directions in stability research naturally lead to the question of whether a single technique can be used for the stability analysis of both stochastic and adversarial queues. In this paper we demonstrate that fluid models can be used for both. Our main result is that the stability of a fluid model implies the stability of an underlying adversarial queueing network. This parallels the result by Dai [8] for stochastic networks. However, the proof technique is different and the result is stated in a worst case sense as opposed to an expected sense. A result similar to ours was established by Hajek [18]. He considers a more general network in which processing times depend on the path and on the edge ("station" in Hajek's terminology). On the other hand he assumes that arrival rates are constant for each path, an assumption we do not require.

We also extend our result to specifically the FIFO policy. Applying Bramson's result on stability of FIFO policies in fluid models [4] we prove that the FIFO policy is stable in adversarial queueing networks when each path has a constant arrival rate.

II. DEFINITIONS AND ASSUMPTIONS

The definitions and assumptions of this paper are adopted from [1]. An adversarial queueing network consists of an undirected graph (V, E), where V is the set of vertices and E is the set of edges. The packets are injected into the network by an adversary; each packet is injected into some node, follows a packet specific path to its destination node, and then is dropped from the network. Although it is not necessary for our analysis, we will assume that all the preassigned paths are simple, so no packet traverses any given edge more than once. Each packet takes a unit of time to traverse a single edge, and only one packet at time can traverse any given edge (in either direction). Packet processing occurs at integer time epochs $t = 0, 1, 2, \cdots$. Packets waiting to traverse an edge e accumulate into a queue.

To describe the dynamics of our network, we introduce some additional notations. Let $\mathcal{P} = \{P_1, P_2, \cdots, P_M\}$ be a set of simple paths in the graph (V, E). Packets will follow paths in $\mathcal{P}$, which might be the set of all simple paths, or might be just a subset of it. For each path $P \in \mathcal{P}$, let $\{e_0^P, e_1^P, \cdots, e_{k(P)}^P\}$ be the set of consecutive edges in P. Let $A_P(t_1, t_2)$ be the total number of packets that are injected during time interval $[t_1, t_2)$ and use path P. For each $P \in \mathcal{P}$ and

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$e \in P$, let $D_{(e, P)}(t_1, t_2)$ be the total number of packets, following path P and traversing edge e within the time interval $[t_1, t_2)$. In particular, $D_{(e, P)}(t, t+1)$ takes value 0 or 1, for each $t = 0, 1, 2, \cdots$. Finally, let $Q_{(e, P)}(t)$ be the total number of packets following path P that are waiting to traverse edge e at time t. We denote by $Q(t)$ the vector $(Q_{(e, P)}(t))_{e \in P, P \in \mathcal{P}}$ of all queue lengths at time t and by $|Q(t)|$ the total number of packets in the network at time t:

$$|Q(t)| = \sum_{e \in P, P \in \mathcal{P}} Q_{(e, P)}(t).$$

For simplicity we denote $A_P(0, t)$ by $A_P(t)$ and $D_{(e, P)}(0, t)$ by $D_{(e, P)}(t)$. Also let $A(t) = (A_P(t))_{P \in \mathcal{P}}$ denote the vector of arrivals up to time t, and let $D(t) = (D_{(e, P)}(t))_{e \in P, P \in \mathcal{P}}$ denote the vector of departures up to time t. The dynamics of the network is described as follows. For each $t = 0, 1, 2, \cdots$ and each path $P \in \mathcal{P}$

$$Q_{(e_0^P, P)}(t) = Q_{(e_0^P, P)}(0) + A_P(t) - D_{(e_0^P, P)}(t) \tag{1}$$

and for all $i = 1, 2, \cdots, k(P)$

$$Q_{(e_i^P, P)}(t) = Q_{(e_i^P, P)}(0) + D_{(e_{i-1}^P, P)}(t) - D_{(e_i^P, P)}(t). \tag{2}$$

For each edge $e \in E$ and each time interval $[t_1, t_2)$ we have the feasibility constraint

$$\sum_{P:\, e \in P} D_{(e, P)}(t_1, t_2) \le t_2 - t_1. \tag{3}$$

In particular, for each $t = 0, 1, 2, \cdots$ and each $e \in E$

$$\sum_{P:\, e \in P} D_{(e, P)}(t, t+1) \le 1.$$

Packets are injected into the system by an adversary subject to the following restriction. There exists a positive real *rate* r and an integer w such that for each edge e and each time interval $[t_1, t_2)$, the total number of packets injected during $[t_1, t_2)$ whose assigned paths contain edge e, is at most $r(t_2 - t_1) + w$. Formally, for each $e \in E$ and $t = 0, 1, 2, \cdots$, we restrict the adversary by

$$\sum_{P:\, e \in P} A_P(t_1, t_2) \le r(t_2 - t_1) + w. \tag{4}$$

Assumption (4) is a generalization of the assumption considered by Cruz in [7], where for each path P the associated arrival process was assumed to be sublinear: for some r_P, b_P and for all $t \ge 0$

$$A_P(t) \le r_P t + b_P. \tag{5}$$

We will say that the network has pathwise constant arrival rates if (5) holds and if, in addition, for some $b'_P \ge 0$ and for all $t \ge 0$

$$A_P(t) \ge r_P t - b'_P. \tag{6}$$

We will be interested in networks, for which the queue sizes at each edge stay bounded over time. A necessary condition for this is

$$r \le 1. \tag{7}$$

As in [1] we will concentrate only on the subcritical case $r < 1$ throughout the paper. A quadruple (V, E, r, w) will be called an *adversarial queueing network* with parameters r and w. Any feasible solution $(Q(t), A(t), D(t))$ to (1)–(4) will be called a realization in the network (V, E, r, w).

So far we have not discussed the *scheduling policies* by which packets are chosen to cross edges. We will consider when an adversarial queueing network is stable under every work-conserving policy and when it is stable specifically under the FIFO policy.

Definition 1: A realization $(Q(t), A(t), D(t))$ in an adversarial queueing network (V, E, r, w) is *work conserving* if whenever there are packets waiting to cross an edge e at time t, at least one of these packets will cross e during interval $[t, t+1)$. Formally, for each $e \in E$ and $t = 0, 1, 2, \cdots$,

$$\sum_{P:\, e \in P} Q_{(e, P)}(t) > 0 \quad \text{implies} \quad \sum_{P:\, e \in P} D_{(e, P)}(t, t+1) = 1. \tag{8}$$

Definition 2: A realization $(Q(t), A(t), D(t))$ in an adversarial queueing network (V, E, r, w) is *stable* if the total number of packets in the system stays bounded over time, i.e., if

$$\sup_{t \in Z_+} |Q(t)| < \infty.$$

An adversarial queueing network (V, E, r, w) is *universally stable*, if every work conserving realization is stable.

Remark 1: Note that, unlike in [1], [3], and [17], our definition of universal stability depends on the arrival rate r and on the set of paths $\mathcal{P}$ used by packets. In this respect our definition is more general.

III. Fluid Models of Adversarial Queueing Networks

In this section we introduce a fluid model which is a continuous-time, continuous-state approximation of an adversarial queueing network.

Given an adversarial queueing network (V, E, r, w) we define an associated fluid model as follows. The fluid model inherits the graph (V, E), the set of paths $\mathcal{P}$, and the arrival rate r. For each path $P \in \mathcal{P}$ in the fluid model there is an external arrival of flow that needs to be processed through the path P. For each positive real t the total amount of flow arriving during the time interval $[0, t]$ and following path P is a nonnegative real value $\overline{A}_P(t) \in \Re_+$. Flow waiting to traverse an edge accumulates into a queue. The flow arrival process $\overline{A}_P(t)$ is an arbitrary continuous nondecreasing function subject to the following load constraint. For all $e \in E$ and $t_1 < t_2 \in \Re_+$

$$\sum_{P:\, e \in P} \left(\overline{A}_P(t_2) - \overline{A}_P(t_1)\right) \le r(t_2 - t_1). \tag{9}$$

In particular, the arrival process is Lipschitz continuous. The total amount of flow following path P and crossing an edge $e \in P$ during the time interval $[0, t]$ is denoted by $\overline{D}_{(e, P)}(t)$, which is also a continuous nondecreasing function. The flow processing rate is equal to one for all edges. In particular, for each edge $e \in E$ and each pair of nonnegative reals $t_1 < t_2$

$$\sum_{P:\, e \in P} \left(\overline{D}_{(e, P)}(t_2) - \overline{D}_{(e, P)}(t_1)\right) \le t_2 - t_1. \tag{10}$$

The total amount of flow following path P and waiting to cross edge e at time $t \in \Re_+$ is some nonnegative real value $Q_{(e,\,P)}(t)$. Then, for each $P \in \mathcal{P}$ and $t \in \Re_+$

$$\overline{Q}_{(e_0^P,\,P)}(t) = \overline{Q}_{(e_0^P,\,P)}(0) + \overline{A}_P(t) - \overline{D}_{(e_0^P,\,P)}(t) \tag{11}$$

and for all $i = 1, 2, \cdots, k(P)$

$$\overline{Q}_{(e_i^P,\,P)}(t) = \overline{Q}_{(e_i^P,\,P)}(0) + \overline{D}_{(e_{i-1}^P,\,P)}(t) - \overline{D}_{(e_i^P,\,P)}(t). \tag{12}$$

Let $\overline{A}(t) = (\overline{A}_P(t))_{P\in\mathcal{P}}$, $\overline{Q}(t) = (\overline{Q}_{(e,\,P)}(t))_{e\in P,\,P\in\mathcal{P}}$, $\overline{D}(t) = (\overline{D}_{(e,\,P)}(t))_{e\in P,\,P\in\mathcal{P}}$. We will assume that each edge e processes a fluid as long as there is some fluid waiting to cross e (work conservation). Namely, for any pair of nonnegative reals $t_1 < t_2$ and for any edge $e \in E$

$$\sum_{P:\,e\in P} \overline{Q}_{(e,\,P)}(t) > 0, \quad \text{for all } t \in [t_1,\,t_2] \text{ implies}$$
$$t_2 - t_1 = \sum_{P:\,e\in P} \left(\overline{D}_{(e,\,P)}(t_2) - \overline{D}_{(e,\,P)}(t_1)\right). \tag{13}$$

This parallels the corresponding definition of work conservation in adversarial queueing networks. Any feasible solution $(\overline{Q}(t), \overline{A}(t), \overline{D}(t))$ to the system of equalities and inequalities (9)–(13) will be called a (work conserving) fluid solution. The fluid network (V, E, r) can be viewed as a continuous time, continuous state analog of the discrete adversarial queueing network (V, E, r, w).

Definition 3: A fluid network (V, E, r) is defined to be globally stable if there exists some time τ, such that any work-conserving fluid solution $(\overline{A}(t), \overline{Q}(t), \overline{D}(t))$ with initial vector of queue lengths $\overline{Q}(0)$ having $|\overline{Q}(0)| = 1$, satisfies $\overline{Q}(t) = 0$, for all $t \geq \tau$.

As we see, the definition of stability for fluid models is somewhat different from the one for adversarial queues. Instead of requiring a bounded number of packets in the network, the fluid stability requires the network to become empty after some finite time. The reason for such definition will become more intuitive in the following section, when we discuss how fluid solutions are obtained as limits of realizations in adversarial queueing networks.

IV. The Connection Between Fluid and Adversarial Stability—Main Result

The main result of the paper is given by the following theorem.

Theorem 1: We are given an adversarial queueing network (V, E, r, w). If the associated fluid network (V, E, r) is globally stable, then the network (V, E, r, w) is universally stable.

We need the following lemma.

Lemma 2: Let $(Q(t), A(t), D(t))$ be any realization in an adversarial queuing network (V, E, r, w). Then for any time $t = 0, 1, 2, \cdots$ and any edge e

$$\left|\sum_{P:\,e\in P} Q_{(e,\,P)}(t+1) - \sum_{P:\,e\in P} Q_{(e,\,P)}(t)\right| \leq w + |\mathcal{P}| \tag{14}$$

and

$$\big|\,|Q(t+1)| - |Q(t)|\,\big| \leq (w + |\mathcal{P}|)|E|.$$

Proof: Note that left-hand side of (14) is equal to the difference between the total number of packets arriving into edge e and departing from e during $[t,\, t+1)$. By constraint (4) the total number of packets injected is at most $\lfloor r + w \rfloor = w$. The total number of packets arriving from previous edges is at most the number of paths containing edge e, which in turn is at most $|\mathcal{P}|$. During the same time interval at most one packet departs, since at most one packet at a time can cross any edge. We conclude that the maximal change of the number of packets in edge e during time interval $[t,\, t+1)$ is at most $\max\{w+|\mathcal{P}|,\, 1\} = w+|\mathcal{P}|$. This proves (14). The total change of the number of packets in the network then satisfies

$$\big|\,|Q(t+1)| - |Q(t)|\,\big| \leq (w + |\mathcal{P}|)|E|.$$

∎

We now state and prove a proposition which is a key to proving Theorem 1. It provides a general technique for building solutions to fluid models from realizations in the underlying adversarial network. Then, in order to prove Theorem 1, we will show that unstable realizations correspond to unstable fluid solutions.

Proposition 1: Given a realization $(Q(t),\, A(t),\, D(t))$ in an adversarial queueing network (V, E, r, w), suppose a nondecreasing sequence of integer times $t_1, t_2, \cdots, t_k, \cdots$ is such that $|Q(t_k)| \leq k + C$ for some constant C. Then there exists a sequence of positive integers $k_1, k_2, \cdots, k_n, \cdots$ with the following properties.

1) The limits

$$\lim_{n\to\infty} \frac{Q(t_{k_n} + tk_n)}{k_n}, \qquad \lim_{n\to\infty} \frac{A(t_{k_n} + tk_n) - A(t_{k_n})}{k_n}$$
$$\lim_{n\to\infty} \frac{D(t_{k_n} + tk_n) - D(t_{k_n})}{k_n} \tag{15}$$

exist for each nonnegative real number t. These limits will be called *fluid limits* and will be denoted by $\overline{Q}(t)$, $\overline{A}(t)$, and $\overline{D}(t)$, respectively.

2) The vector-valued function $(\overline{Q}(t), \overline{A}(t), \overline{D}(t))$ is a fluid solution of the fluid model (V, E, r). If the realization $(Q(t), A(t), D(t))$ is work conserving, then the fluid solution $(\overline{Q}(t), \overline{A}(t), \overline{D}(t))$ is also work conserving.

Proof of Proposition 1: Let $\mathcal{Q}_+ = \{q_1, q_2, \cdots, q_n, \cdots\}$ denote the (countable) set of nonnegative rational numbers. We first find a sequence $k_1^{(1)}, k_2^{(1)}, \cdots, k_n^{(1)}, \cdots$ such that limits (15) exist for $t = q_1$. Consider all integers $k > 0$ such that kq_1 is also an integer. Applying Lemma 2 to the differences $Q(t_k+1) - Q(t_k)$, $Q(t_k+2) - Q(t_k+1), \cdots, Q(t_k + q_1 k) - Q(t_k + q_1 k - 1)$, we obtain

$$\frac{|Q(t_k + q_1 k)|}{k} \leq \frac{k + C + q_1 k (w + |\mathcal{P}|)|E|}{k}$$
$$= 1 + q_1(w + |\mathcal{P}|)|E| + \frac{C}{k}.$$

Also, note from (4) and (3) that for each $P \in \mathcal{P}$ and $e \in P$

$$\frac{A_P(t_k + q_1 k) - A_P(t_k)}{k} \leq \frac{rq_1 k + w}{k} = rq_1 + \frac{w}{k}$$

and

$$\frac{D_{(e,\,P)}(t_k + q_1 k) - D_{(e,\,P)}(t_k)}{k} \leq \frac{q_1 k}{k} = q_1.$$

In particular, all the ratios are bounded. Then there exists an infinite sequence $k_n^{(1)}$, $n = 0, 1, 2, \cdots$ such that the limits

$$\lim_{n\to\infty} \frac{Q(t_{k_n^{(1)}} + q_1 k_n^{(1)})}{k_n^{(1)}}, \qquad \lim_{n\to\infty} \frac{A(t_{k_n^{(1)}} + q_1 k_n^{(1)}) - A(t_{k_n^{(1)}})}{k_n^{(1)}}$$
$$\lim_{n\to\infty} \frac{D(t_{k_n^{(1)}} + q_1 k_n^{(1)}) - D(t_{k_n^{(1)}})}{k_n^{(1)}} \tag{16}$$

exist. We denote these limits by $\overline{Q}(q_1)$, $\overline{A}(q_1)$, and $\overline{D}(q_1)$, respectively.

By a similar argument, we can construct a subsequence $k_n^{(2)}$ of the sequence $k_n^{(1)}$ such that the limits

$$\lim_{n\to\infty} \frac{Q(t_{k_n^{(2)}} + q_2 k_n^{(2)})}{k_n^{(2)}}, \qquad \lim_{n\to\infty} \frac{A(t_{k_n^{(2)}} + q_2 k_n^{(2)}) - A(t_{k_n^{(2)}})}{k_n^{(2)}}$$
$$\lim_{n\to\infty} \frac{D(t_{k_n^{(2)}} + q_2 k_n^{(2)}) - D(t_{k_n^{(2)}})}{k_n^{(2)}} \tag{17}$$

exist. We denote these limits by $\overline{Q}(q_2)$, $\overline{A}(q_2)$, $\overline{D}(q_2)$. Since $k_n^{(2)}$ is a subsequence of $k_n^{(1)}$, (16) exists when $k_n^{(2)}$ is substituted for $k_n^{(1)}$. Continuing, we build a series of sequences $k_n^{(m)}$, $m = 1, 2, \cdots$ with the following properties:

1) for each m, $k_n^{(m)}$ is a subsequence of $k_n^{(m-1)}$;
2) for each l and each $m \geq l$ the limits

$$\lim_{n\to\infty} \frac{Q(t_{k_n^{(m)}} + q_l k_n^{(m)})}{k_n^{(m)}}$$
$$\lim_{n\to\infty} \frac{A(t_{k_n^{(m)}} + q_l k_n^{(m)}) - A(t_{k_n^{(m)}})}{k_n^{(m)}}$$
$$\lim_{n\to\infty} \frac{D(t_{k_n^{(m)}} + q_l k_n^{(m)}) - D(t_{k_n^{(m)}})}{k_n^{(m)}}$$

exist and are independent of m. Denote these limits by $\overline{Q}(q_l)$, $\overline{A}(q_l)$, $\overline{D}(q_l)$.

Now, consider a diagonal sequence $k_n = k_n^{(n)}$, $n = 0, 1, 2, \cdots$. Each $k_n^{(m)}$ has k_n as a subsequence, disregarding finitely many initial terms of k_n. It follows that for each rational q_l the limits

$$\lim_{n\to\infty} \frac{Q(t_{k_n} + q_l k_n)}{k_n}, \qquad \lim_{n\to\infty} \frac{A(t_{k_n} + q_l k_n) - A(t_{k_n})}{k_n}$$
$$\lim_{n\to\infty} \frac{D(t_{k_n} + q_l k_n) - D(t_{k_n})}{k_n}$$

exist and are equal to $\overline{Q}(q_l)$, $\overline{A}(q_l)$, $\overline{D}(q_l)$, respectively.

We have proved the existence of limits (15) for the sequence k_n and for all nonnegative rational values of t. Let us now prove that the functions $\overline{Q}_{(e,P)}(t)$, $\overline{A}_P(t)$, $\overline{D}_{(e,P)}(t)$ are Lipschitz continuous on $\mathcal{Q}_+$. Fix a pair of nonnegative rational numbers $t' < t''$. For each edge e and path P

$$\begin{aligned}\overline{A}_P(t'') - \overline{A}_P(t') &\leq \sum_{P:\, e\in P} \left(\overline{A}_P(t'') - \overline{A}_P(t')\right) \\ &= \lim_{n\to\infty} \sum_{P:\, e\in P} \frac{A_P(t_{k_n} + t'' k_n) - A_P(t_{k_n} + t' k_n)}{k_n} \\ &\leq \lim_{n\to\infty} \frac{r(t''-t')k_n + w}{k_n} = r(t''-t')\end{aligned} \tag{18}$$

where the second inequality follows from (4). The Lipschitz continuity of $\overline{D}_{(e,P)}(t)$ and $\overline{Q}_{(e,P)}(t)$ is proved similarly, using (3), (1), and (2). In particular, we obtain that for each edge e, path P, and any pair of rational values $t' < t''$

$$\sum_{P:\, e\in P} \left(\overline{D}_{(e,P)}(t'') - \overline{D}_{(e,P)}(t')\right) \leq t'' - t' \tag{19}$$

$$\begin{aligned}\overline{Q}_{(e_0^P,P)}(t'') = {} & \overline{Q}_{(e_0^P,P)}(t') + \overline{A}_P(t'') - \overline{A}_P(t') \\ & - \left(\overline{D}_{(e_0^P,P)}(t'') - \overline{D}_{(e_0^P,P)}(t')\right)\end{aligned} \tag{20}$$

and

$$\begin{aligned}\overline{Q}_{(e_i^P,P)}(t'') = {} & \overline{Q}_{(e_i^P,P)}(t') + \overline{D}_{(e_{i-1}^P,P)}(t'') - \overline{D}_{(e_{i-1}^P,P)}(t') \\ & - \left(\overline{D}_{(e_i^P,P)}(t'') - \overline{D}_{(e_i^P,P)}(t')\right)\end{aligned} \tag{21}$$

for $i = 1, 2, \cdots, k(P)$. From Lipschitz continuity of the functions $\overline{Q}(t)$, $\overline{A}(t)$, $\overline{D}(t)$ on $\mathcal{Q}_+$, it follows that there exists a unique Lipschitz continuous extension of these functions to the $\Re_+$.

For all $t \in \Re_+$ the extension satisfies

$$\overline{Q}(t) = \lim_{q\to t,\, q\in\mathcal{Q}_+} \overline{Q}(q), \quad \overline{A}(t) = \lim_{q\to t,\, q\in\mathcal{Q}_+} \overline{A}(q)$$
$$\overline{D}(t) = \lim_{q\to t,\, q\in\mathcal{Q}_+} \overline{D}(q). \tag{22}$$

The limits (15) then hold for any nonnegative real value of t, because for any nonnegative real t and any rationals t', t'' such that $t' \leq t \leq t''$, we have

$$\frac{A_P(t_{k_n} + t' k_n)}{k_n} \leq \frac{A_P(t_{k_n} + t k_n)}{k_n} \leq \frac{A_P(t_{k_n} + t'' k_n)}{k_n}$$

and

$$\frac{D_{(e,P)}(t_{k_n} + t' k_n)}{k_n} \leq \frac{D_{(e,P)}(t_{k_n} + t k_n)}{k_n} \leq \frac{D_{(e,P)}(t_{k_n} + t'' k_n)}{k_n}.$$

This completes the proof of the first part of Proposition 1.

To prove the second part, we need to show that the constructed process $(\overline{Q}(t), \overline{A}(t), \overline{D}(t))$, $t \in \Re_+$ satisfies constraints (9)–(13) of the fluid model (V, E, r). From (18)–(22) it follows that (9)–(12) are satisfied. We now show that the fluid solution is work conserving if the realization $(Q(t), A(t), D(t))$ is work conserving. Consider any interval $[t_1, t_2]$ and edge $e \in E$ such that for all $t \in [t_1, t_2]$

$$\sum_{P:\, e\in P} \overline{Q}_{(e,P)}(t) > 0. \tag{23}$$

From (17) we have

$$\sum_{P:\, e\in P} \left(\overline{D}_{(e,\,P)}(t_2) - \overline{D}_{(e,\,P)}(t_1)\right) = \lim_{n\to\infty} \sum_{P:\, e\in P} \frac{D_{(e,\,P)}(t_{k_n}+t_2k_n) - D_{(e,\,P)}(t_{k_n}+t_1k_n)}{k_n}.$$

If

$$\sum_{P:\, e\in P} \left(D_{(e,\,P)}(t_{k_n}+t_2k_n) - D_{(e,\,P)}(t_{k_n}+t_1k_n)\right) = (t_2-t_1)k_n$$

for all sufficiently large n, then

$$\sum_{P:\, e\in P} \left(\overline{D}_{(e,\,P)}(t_2) - \overline{D}_{(e,\,P)}(t_1)\right) = t_2 - t_1$$

and we are done. Otherwise, the sequence k_n has an infinite subsequence $\hat{k}_n$, for which

$$\sum_{P:\, e\in P} \left(D_{(e,\,P)}(t_{\hat{k}_n}+t_2\hat{k}_n) - D_{(e,\,P)}(t_{\hat{k}_n}+t_1\hat{k}_n)\right) < (t_2-t_1)\hat{k}_n.$$

From the work conservation constraint (8) for the underlying adversarial queueing network, it follows that for each n, there exists $z_n \in [t_{\hat{k}_n}+t_1\hat{k}_n,\, t_{\hat{k}_n}+t_2\hat{k}_n]$ such that

$$\sum_{P:\, e\in P} Q_{(e,\,P)}(z_n) = 0. \tag{24}$$

Note that the sequence $(z_n - t_{\hat{k}_n})/\hat{k}_n$ is contained in $[t_1,\, t_2]$ and therefore has some accumulation point $z_0 \in [t_1,\, t_2]$. We may assume that z_0 is in fact a limit point (if not replace the sequence with a convergent subsequence). We now argue that $\sum_{P:\, e\in P} \overline{Q}_{(e,\,P)}(z_0) = 0$. From (17)

$$\sum_{P:\, e\in P} \overline{Q}_{(e,\,P)}(z_0) = \lim_{n\to\infty} \sum_{P:\, e\in P} \frac{Q_{(e,\,P)}\left(t_{\hat{k}_n}+z_0\hat{k}_n\right)}{\hat{k}_n}.$$

Applying Lemma 2

$$\left|\sum_{P:\, e\in P} Q_{(e,\,P)}\left(t_{\hat{k}_n}+z_0\hat{k}_n\right) - \sum_{P:\, e\in P} Q_{(e,\,P)}(z_n)\right| \le \left|t_{\hat{k}_n}+z_0\hat{k}_n - z_n\right|(w+|\mathcal{P}|).$$

Then, using (24)

$$\lim_{n\to\infty} \left|\sum_{P:\, e\in P} \frac{Q_{(e,\,P)}\left(t_{\hat{k}_n}+z_0\hat{k}_n\right)}{\hat{k}_n}\right| \le \lim_{n\to\infty} \frac{\left|t_{\hat{k}_n}+z_0\hat{k}_n - z_n\right|(w+|\mathcal{P}|)}{\hat{k}_n} = 0$$

where the last equality follows from the definition of z_0. We conclude that $\sum_{P:\, e\in P} \overline{Q}_{(e,\,P)}(z_0) = 0$, which contradicts (23). Thus, the constructed solution $(\overline{Q}(t),\, \overline{A}(t),\, \overline{D}(t)),\, t \in \Re_+$ is work conserving. This completes the proof of Proposition 1. ■

We now have the necessary tools for proving our main result.

Proof of Theorem 1: We will prove the result by contradiction. Suppose there exists an unstable realization $(Q(t),\, A(t),\, D(t)),\, t = 0,\, 1,\, 2,\, \cdots$ in the adversarial network $(V,\, E,\, r,\, w)$. Let τ be the emptying time specified by Definition 3. In particular, any fluid solution satisfying $|\overline{Q}(0)| = 1$, must also satisfy $\overline{Q}(\tau) = 0$. We will construct a particular fluid solution $(\overline{Q}(t),\, \overline{A}(t),\, \overline{D}(t))$ satisfying $|Q(0)| = 1$ and $\overline{Q}(\tau) \ne 0$, thereby obtaining a contradiction.

Lemma 3: Given an unstable realization $(Q(t),\, A(t),\, D(t)),\, t = 0,\, 1,\, 2,\, \cdots$ there exists a nondecreasing sequence of times $t_1,\, t_2,\, \cdots,\, t_k,\, \cdots$ such that $k \le |Q(t_k)| \le k + (w+|\mathcal{P}|)|E|$ and $|Q(t_k+\tau k)| \ge k$ for all $k \ge |Q(0)|$.

Proof: Put $t_i = 0$ for all $i = 1,\, 2,\, \cdots |Q(0)|-1$. Fix $k \ge |Q(0)|$. We now construct the kth member of the required sequence, assuming that t_i are constructed for $i \le k-1$. Let $\hat{t}_1 \ge t_{k-1}$ be the smallest time such that $|Q(\hat{t}_1)| \ge k$. (Such a time exists since the realization is unstable). From Lemma 2 $|Q(\hat{t}_1)| \le k + (w+|\mathcal{P}|)|E|$ and $|Q(t)| \le k + (t - \hat{t}_1 + 1)(w+|\mathcal{P}|)|E|$ for $t = \hat{t}_1,\, \hat{t}_1+1,\, \cdots,\, \hat{t}_1+\tau k$. If $|Q(\hat{t}_1+\tau k)| \ge k$, then put $t_k = \hat{t}_1$ and we are done. Otherwise, let $\hat{t}_2$ be the first time after $\hat{t}_1 + \tau k$, for which $|Q(\hat{t}_2)| \ge k$. Again $|Q(\hat{t}_2)| \in [k,\, k+(w+|\mathcal{P}|)|E|]$. If $|Q(\hat{t}_2+k\tau)| \ge k$ then put $t_k = \hat{t}_2$. Otherwise, find $\hat{t}_3$, and so on. If this procedure never stops, then we obtain a sequence $\hat{t}_1,\, \hat{t}_2,\, \cdots,\, \hat{t}_n,\, \cdots$ such that for all n, $|Q(t)| \in [k,\, k+(\tau k+1)(w+|\mathcal{P}|)|E|]$ for $t \in [\hat{t}_n,\, \hat{t}_n+\tau k-1]$, and $|Q(t)| < k$ for $t \in [\hat{t}_n+\tau k,\, \hat{t}_{n+1}-1]$. In particular, the realization $Q(t),\, A(t),\, D(t)$ is stable, contradicting the instability assumption. ■

Now apply Proposition 1 to the given unstable realization and the sequence $t_1,\, t_2,\, \cdots,\, t_k,\, \cdots$, constructed in Lemma 3. From Proposition 1 there is a subsequence k_n for which the fluid limit $\overline{Q}(t)$ satisfies

$$|\overline{Q}(0)| = \lim_{n\to\infty} \frac{|Q(t_{k_n})|}{k_n} \le \lim_{n\to\infty} \frac{k_n + (w+|\mathcal{P}|)|E|}{k_n} = 1$$

where the inequality follows from $|Q(t_{k_n})| \le k_n + (w+\mathcal{P})|E|$. Similarly, $|\overline{Q}(0)| \ge 1$. Thus $|\overline{Q}(0)| = 1$. Also

$$\overline{Q}(\tau) = \lim_{n\to\infty} \frac{Q(t_{k_n}+\tau k_n)}{k_n} \ge 1$$

since $|Q(t_{k_n}+\tau k_n)| \ge k_n$. We conclude that our fluid limit satisfies $\overline{Q}(\tau) \ne 0$, contradicting global stability. ■

V. Extension to the First-In-First-Out Scheduling Policy

We now establish an analog of Theorem 1 for networks operating specifically under the FIFO policy. We begin by constructing fluid models corresponding to the FIFO policy. (Such fluid models have been considered previously; see [4] and [6]). In analogy with Theorem 1, we then prove that the stability of fluid solutions corresponding to the FIFO policy implies stability of the FIFO policy in the underlying adversarial queueing network.

Let $Q_e(t)$ denote the total number of packets waiting to cross edge e at time t, $Q_e(t) = \sum_{P:\, e\in P} Q_{(e,\,P)}(t)$. Since the service rate is one, the time it takes to process these $Q_e(t)$ packets is exactly $Q_e(t)$. Under the FIFO policy, any packet arriving at e before t will be processed before $t+Q_e(t)$. Also, any packet arriving at e after t will be processed after $t+Q_e(t)$. Thus, for each path P

$$D_{(e_0^P,\, P)}\left(t + Q_{e_0^P}(t)\right) = Q_{(e_0^P,\, P)}(0) + A_P(t) \tag{25}$$

and for all $i = 1, 2, \cdots, k(P)$

$$D_{(e_i^P, P)}\left(t + Q_{e_0^P}(t)\right) = Q_{(e_i^P, P)}(0) + D_{(e_{i-1}^P, P)}(t). \quad (26)$$

It is this property that we adopt to fluid solutions corresponding to FIFO policy. A solution $(\overline{Q}(t), \overline{A}(t), \overline{D}(t)$ to (9)–(13) is defined to be FIFO if it satisfies (25) and (26). The following theorem is an analog of Theorem 1 for networks operating under FIFO policy.

Theorem 4: Given an adversarial queueing network (V, E, r, w) and associated fluid model (V, E, r). Suppose there exists $\tau > 0$ such that all FIFO fluid solutions with $|\overline{Q}(0)| = 1$ have $\overline{Q}(t) = 0$ for all $t \geq \tau$. Then the FIFO policy is stable in the adversarial queueing network (V, E, r, w). That is, if all FIFO fluid solutions are stable, then the FIFO policy is stable in the underlying adversarial queueing network.

The proof of this theorem is similar to that of Theorem 1. In the interest of space it is omitted, but can be found in [16]. Now consider adversarial queueing networks with path-wise constant arrival rates. Recall that network has path-wise constant arrival rates if for each path P there exists $r_P \geq 0$ such that for all edges e, $\sum_{P:\, e \in P} r_P < 1$, and (5) and (6) are satisfied. Using the Lyapunov function technique, Bramson proved that FIFO solutions are stable in fluid networks with path-wise constant arrival rates [4]. The emptying time τ can be expressed in terms of the parameters of the Lyapunov function. Combining this result with Theorem 4 we obtain the following theorem.

Theorem 5: The FIFO scheduling policy is stable in adversarial queueing networks with path-wise constant arrival rates.

In contrast, if arrival rates are not path-wise constant, the FIFO policy can be unstable [1].

VI. Conclusion

We have proposed a fluid model approximation of adversarial queueing networks for the purposes of stability analysis. We have proved that universal stability of an adversarial queueing network is implied by a global stability of an associated fluid model.

This result opens up an opportunity for using methods from continuous-time, continuous-state processes for stability analysis of adversarial networks. Such methods include Lyapunov functions [5], [8], [10], [11], [13] and trajectory decomposition [2], [14].

A number of interesting questions remain open. What are necessary and sufficient condition for universal stability of any given network? Networks, which are universally stable for all $r < 1$ are characterized in [15] and [17]. But question which networks are universally stable for specific $r < 1$ and specific set of path requested, remains unanswered. Note that constructing networks unstable for given r is progressively harder as r gets smaller. Moreover, it has been shown by Borodin *et al.* [3] that for an arbitrary small rates $r > 0$ there exist r-unstable networks. Characterizing exactly the set of r-unstable networks is another interesting open problem.

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