Minimality of the Group AUT(C)

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Abstract

We investigate the minimality property of the group of homeomorphisms $AUT(C)$ of some compact set C. In particular we show that for $C = [0,1]^n$ the corresponding group is minimal iff $n = 1$. Also we prove the minimality of $AUT(D^\infty)$, where D^∞ is a Cantor cube with a countable weight.

These results partially answer the general question raised by Prodanov and Stojanov ([2]) , Dierolf et.al. ([3]) and ([4]).

1 Preliminaries

A Hausdorff topological group (G, τ_0) is called *minimal* if no Hausdorff group topology τ on G is strictly coarser than τ_0.

Let (G, τ) be a topological group acting on some space X with uniformity Λ :

$$\alpha : G \times X \longrightarrow X$$

Let $N_e(\tau)$ be set of all τ -neighborhoods of the identity element $e \in G$.

Definition. Action α is called *quasibounded* if for every $P \in \Lambda$ there exist $Q_P \in \Lambda$ and $U_p \in N_e(\tau)$ such that whenever $(x, y) \in Q_P$ and $g \in N_e(\tau)$, then $(gx, gy) \in P$. Also a Hausdorff group topology τ on the group $AUT(K)$ of homeomorphisms of a compact set K is called quasibounded if the natural action

$$AUT(K) \times K \longrightarrow K$$

is quasibounded ([5]).

2 Minimality within the class of quasibounded topologies

In this section we show that the compact-open topology on the group AUT(K) is minimal among all the quasibounded topologies. First we recall the following characteristic property of compacts:

Lemma 2.1. *Let $\Im$ be an ultrafilter on the set X and let ϕ be an mapping from X into a compact set K. Then there exists one and only one point $\tilde{x}$ such that each nbd $O(\tilde{x})$ containing $\tilde{x}$ satisfies*

$$\phi^{-1}(O(\tilde{x})) \in \Im$$

In other words, $\tilde{x}$ is a limit point of ϕ under $\Im$.

We now state and prove the main result of this section

Proposition 2.1 *Let $AUT(K)$ be a group of homeomorphisms of a compact set K. The compact-open topology τ_0 is minimal within the class of quasibounded topologies on $AUT(K)$. Namely, if τ is a quasibounded Hausdorff topology on $AUT(K)$ such that $\tau \subset \tau_0$, then $\tau = \tau_0$.*

Proof: Let τ be a quasibounded Hausdorff topology on $AUT(K)$ which is strictly coarser than compact-open topology τ_0.

It can be easily verified that if the action

$$AUT(K) \times K \longrightarrow K$$

is continuous from left, i.e. if orbit maps

$$AUT(K) \longrightarrow K$$

given by

$$g \longrightarrow gk_0,$$

are continuous, then $\tau = \tau_0$. Thus, there exist $x_0 \in K$ and $P_0 \in \Lambda$ (Λ-natural uniformity on K) such that for every $U \in N_e(\tau)$ there exists $g_U \in U$ for which

$$(g_U x_0, x_0) \notin P_0 \tag{1}$$

For any given $U \in N_e(\tau)$, let

$$F(U) = \{V \in N_e(\tau) : V \subset U\}$$

Consider a filter on the set $N_e(\tau)$ with the filter base $\{F(U)\}_{U \in N_e(\tau)}$. Let $\Im$ be a ultrafilter containing this one. For each $x \in K$ we have a map

$$N_e(\tau) \longrightarrow K$$

defined by the rule

$$U \longrightarrow g_U x$$

Let $\tilde{x}$ be a point given by Lemma 2.1. Namely, $\tilde{x}$ is a limit point of this map under the filter $\Im$. This point exists due to the compactness of K. This, in its order, defines some trasformation

$$h : K \longrightarrow K, \;\; h(x) = \tilde{x}$$

Lemma 2.2. *h is a non-trivial homeomorphism of K.*

Proof of Lemma: First we show that transformation h is one-to-one. Assume for the purposes of contradiction that there exist $x, y, z \in K$ such that $h(x) = h(y) = z$ and $x \neq y$. Find $P \in \Lambda$ such that $(x, y) \notin P$.

Choose $U_P \in N_e(\tau),\;\; Q_P \in \Lambda$ from the definition of quasibounded topologies. Choose also $Q \in \Lambda$ satisfying $Q^2 \subset Q_P$.

Since by the definition of h, $z = h(x)$ is a limit of $g_U x$ under the filter $\Im$ then we have

$$A(x) = \{U \in \Lambda : (g_U x, z) \in Q\} \in \Im$$

Similarly,

$$A(y) = \{U \in \Lambda : (g_U y, z) \in Q\} \in \Im$$

Also by construction of $\Im$, $F(U_P^{-1}) \in \Im$. Hence, the intersection

$$A(x) \cap A(y) \cap F(U_P^{-1})$$

is non-empty. Let U_0 be any element of this intersection. Then in particular

$$g_{U_0} \in U_0 \subset U_P^{-1}, \;\; (g_{U_0} x, z) \in Q, \;\; (g_{U_0} y, z) \in Q$$

Therefore

$$(g_{U_0} x, g_{U_0} y) \in Q^2 \subset Q_P$$

But

$$(g_{U_0}^{-1} g_{U_0} x, g_{U_0}^{-1} g_{U_0} y) = (x, y) \notin P$$

Since $g_{U_0}^{-1} \in U_P$ this is a contradiction to the choice of Q_p and U_p. This proves that transformation h is one-to-one.

Now we show that transformation h is onto. Suppose y is an aritrary element of K. Consider the mapping

$$N_e(\tau) \longrightarrow K$$

given by

$$U \longrightarrow g_U^{-1} y$$

Let x be a limit of this mapping under filter $\Im$ as defined in Lemma 2.1. We will prove $h(x) = y$. For that purpose we need to show that y is the limit of $g_U x$ under the filter $\Im$. Let $P \in \Lambda$ be an arbitrary member of the uniformity class. Choose $U_P, \; Q_P$ from the definition of quasibounded topology. Since x is a limit of $g_U^{-1} y$ then

$$A(x) = \{U \in N_e(\tau) : (g_U^{-1} y, x) \in Q_P\} \in \Im$$

On the other hand $F(U_P)$ is also a member of filter $\Im$. Let

$$U \in A(x) \cap F(U_P)$$

Then in particular $U \subset U_P$ and $(g_U^{-1}y, x) \in Q_P$. But then by the choice of U_P and Q_P and from $g_U \in U \subset U_P$

$$(y, g_U x) \in P \tag{2}$$

We have shown that any member of the intersection

$$A(x) \cap F(U_P)$$

satisfies (2). Therefore

$$\{U \in N_e(\tau) : (y, g_U x) \in P\} \in \Im$$

This is true for any choice of P. So y is a limit of $g_U x$ under the filter $\Im$, which means $h(x) = y$. Thus transformation h is into and is a bijection.

Finally, to conclude the proof of the Lemma we show that h is continuous. Since K is a compact set this would also imply the continuity of h^{-1}. Let $P_0 \in \Lambda$ be arbitrary. Choose $P \in \Lambda$ such that $P^3 \subset P_0$. For P find the corresponding Q_P and U_P from the definition of quasibounded topologies. We will show that whenever

$$(x, y) \in Q_P \tag{3}$$

then

$$(h(x), h(y)) \in P_0$$

and therefore h is uniformly continuous. Suppose for some pair $(x, y) \in Q_P$, $(h(x), h(y)) \notin P_0$. This means if t', t'' satisfy

$$(h(x), t') \in P, \ \ (h(y), t'') \in P$$

then

$$(t', t'') \notin P \tag{4}$$

Since $h(x)$ is a limit of $g_U x$, $U \in N_e(\tau)$ then

$$A(x) = \{U \in N_e(\tau) : (g_U x, h(x)) \in P\} \in \Im$$

By a similar reason

$$A(y) = \{U \in N_e(\tau) : (g_U y, h(y)) \in P\} \in \Im$$

Also $F(U_P) \in \Im$. Let

$$V \in A(x) \cap A(y) \cap F(U_P)$$

In particular

$$(g_V x, h(x)) \in P$$

and

$$(g_V y, h(y)) \in P$$

Then from (4) we obtain

$$(g_V x, g_V y) \notin P$$

But $V \subset U_P$, so $g_V \in U_P$. Also recall from (3) $(x, y) \in Q_P$. This is a contradiction to the choice of U_P and Q_P. We conclude that h is continuous and thus it is a homeomorphism.

From (1) we have some $x_0 \in K$ and $P_0 \in \Lambda$ such that for any $U \in N_e(\tau)$

$$(g_U x_0, x_0) \notin P_0$$

This implies $h(x_0) \neq x_0$ and transformation h is non-trivial. □

The next Lemma shows that topology τ is not Hausdorff.

Lemma 2.3 *For every $U \in N_e(\tau),\ h \in U$.*

Proof of Lemma: For every $g \in AUT(K)$ and every $P \in \Lambda$ let

$$\tilde{P}(g) = \{g' \in AUT(K) : \forall\, x \in K,\ (gx, g'x) \in P\}$$

It is well-known that

$$\{\tilde{P}(g) : P \in \Lambda,\ g \in AUT(K)\}$$

form a base of the compact-open topology τ_0 on $AUT(K)$.

In order to prove our statement it suffices to show that for every $U_0 \in N_e(\tau)$ and every $P \in \Lambda$, we have

$$h \in [\tilde{P}^3(e)]^{-1} U_0$$

Indeed, for any $U \in N_e(\tau)$ we can find $U_0 \in N_e(\tau)$ such that $U_0^2 \subset U,\ U_0^{-1} = U_0$. But $\tau \subset \tau_0$,, hence there exists $P \in \Lambda$ with $\tilde{P}^3(e) \subset U_0$ Therefore

$$[P^3(e)]^{-1} U_0 \subset U_0^{-1} U_0 \subset U$$

Let $Q_P \in \Lambda$ and $U_P \in N_e(\tau)$ be chosen for $P \in \Lambda$ according to the definition of quasibounded topology. Suppose $x \in K$ is arbitrary. Define again

$$A(x) = \{U \in N_e(\tau) : (g_U h^{-1}(x), x) \in P\}$$

Since $h(h^{-1}(x)) = x$ then from the definition of h, $A(x) \in \Im$.

Let us choose $R \in \Lambda$, $R \subset P$, such that

$$(t', t'') \in R \ => \ (h^{-1}(t)', h^{-1}(t'')) \in Q_P \tag{5}$$

Consider an R-net of K i.e. a finite subset $\{x_1, ..., x_n\} \subset K$ such that for every $x \in K$ there exists an i, $1 \leq i \leq n$,for which $(x, x_i) \in R$. Suppose now

$$U \in (\cap_{i=1}^n A(x_i)) \cap F(U_P \cap U_0) \tag{6}$$

For any $x \in K$ by the choice of R we can find some i such that $(x, x_i) \in R$ and so from (5)

$$(h^{-1}(x), h^{-1}(x_i)) \in Q_P$$

Since $U \subset U_P$ then by the choice of U_p and Q_p

$$(g_U h^{-1}(x), g_U h^{-1}(x_i)) \in P$$

Also from $U \in A(x_i)$ it follows

$$(g_U h^{-1}(x_i), x_i) \in P$$

Recall that $(x, x_i) \in R \subset P$. Combining, we obtain

$$(g_U h^{-1}(x), x) \in P^3$$

therefore

$$g_U h^{-1} \in \tilde{P}^3(e)$$

But $g_U \in U \subset U_0$, thus

$$h \in \tilde{[P^3(e)]}^{-1} U_0$$

and the proof of the Lemma is completed. □

Lemma 2.2 and 2.3 jointly prove the proposition. □

3 Minimality of the group $AUT([0, 1]^n)$.

In this section we prove that the group $AUT([0, 1]^n)$ under compact-open topology is minimal if $n = 1$ and is not minimal if $n > 1$.

Theorem 3.1 *Let $I = [0, 1]$ and $n \in Z_+$. The group $AUT(I^n, \tau_0)$ of homeomorphisms of I^n with the compact-open topology τ_0 is minimal iff $n = 1$.*

Proof for the case $n = 1$:

Let τ be a Hausdorff group topology, strictly coarser than τ_0. From Proposition 2.1 we conclude that topology τ is not quasibounded. Therefore, there exists $c > 0$ such that for every pair $\alpha = (n, U) \in Z_+ \times N_e(\tau)$ there exist $x_\alpha, y_\alpha \in I$ and $g_\alpha \in U$ such that

$$|x_\alpha - y_\alpha| < 1/n \tag{7}$$

but

$$|g_\alpha x_\alpha - g_\alpha y_\alpha| \geq c \tag{8}$$

Consider four mappings from $Z_+ \times N_e(\tau))$ to I given by the rules:

$$\alpha \longrightarrow x_\alpha \ \alpha \longrightarrow y x_\alpha \ \alpha \longrightarrow g_\alpha x_\alpha \ \alpha \longrightarrow g_\alpha y_\alpha$$

Let z_1, z_2, x_0, y_0 be the limits of these mappings as defined in Lemma 1.1. From (7) and (8) it follows, $z_1 = z_2 = z$ and $|x_0 - y_0| \geq c$. Without the loss of generality we may assume $x_0 < y_0$. Take any $t_1, t_2 \in I$ satisfying

$$x_0 < t_1 < t_2 < y_0$$

Fix aOBny homeomorphism $\psi \in AUT(I)$ which is trivial on the set $I - (t_1, t_2)$, namely, for all $t \leq t_1$ and all $t \geq t_2$, there holds $\psi(t) = t$.

The following lemma proves that topology τ is not Hausdorff and thus completes the $n = 1$ part of the theorem.

Lemma 3.1 *For every $U \in N_e(\tau)$ there holds $\psi \in U$.*

Proof: It suffices to prove that for every $U_0 \in N_e(\tau)$ and $\epsilon > 0$ there holds

$$\psi \in U_0 O_\epsilon U_0^{-1} \tag{9}$$

where

$$O_\epsilon(e) = \{g \in AUT(I) | \forall\ x \in I, |gx - x| < \epsilon\}$$

For each $\alpha_0 = (n_0, U_0) \in Z_+ \times N_e(\tau)$ let

$$F(\alpha_0) = \{(n, U) \in N : n \geq n_0, U \subset U_0\}$$

Then $\{F(\alpha)\}_{\alpha \in N}$ is a filter base on the set $Z_+ \times N_e(\tau))$. Let $\Im$ be an ultrafilter containing this filter base. Since $[0, t_1)$ and $(t_2, 1]$ are neighborhoods of x_0 and y_0 we have from Lemma 2.1

$$A(t_1) = \{\alpha \in N | g_\alpha x_\alpha < t_1 \in \Im\}$$

and

$$A(t_2) = \{\alpha \in N | g_\alpha y_\alpha > t_2 \in \Im\}$$

Choose n_0 satisfying $1/n_0 < \epsilon$ and set $\alpha_0 = (n_0, U_0)$. Since $F(\alpha_0) \in \Im$ then

$$F(\alpha_0) \cap A(t_1) \cap A(t_2) \neq \emptyset$$

Let $\bar{\alpha} \equiv (\bar{n}, \bar{U})$ belong to this intersection. Then in particular we have

$$|x_{\bar{\alpha}} - y_{\bar{\alpha}}| < 1/\bar{n} < 1/n_0 < \epsilon. \tag{10}$$

Also

$$g_{\bar{\alpha}} \in \bar{U} \subset U_0 \tag{11}$$

Finally, since $\bar{\alpha} \in A(t_1)$, $\bar{\alpha} \in A(t_2)$ then

$$g_{\bar{\alpha}} x_{\bar{\alpha}} < t_1, \;\; g_{\bar{\alpha}} y_{\bar{\alpha}} > t_2$$

Suppose now $t \notin (x_{\bar{\alpha}}, y_{\bar{\alpha}})$ (or $t \notin (y_{\bar{\alpha}}, x_{\bar{\alpha}})$).Then from above and by the monotonicity of the homeomorphism $g_{\bar{\alpha}}$ we have $g_{\bar{\alpha}}t < t_1$, *or* $g_{\bar{\alpha}}t > t_2$ and ψ is trivial outside of (t_1, t_2) :

$$\forall\, t \notin (x_{\bar{\alpha}}, y_{\bar{\alpha}})\ (or\ t \notin (y_{\bar{\alpha}}, x_{\bar{\alpha}})) : g_{\bar{\alpha}}^{-1}\psi g_{\bar{\alpha}}(t) = t$$

namely, $g_{\bar{\alpha}}^{-1}\psi g_{\bar{\alpha}}$ is trivial outside of $(x_{\bar{\alpha}}, y_{\bar{\alpha}})$ or $(y_{\bar{\alpha}}, x_{\bar{\alpha}})$. Recall, however, that from (10)

$$|x_{\bar{\alpha}} - y_{\bar{\alpha}}| < \epsilon$$

therefore

$$g_{\bar{\alpha}}^{-1}\psi g_{\bar{\alpha}} \in O_\epsilon(e)$$

From (11) it follows

$$\psi \in U_0 O_\epsilon U_0^{-1}$$

which is (9), and the statement of the lemma is proven. This concludes the proof of $n = 1$ part of the theorem.□

Proof for the case $n > 1$:

Now, let us prove the second part of the theorem. We will construct some Hausdorff topology τ which will be strictly coarser than the compact open topology τ_0.

Let F be a boundary of I^n, and $|\cdot|$ be a natural norm in I^n. For any $\epsilon > 0$ define

$$O_\epsilon(e) = \{g \in AUT(I^n) : \forall\, x \in I^n, |gx - x| < \epsilon,\ |g^{-1}x - x| < \epsilon\}$$

Sets O_ϵ form the neighborhood base of the identity in compact-open topology. Let

$$F_\epsilon = \{x \in I^n : \exists\, y \in F,\ s.t. |x - y| < \epsilon\}$$

F_ϵ is ϵ-neighborhood of the boundary F. Finally, let

$$\tilde{O}_\epsilon(e) = \{g \in AUT(I^n) : \forall\, x \notin F_\epsilon\ |gx - x| < \epsilon,\ |g^{-1}x - x| < \epsilon\}$$

The next lemma shows that topology τ formed by O_ϵ is a Hausdorff group topology that is strictly coarser than the compact-open topology τ_0.

Lemma 3.2 *The family* $\{\tilde{O}_\epsilon(e)\}_{\epsilon>0}$ *is a neighborhood base of identity in some Hausdorff group topology* $\tau \subset \tau_0$.

Proof of Lemma: We will show that the family $\{\tilde{O}_\epsilon(e)\}_{\epsilon>0}$ satisfies the axioms of neighborhood base of identity.

1. For every $\epsilon_1, \epsilon_2 > 0$ clearly

$$\tilde{O}_{\epsilon_1}(e) \cap \tilde{O}_{\epsilon_2}(e) \supset \tilde{O}_\delta(e)$$

for any $\delta < \min(\epsilon_1, \epsilon_2)$.

2. Suppose $\epsilon > 0$, $\delta < \epsilon/2$. Let $g_1, g_2 \in \tilde{O}_\delta(e)$. For arbitrary $x \notin F_\epsilon$ there holds $x \notin F_\delta$, so

$$|g_1 x - x| < \delta$$

and therefore $g_1 x \notin F_\delta$. This in its order implies

$$|g_2 g_1 x - g_1 x| < \delta$$

and therefore

$$|g_2 g_1 x - x| < 2\delta < \epsilon$$

By a similar reasoning we show that for arbitrary $x \notin F_\epsilon$

$$|g_2^{-1} g_1^{-1} x - x| < 2\delta < \epsilon$$

We conclude that

$$\tilde{O}_\delta(e) \tilde{O}_\delta(e) \subset \tilde{O}_\epsilon(e)$$

3. Straight from the definition we have

$$\tilde{O}_\epsilon^{-1}(e) = \tilde{O}_\epsilon(e)$$

4. Let $\epsilon > 0$ and g be arbitrary element of $\tilde{O}_\epsilon(e)$. We will show that for some $\delta > 0$ there holds

$$g \tilde{O}_\delta(e) \subset \tilde{O}_\epsilon(e) \tag{12}$$

Introduce

$$\epsilon_1 \equiv \max_{x \notin F_\epsilon} (|gx - x|, |g^{-1}x - x|) \tag{13}$$

Clearly, $\epsilon_1 \leq \epsilon$. It is not hard to see that in fact $\epsilon_1 < \epsilon$. Let $\epsilon_2 = \epsilon - \epsilon_1$. Let us find some $\delta < \epsilon_2$ such that

$$|t' - t''| < \delta \;=>\; |gt' - gt''| < \epsilon_2, \;\; |g^{-1}t' - g^{-1}t''| < \epsilon_2 \tag{14}$$

For any $g' \in \tilde{O}_\delta(e)$ and any $x \notin F_\epsilon$ we have $x \notin F_\delta$ and therefore $|g'x - x| < \delta$. Consequently from (14) it follows

$$|gg'x - gx| < \epsilon_2$$

and therefore

$$|gg'x - x| \leq |gg'x - gx| + |gx - x| < \epsilon_2 + \epsilon_1 = \epsilon \tag{15}$$

Also if $x \notin F_\epsilon$ then from (13)

$$|g^{-1}x - x| \leq \epsilon_1$$

so $g^{-1}x \notin F_{\epsilon_2}$ and therefore $g^{-1}x \notin F_\delta$. But then it follows

$$|g'^{-1}g^{-1}x - g^{-1}x| < \delta < \epsilon_2$$

Combining, we obtain

$$|g'^{-1}g^{-1}x - x| < \epsilon$$

This together with (15) proves (12).

5. We now show that for any $\epsilon > 0$ and $g_0 \in Aut(I^n)$, there exists $\delta > 0$ for which

$$g_0^{-1}\tilde{O}_\delta g_0 \subset \tilde{O}_\epsilon(e)$$

Let $\epsilon > 0$ and fix any $g_0 \in AUT(I^n)$. We can find some $\delta > 0$ which satisfies two conditions

(i) $F_\delta \subset g_0(F_\epsilon)$

(ii) $|t' - t''| < \delta \implies |g_0^{-1}t' - g_0^{-1}t''| < \epsilon$

Let now $x \notin F_\epsilon$. Then $gx_0 \notin F_\delta$. Therefore for each $g \in \tilde{O}_\delta(e)$ there holds:

$$|gg_0x - g_0x| < \delta,\ |g^{-1}g_0x - x| < \delta$$

From condition (ii) then

$$|g_0^{-1}gg_0x - x| < \epsilon,\ |g_0^{-1}g^{-1}g_0x - x| < \epsilon$$

Thus

$$g_0^{-1}\tilde{O}_\delta(e)g_0 \in \tilde{O}_\epsilon(e)$$

Conditions 1-5 show that the family $\{\tilde{O}_\epsilon(e)\}_{\epsilon>0}$ describes some topology $\tau \subset \tau_0$ on the group $AUT(I^n)$. To conclude the proof of the lemma we need to show that topology τ is Hausdorff.

Suppose $g \neq e$. The set

$$U = \{x \in I^n : gx \neq x\}$$

is an open set in I^n and hence $U - F \neq \emptyset$. Let $x_0 \in U - F$. Choose $\epsilon > 0$ such that $\epsilon < |gx_0 - x_0|$ and for all $x \in F$, $\epsilon < |x_0 - x|$. But then $x_0 \notin F_\epsilon$ and so $g \notin \tilde{O}_\epsilon(e)$. Therefore topology τ is Hausdorff. $\square$

To complete the proof of the theorem we show that topology τ is strictly coarser than compact-open topology τ_0.

Fix a and b, two distinct points of the boundary F. It can be easily verified that for every $\epsilon > 0$ there exists $g \in \tilde{O}_\epsilon(e)$ such that $ga = b$. Thus if we set $\epsilon_0 = |a - b|$ then for every $\epsilon > 0$ the condition

$$\tilde{O}_\epsilon(e) \subset O_\epsilon(e)$$

does not hold and topology τ is strictly coarser than compact-open topology τ_0.

This completes the proof of the theorem.□

4 Minimality of the group of homeomorphisms of the Cantor cube

In this section we will prove the minimality of the group $AUT(D^\infty)$ under the compact-open topology τ_0, where D^∞ is a Cantor cube with a countable weight.

Let $K = D^\infty$ be a Cantor cube with a countable weight and metric d. Define d as

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n|,\ \ x,y \in D^\infty$$

Let also τ be a Hausdorff topology on $AUT(D^\infty)$ which is strictly coarser than the compact-open topology τ_0.

Lemma 4.1 *For every $U \in N_\tau(e)$ and every $x, y \in D^\infty$ there exists some $\delta > 0$ such that every element $g \in AUT(D^\infty)$ that is trivial on $D^\infty - (B(x,\delta) \cup B(y,\delta))$ satisfies $g \in U$.*

Proof of Lemma: From theorem 2.1 we conclude that topology τ is not quasibounded on $AUT(D^\infty)$ and therefore there exists some positive constant $c > 0$ such that for every $\alpha = (n,U)$ there exist some $x_\alpha, y_\alpha \in D^\infty$and $g_\alpha \in U$ with

$$d(x_\alpha, y_\alpha) < 1/n \tag{16}$$

and

$$d(g_\alpha x_\alpha, g_\alpha y_\alpha) \geq c \tag{17}$$

As before for each $\alpha_0 = (n_0, U_0) \in Z_+ \times N_e(\tau)$ let

$$F(\alpha_0) = \{(n,U) \in N : n \geq n_0, U \subset U_0\}$$

Then $\{F(\alpha)\}_{\alpha \in N}$ is a filter base on the set $Z_+ \times N_e(\tau)$. Let $\Im$ be an ultrafilter containing this filter base.

Consider four mappings from $Z_+ \times N_e(\tau)$ to D^∞ given by the rules

$$\alpha \longrightarrow x_\alpha\ \alpha \longrightarrow y_\alpha\ \alpha \longrightarrow g_\alpha x_\alpha\ \alpha \longrightarrow g_\alpha y_\alpha$$

Let z_1, z_2, x_0, y_0 be the limits of these mappings in the sense described before. Then $z_1 = z_2$ and $d(x_0, y_0) \geq c$.

First, we will prove the lemma for the case $x = x_0, y = y_0$.

Suppose $U_0 \in N_\tau(e)$ is symmetric (i.e. $U_0^{-1} = U_0$.) and $\epsilon > 0$ are arbitrary. Define again

$$O_\epsilon(e) = \{g \in Aut(D^\infty) : \forall\, x \in D^\infty \ |gx - x|, g^{-1}x - x| < \epsilon\}$$

We will show that there exists $\delta > 0$ such that every $g \in AUT(D^\infty)$ that is trivial on $D^\infty - (B(x_0, \delta) \cup B(y_0, \delta))$ belongs to

$$O_{2\epsilon}(e)U_0O_{2\epsilon}(e)U_0O_{2\epsilon}(e)$$

Let

$$A(x_0) = \{\alpha \in Z_+ \times N_e(\tau) : d(g_\alpha x_\alpha, x_0) < \epsilon\}$$

$$A(y_0) = \{\alpha \in N : d(g_\alpha y_\alpha, y_0) < \epsilon\}$$

Both of these sets belong to ultrafilter $\Im$ by the definition of the limits x_0, y_0.

Select $n_0 \in Z_+$ satisfying $1/n_0 < \epsilon/3$. Set $\alpha_0 = (n_0, U_0)$ and consider non-empty intersection

$$A(x_0) \cap A(y_0) \cap F(\alpha_0) \in \Im$$

Let $\alpha_1 = (n_1, U_1)$ be an arbitrary member of this intersection

$$\alpha_1 = (n_1, U_1) \in A(x_0) \cap A(y_0) \cap F(\alpha_0)$$

In particular,

$$d(g_{\alpha_1}x_{\alpha_1}, x_0) < \epsilon,\ d(g_{\alpha_1}y_{\alpha_1}, y_0) < \epsilon,\ n_1 \geq n_0,\ U_1 \subset U_0 \tag{18}$$

Since $d(g_{\alpha_1}x_{\alpha_1}, x_0) < \epsilon,\ d(g_{\alpha_1}y_{\alpha_1}, y_0) < \epsilon$, there exists $\bar{g} \in O_\epsilon(e)$ such that $\bar{g}(g_{\alpha_1}x_{\alpha_1}) = x_0$, $\bar{g}(g_{\alpha_1}y_{\alpha_1}) = y_0$. Let $\hat{g} = \bar{g}g_{\alpha_1}$. Since $g_{\alpha_1} \in U_1 \subset U_0$, we have

$$\hat{g} \in O_\epsilon(e)U_0 \tag{19}$$

Let δ be chosen such that $d(t, t') < \delta$ implies $d(\hat{g}^{-1}t, \hat{g}^{-1}t') < \epsilon/3$ for all $t, t' \in D^\infty$.

Since $\hat{g}^{-1}x_0 = x_{\alpha_1}$ and $\hat{g}^{-1}y_0 = y_{\alpha_1}$, we have

$$\hat{g}^{-1}(B(x_0, \delta)) \subset B(x_{\alpha_1}, \epsilon/3)$$

and

$$\hat{g}^{-1}(B(y_0, \delta)) \subset B(y_{\alpha_1}, \epsilon/3)$$

Recall from (16) and (18)

$$d(x_{\alpha_1}, y_{\alpha_1}) < 1/n_1 \leq 1/n_0 < \epsilon/3$$

so there exists a ball B with diameter ϵ such that

$$\hat{g}^{-1}(B(x_0, \delta)) \cup \hat{g}^{-1}(B(y_0, \delta)) \subset B$$

Let $g \in Aut(D_\infty)$ be trivial outside of $B(x_0, \delta) \cup B(y_0, \delta)$. Then $\hat{g}^{-1} g \hat{g}$ is trivial outside of B. In particular, $\hat{g}^{-1} g \hat{g} \in O_{2\epsilon}(e)$. From (19) $g \in O_{2\epsilon}(e) U_0 O_{2\epsilon}(e) U_0 O_{2\epsilon}(e)$. This completes the proof for the case $x = x_0, y = y_0$.

Now let $x, y \in D^\infty,\ U \in N_e(\tau)$ be arbitrary. Let us find $g_0 \in Aut(D^\infty)$ and $V \in N_e(\tau)$ such that

$$g_0 x_0 = x,\ g_0 y_0 = y,\ g_0 V g_0^{-1} \subset U \tag{20}$$

As we have just shown, there exists $\delta_1 > 0$ such that any element $g \in Aut(D^\infty)$ that is trivial on $D^\infty - B(x_0, \delta_1) \cup B(y_0, \delta_1)$ belongs to V. Find $\delta > 0$ such that $d(t, t') < \delta$ implies $d(g_0^{-1} t, g_0^{-1} t') < \delta_1$. Suppose $g \in Aut(D^\infty)$ is trivial on $D^\infty - B(x, \delta) \cup B(y, \delta)$ Since $g_0^{-1}(B(x, \delta)) \subset B(x_0, \delta_1)$ and $g_0^{-1}(B(y, \delta)) \subset B(y_0, \delta_1)$, then $g_0^{-1} g g_0$ is trivial on $D^\infty - B(x_0, \delta_1) \cup B(y_0, \delta_1)$. Therefore, $g_0^{-1} g g_0 \in V$. But then from (20) $g \in g_0 V g_0^{-1} \subset U$.

This completes the proof of the lemma.□

We now proceed to proving the main result of this section.

Theorem 4.1. *The group of homeomorphisms of a Cantor cube $Aut(D^\infty)$ with the compact-open topology τ_0 is minimal.*

Proof: We represent D^∞ as a set of zero-one sequences $(x_1, x_2, ..., x_n, ...)$. For $a = 0 (a = 1)$ define $\bar{a} = 1 (\bar{a} = 0)$. For a sequence $x = (x_1, x_2, ..., x_n, ...) \in D^\infty$ define $\bar{x} = (\bar{x}_1, x_2, ...x_n, ...)$. Let us consider $\Delta : D^\infty \to D^\infty$ defined by the rule $\Delta(x) = \bar{x}$. Let τ be any topology on $Aut(D^\infty)$ that is strictly coarser than the compact-open topology τ_0. The folowing Lemma shows that τ topology is not Hausdorff which proves the Theorem.

Lemma 4.2. *For each $U \in N_e(\tau)$ there holds $\Delta \in U$.*

Proof: For any $\epsilon > 0$ let $V_\epsilon(e)$ denote the set

$$V_\epsilon(e) \equiv \{g \in Aut(D^\infty) : \forall x \in D^\infty |gx - x|, |g^{-1}x - x| < \epsilon\}$$

Let U_0 be any element of $N_e(\tau)$. We will show that $\Delta \in U_0$. Select $U \in N_e(\tau)$ such that $U^6 \subset U_0$, and choose $\epsilon > 0$ so that

$$V_\epsilon^2(e) \subset U \tag{21}$$

Let us find positive integer n such that

$$\sum_{k=n}^{\infty} \frac{1}{2^k} < \epsilon \tag{22}$$

Let $A = \{a^1, a^2, ..., a^{2^n}\} = \{x \in D^\infty : x_{n+1} = x_{n+2} = ... = 0\}$ i.e. A is a finite ϵ-net in D^∞, consisting of 2^n elements. Let $W \in N_e(\tau)$ satisfy

$$W^{2^n} \subset U \tag{23}$$

and choose $\delta > 0$ small enough, so that the conditions of the Lemma 4.1 hold for pairs
$(a^1, \bar{a}^1), ..., (a^{2^n}, \bar{a}^{2^n})$ and W.

Select a positive integer $N > n$ satisfying

$$\sum_{k=N}^{\infty} \frac{1}{2^k} < \delta \tag{24}$$

For every $i = 1, 2, ..., 2^n$ define $\psi_i \in Aut(D^\infty)$ by the following rule

$$\psi_i(x) = \begin{cases} \bar{x} & \text{if } x_1 = a_1^i, x_2 = a_2^i, ..., x_N = a_N^i \quad \text{(in particular } x_{n+1} = ... = x_N = 0) \\ x & \text{otherwise} \end{cases}$$

Notice from (24) that ψ_i is trivial outside of $B(a^i, \delta) \cup B(\bar{a}^i, \delta)$, therefore by Lemma 4.1 and by the choice of δ, $\psi_i \in W$. From (23)

$$\psi \equiv \psi_1 \psi_2 ... \psi_{2^n} \in W^{2^n} \subset U \tag{25}$$

It can be seen that ψ flips the δ-neighborhoods of the points $a^1, ..., a^{2^n}$ with their matches $\bar{a}^1, ..., \bar{a}^{2^n}$. To be precise

$$\psi(x) = \begin{cases} \bar{x} & \text{if } x_{n+1} = ... = x_N = 0 \\ x & \text{otherwise} \end{cases} \tag{26}$$

We will use this and (25) to show that homeomorphism that flips every point $x \in D^\infty$ with its match (that is homeomorphism Δ) belongs to U_0.

For that purpose, divide the space D^∞ into four subsets

$$D_1 = \{x \in D^\infty : x_{n+1} = 0, \exists\, i, n+1 < i \leq N, \, x_i = 1\}$$

$$D_2 = \{x \in D^\infty : x_{n+1} = x_{n+2} = ... = x_N = 0\}$$

$$D_3 = \{x \in D^\infty : x_{n+1} = 1, \exists\, i, n+1 < i \leq N, \, x_i = 1\}$$

$$D_4 = \{x \in D^\infty : x_{n+1} = 1, x_{n+2} = ... = x_N = 0\}$$

On these sets define three mappings

$$\phi_1 : D_1 \cup D_2 \to D_2, \;\; \phi_1(x) = y,$$

$$\textit{where } y_1 = x_1, ..., y_n = x_n, y_{n+1} = ... = y_N = 0, y_{N+i} = x_{n+1+i}, \; i \geq 1$$

$$\phi_2 : D_3 \to D_2, \;\; \phi_2(x) = y,$$

$$\textit{where } y_{n+1} = 0, \; y_i = x_i, \; for\, i \neq n+1$$

$$\phi_3 : D_4 \to D_3 \cup D_4, \;\; for\, x \in D_4, \phi_3(x) = y,$$

$$where\ y_1 = x_1, ..., y_n = x_n, y_{n+1+i} = x_{N+i},\ i \geq 1$$

Let us consider the homeomorphism ϕ which combines ϕ_1, ϕ_2, ϕ_3 on the total space D^∞. We see that ϕ roughly acts as follows. It shrinks the half-space of D^∞ (points x-s with $x_{n+1} = 0$) into δ-neighborhood of the net $a^1, ..., a^{2^n}$, it maps the second half space (x-s with $x_{n+1} = 1$) into the first falf-space, and it "de-shrinks" the intersection of the second half-space and δ-neighborhood of the net into whole second half-space. We will use this in combination with mapping ψ to prove $\Delta \in U_0$.

Observe that ϕ does not change the first n coordinates of the point, so from (22) $\phi \in V_\epsilon(e)$. Let us consider $\phi^{-1}\psi\phi(x)$ for $x \in D^\infty$. Note from (25) and (21)

$$\phi^{-1}\psi\phi(x) \in V_\epsilon U V_\epsilon \subset U^3 \tag{27}$$

Also,

- 1. If $x_{n+1} = 0$ then $x \in D_1 \cup D_2$, and $\phi(x) = \phi_1(x) \in D_2$. From (26), $\psi(\phi(x)) = \bar{\phi}(x) = \phi(\bar{x})$. Therefore, $\phi^{-1}\psi\phi(x) = \phi^{-1}\phi(\bar{x}) = \bar{x}$.
- 2. If $x_{n+1} = 1$, and for some $i = n+2, ..., N$, $x_i = 1$, then ϕ just changes coordinate $n+1$ to 0 in x and all the other coordinates remain unchanged. Therefore, for $\phi(x) = \phi_2(x)$, $(\phi(x))_i = 1$, for some $i, n+2 \leq i \leq N$. From (26) $\psi(\phi(x) = \phi(x)$ and $\phi^{-1}\psi\phi(x) = x$.
- 3. If $x_{n+1} = 1$, $x_{n+2} = ... = x_N = 0$, then $\phi(x) = \phi_3(x)$ has $n+1$-st coordinate equal to 1: $(\phi(x))_{n+1} = 1$. Therefore $\psi(\phi(x)) = \phi(x)$ and $\phi^{-1}\psi\phi(x) = x$.

We conclude that $\phi^{-1}\psi\phi(x) = \bar{x}$, if $x_{n+1} = 0$ and $\phi^{-1}\psi\phi(x) = x$, otherwise.

Now, let $\tilde{\phi}$ be a homeomorphism which flips the $n+1$-st coordinate. That is

$$(\tilde{\phi}(x))_{n+1} = \bar{x}_{n+1},\ (\tilde{\phi}(x))_i = x_i,\ i \neq n+1$$

From (22) $\tilde{\phi} \in V_\epsilon(e)$ and

$$\hat{\phi} \equiv (\phi\tilde{\phi})^{-1}\psi(\phi\tilde{\phi}) \in V_\epsilon^2(e) U V_\epsilon^2(e) \subset U^3$$

By argiung as above we can check that $\hat{\phi}(x) = x$ if $x_{n+1} = 0$, and $\hat{\phi}(x) = \bar{x}$ if $x_{n+1} = 1$. From (27)

$$\hat{\phi}(\phi^{-1}\psi\phi) \in U^3U^3 \subset U_0$$

Note, $\hat{\phi}(\phi^{-1}\psi\phi)(x) = \bar{x}$ for all $x \in D^\infty$. Namely, $\hat{\phi}(\phi^{-1}\psi\phi) = \Delta$. Thus $\Delta \in U_0$ and the proof of the lemma is completed. □

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