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CHARACTERIZATION AND MEASUREMENT OF TIME- AND FREQUENCY-SPREAD CHANNELS

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ABSTRACT

This report deals with the characterization and measurement of channels in which the input waveform is subject to both additive noise and to time and frequency spreading. Part of the report is tutorial in nature and attempts to relate the wealth of mathematically oriented literature on spread channels, with both the underlying physical machanisms and with the engineering concepts concerning spread-channel communication.

Many spread channels are adequately characterized by any one of three functions of two variables: the scattering function, the tap-gain correlation function, and the two-frequency correlation function. These quantities are carefully defined and related by Fourier transform relationships. A number of interpretations are given for these functions, and the physical circumstances in which the functions provide meaningful characterizations of the channel are discussed.

Techniques are given for measuring each of the above three functions. It is shown that the variance of these measurements approaches zero with the reciprocal of the measurement time. One of these techniques, using a chirp input signal to measure the two-frequency correlation function, appears to have definite advantages over the others, in terms of required measurement time and in ease of implementation. The analysis clears up some earlier paradoxes about measuring overspread channels and gives some insight into the relative merits of different input signals.

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I. INTRODUCTION

Many communication channels in use today are subject not only to additive noise but also to time and frequency spreading. Time spreading, often called multipath smear or dispersion, manifests itself most clearly when a narrow pulse at the channel input is converted into an output that is spread out over a significant period of time. Frequency spreading, often called doppler spreading, manifests itself most clearly when a sinusoidal input is converted into an output that is spread out over a significant band of frequencies. Both these phenomena together distort arbitrary waveforms in unusual and ever-changing ways; the problems created by this distortion are often considerably more serious for communication purposes than the problems created by additive noise.

Before discussing the modeling and characterization of such channels, it may be helpful to describe several time- and frequency-spread channels to examine the kinds of mechanisms which give rise to the spreading.

As a first example, let us consider an orbital dipole channel.[1] Such a channel can be generated by forming around the earth an orbiting belt comprised of metallic dipoles tuned to about 5 or 10 kMc; communication can then take place from one antenna to another via reflection from the belt. A typical order of magnitude for the width of such a belt is about 80 miles. When one transmits a narrow pulse over the channel, the return is spread out over about 100 μsec due to the different path lengths to each reflector. On the other hand, the impulse response time of a single dipole has a duration of only about 10^{-9} sec. The multipath smear (100 μsec in this case) is an important parameter for communication and plays a prominent role in the channel characterizations to be discussed later. The impulse response duration of a single dipole is of less interest and will not appear in our channel characterizations. The importance of the dipole response duration lies in determining the over-all bandwidth of the channel. This is certainly of communication importance but not of importance in describing the particular phenomena associated with time and frequency spreading.

In addition to time spreading, there is doppler spreading due to the motion of the dipoles. This doppler spreading has a typical order of magnitude of 1 kc. The motion of dipoles relative to each other also produces alternating constructive and destructive interference and thus causes fading. The relationship of doppler spreading to fading intervals and coherence times will be discussed in more detail later.

As a second example, consider chaff channels. These channels, formed by placing a cloud of dipoles several miles up in the atmosphere, provide short-range communication by reflection from the cloud. The channels are very similar to the orbiting dipole channel except for the orders of magnitude involved — 5 μsec for multipath smear and 100 cps for doppler spreading. Naturally, because of the short range, the power limitations are much less severe than with orbital scatter.

As a third example, consider communication by reflection from the moon. Here again the reflecting body is so large that there is a multipath smear of about 10 msec. Similarly, the differential velocities of different parts of the moon relative to each other give rise to a doppler spread of about 10 cps.

The examples discussed here, as well as most other well-known time- and frequency-spread channels, can be represented as linear time-varying filters with additive noise at the receiver. If the reflecting clouds or bodies were stationary, the channel could be represented as a linear time-invariant filter with noise. The motion of the reflections simply causes the response to change in time without affecting the linearity.

In Sec. II we shall discuss linear time-varying filters, and in Sec. IV we shall discuss a statistical characterization of such filters that is applicable to channels such as those discussed in these examples.

II. LINEAR TIME-VARYING FILTERS

A linear time-varying filter has the following property: let $x_1(t)$ and $x_2(t)$ be any two possible inputs to the filter, and let $y_1(t)$ and $y_2(t)$ be the corresponding outputs; then for any two constants, a_1 and a_2, the input $a_1x_1(t) + a_2x_2(t)$ gives rise to the output $a_1y_1(t) + a_2y_2(t)$. A linear time-varying filter differs from the more usual linear time-invariant filter in that if $x(t)$ gives rise to $y(t)$, $x(t+\tau)$ does not necessarily give rise to $y(t+\tau)$.

There is considerable literature on the characterizations of linear time-varying (LTV) filters, much of which is summarized and referenced in Kailath.[2] The only results of this theory that we need here are that LTV filters can be completely characterized by either of two quantities: one, the frequency response function $H(f,t)$, and the other, the impulse response function $h(\tau,t)$. The function $H(f,t)$, introduced by Zadeh,[3] is defined as the ratio of channel output to input $y(t)/x(t)$ when the input as a function of t is $\exp[j2\pi ft]$. This definition is somewhat unsatisfying operationally, since it takes an infinite time to measure $H(f,t)$ for a single value of f. However, it is a satisfying definition in the sense that, if we happen to know $H(f,t)$ for all f and t, we can find the response to an arbitrary input. To see this, let $x(t)$ be an arbitrary input with Fourier transform $X(f)$.

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp[-j2\pi ft]\, dt \tag{1}$$

and

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp[j2\pi ft]\, df \quad . \tag{2}$$

From the definition of $H(f,t)$, the response to $X(f)\exp[j2\pi ft]$ is $X(f)\,H(f,t)\exp[j2\pi ft]$. From the assumption of linearity, the response to the integral in Eq. (2) is therefore

$$y(t) = \int_{-\infty}^{\infty} X(f)\, H(f,t) \exp[j2\pi ft]\, df \quad . \tag{3}$$

Thus the output from an arbitrary Fourier transformable input is specified in terms of $H(f,t)$.

Next let $h(\tau, t)$ be the response at time t to a unit impulse input applied at time $t - \tau$. From Eq. (1), the Fourier transform of a unit impulse at $t - \tau$ is $X(f) = \exp[-j2\pi f(t - \tau)]$. Substituting this into Eq. (3), we have

$$h(\tau, t) = \int_{-\infty}^{\infty} \exp[-j2\pi f(t - \tau)]\, H(f, t) \exp[j2\pi ft]\, df = \int_{-\infty}^{\infty} H(f, t) \exp[j2\pi f\tau]\, df \quad . \tag{4}$$

Thus $h(\tau, t)$ is the Fourier transform of $H(f, t)$, and the one specifies the other. Finally, we can relate $y(t)$ directly to $x(t)$ and $h(\tau, t)$. Applying the convolution theorem to Eq. (3), we have†

$$y(t) = \int x(t - \tau)\, h(\tau, t)\, d\tau \quad . \tag{5}$$

Thus we have related $h(\tau, t)$ to $H(f, t)$ and showed that either can specify the filter output from an arbitrary input.

Kailath[2] shows that in many cases $h(\tau, t)$ can be measured, at least in principle. If $h(\tau, t)$ is zero for τ outside some range, say $0 \leqslant \tau \leqslant L$, one can use an input of a periodic train of impulses separated by L and measure $h(\tau, t)$ at values of t separated by L. If $h(\tau, t)$ considered as a function of t is limited to frequencies below $1/2L$, $h(\tau, t)$ can be reconstructed from these measured values by the sampling theorem.

We shall not be concerned here, however, with the measurement of $h(\tau, t)$ or $H(f, t)$. For the channels we are concerned with, $h(\tau, t)$ and $H(f, t)$ are in some sense statistically varying quantities, and we shall be concerned with measuring the statistics of these functions rather than the actual functions themselves.

III. COMPLEX LOW-PASS REPRESENTATION OF WAVEFORMS

In most of this report we shall be dealing with bandpass waveforms and bandpass filters. It is analytically convenient to represent such waveforms and filter responses by equivalent, complex, low-pass waveforms. This is standard procedure in the literature, and one can convolve, correlate, multiply, and so forth, in the low-pass representation in much the same way as with the actual waveforms. Unfortunately, there are a number of annoying details such as complex conjugates and factors of $\frac{1}{2}$ that enter into operations on the low-pass representations. There appears to be no standard way of handling such details. (Woodward[4] is an excellent reference here, but his treatment is too brief for our present purposes.)

In this section we shall define complex low-pass representations and summarize the formulas we shall use for convolution, correlation, spectral density, and multiplication in the low-pass representation. The reader who is familiar with this material may omit the section and simply refer to the appropriate relationship when it is used subsequently.

Let $x(t)$ be a waveform with Fourier transform $X(f)$:

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp[-j2\pi ft]\, dt \tag{6}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp[j2\pi ft]\, df \quad . \tag{7}$$

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† The fact that $H(f,t)$ is a function of t does not affect the convolution theorem since t can be considered as simply a parameter on the right sides of Eqs. (3) and (5).

We are interested primarily in waveforms that are band-limited in the sense that $X(f)$ is essentially zero except in the vicinity of some carrier frequency f_o and $-f_o$. Our mathematical definitions do not involve this bandwidth, however, so we need not be fussy about the meaning of "essentially zero."

We define the quantity $\underline{X}(f)$ by

$$\begin{aligned} \underline{X}(f) &= 2X(f + f_o) \qquad f > -f_o \\ &= 0 \qquad\qquad\qquad f < -f_o \quad . \end{aligned} \tag{8}$$

Thus $\underline{X}(f)$ is twice the positive frequency part of $X(f)$ shifted down by f_o. Since $x(t)$ is real, $X(f) = X^*(-f)$, and we have

$$X(f) = \frac{1}{2}\underline{X}(f - f_o) + \frac{1}{2}\underline{X}^*(-f - f_o) \quad . \tag{9}$$

The first term in Eq. (9) is the positive frequency part of $X(f)$, and the second term is the negative frequency part.

Now we define the complex low-pass representation of $x(t)$ denoted $\underline{x}(t)$ by

$$\underline{x}(t) = \int_{-\infty}^{\infty} \underline{X}(f) \exp[j2\pi ft] \, df \tag{10}$$

$$\underline{X}(f) = \int_{-\infty}^{\infty} \underline{x}(t) \exp[-j2\pi ft] \, dt \quad . \tag{11}$$

(Throughout this report we shall underline all complex low-pass representations.) Taking the inverse Fourier transform of each term in Eq. (9),

$$x(t) = \frac{1}{2}\underline{x}(t) \exp[j2\pi f_o t] + \frac{1}{2}\underline{x}^*(t) \exp[-j2\pi f_o t] \tag{12}$$

$$= \text{Re}\left\{\underline{x}(t) \exp[j2\pi f_o t]\right\} \tag{13}$$

$$= \text{Re}[\underline{x}(t)] \cos 2\pi f_o t - \text{Im}[\underline{x}(t)] \sin 2\pi f_o t \quad . \tag{14}$$

Thus the real and imaginary parts of $\underline{x}(t)$ are interpreted as amplitude modulations of cosine and sine carriers.

Next let $S_x(f)$ be the energy spectral density of $x(t)$:

$$S_x(f) = X(f) \, X^*(f) \quad . \tag{15}$$

Also let $R_x(\tau)$ be the autocorrelation function of $x(t)$:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t) \, x(t + \tau) \, dt \quad . \tag{16}$$

Substituting Eq. (6) into (15), and performing some manipulation, it follows that $S_x(f)$ is the Fourier transform of $R_x(\tau)$. We define the low-pass spectral density of $x(t)$ as

$$\begin{aligned} \underline{S}_x(f) &= 2S_x(f + f_o) \qquad f > -f_o \\ &= 0 \qquad\qquad\qquad f < -f_o \quad . \end{aligned} \tag{17}$$

Since $S_x(f) = S_x(-f)$, we have

$$S_x(f) = \frac{1}{2}\,\underline{S}_x(f - f_o) + \frac{1}{2}\,\underline{S}_x(-f - f_o) \quad . \tag{18}$$

Substituting Eqs. (15) and (8) into Eq. (17), we have

$$\underline{S}_x(f) = \frac{1}{2}\,\underline{X}(f)\,\underline{X}^*(f) \quad . \tag{19}$$

Note that Eqs. (15) and (19) differ by a factor of $\frac{1}{2}$.

We define the complex low-pass autocorrelation function as

$$\underline{R}_x(\tau) = \int_{-\infty}^{\infty} \underline{S}_x(f)\,\exp[j2\pi f\tau]\,df \tag{20}$$

$$\underline{S}_x(f) = \int_{-\infty}^{\infty} \underline{R}_x(\tau)\,\exp[-j2\pi f\tau]\,d\tau \quad . \tag{21}$$

Taking the Fourier transform of Eq. (18),

$$R_x(\tau) = \frac{1}{2}\,\underline{R}_x(\tau)\,\exp[j2\pi f_o\tau] + \frac{1}{2}\,\underline{R}_x^*(\tau)\,\exp[-j2\pi f_o\tau] \tag{22}$$

$$R_x(\tau) = \mathrm{Re}\left\{\underline{R}_x(\tau)\,\exp[j2\pi f_o\tau]\right\} \quad . \tag{23}$$

Finally, upon substituting Eq. (19) into (20), we have, after some manipulation,

$$\underline{R}_x(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \underline{x}^*(t)\,\underline{x}(t+\tau)\,dt \quad . \tag{24}$$

Note that Eq. (24) differs from Eq. (16) by a factor of $\frac{1}{2}$ and also by involving a complex conjugate. Although $\underline{R}_x(\tau)$ is not necessarily real, it follows immediately from Eq. (24) that

$$\underline{R}_x^*(-\tau) = \underline{R}_x(\tau) \quad . \tag{25}$$

From Eqs. (16), (23), and (24), we see that the energy E of the waveform is given by

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = R_x(0) = \underline{R}_x(0) = \frac{1}{2}\int |\underline{x}(t)|^2\,dt \quad . \tag{26}$$

Next we wish to investigate convolution in the complex low-pass representation. Let $x(t)$, $h(t)$, and $y(t)$ be three waveforms, and let

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau \quad .$$

From the convolution theorem, the Fourier transforms of these quantities are related by

$$Y(f) = X(f)\,H(f) \quad . \tag{27}$$

Using Eq. (8) on all three terms above,

$$\underline{Y}(f) = \frac{1}{2}\,\underline{X}(f)\,\underline{H}(f) \quad . \tag{28}$$

Applying the convolution theorem to Eq. (28),

$$\underline{y}(t) = \frac{1}{2} \int \underline{x}(t-\tau)\, \underline{h}(\tau)\, d\tau \quad . \tag{29}$$

Note the factor of $\frac{1}{2}$ and the absence of any conjugate in Eq. (29).

Finally, let $h(\tau, t)$ be the response at t to an impulse τ seconds earlier of a linear time-varying filter. Treating t as a parameter, $h(\tau, t)$ is a finite energy waveform in τ, and we define $\underline{h}(\tau, t)$ as its complex low-pass representation. That is,

$$\begin{aligned} \underline{H}(f,t) &= 2H(f+f_o, t) \qquad f > -f_o \\ &= 0 \qquad\qquad\qquad\quad f < -f_o \end{aligned} \tag{30}$$

$$\underline{h}(\tau,t) = \int \underline{H}(f,t) \exp[j2\pi f\tau]\, df \quad . \tag{31}$$

This notation will cause no confusion since the linear time-varying filters we consider will always be bandpass functions of τ and slowly varying functions of t. If $x(t)$ is the input, and $y(t)$ the output from a filter with a response $h(\tau, t)$, we can now immediately apply Eq. (29):

$$\underline{y}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \underline{x}(t-\tau)\, \underline{h}(\tau,t)\, d\tau \quad . \tag{32}$$

IV. CHARACTERIZATION OF TIME- AND FREQUENCY-SPREAD CHANNELS

In this section we shall discuss three functions: the two-frequency correlation function, the tap-gain correlation function, and the scattering function, all of which have been proposed as characterizations of time- and frequency-spread channels. We shall show how these functions are related as Fourier transforms and how they are related to the frequency and impulse responses of the channel. Examples will be given to show the relevance of the functions to communication problems, and, where possible, these functions will be related to the physical processes in the channel. The problem of measuring these functions will be postponed until Sec. V.

We have stated that the transmission characteristics of a time- and frequency-spread channel are specified by the time-varying frequency response $H(f, t)$ and its inverse Fourier transform $h(\tau, t)$. Even if these functions could be measured, using them to characterize a given channel would be somewhat akin to using a sample function of a random process to characterize the process. In other words, these functions contain too much data to be useful without processing; we shall be interested only in finding various averages of these quantities.

We shall make the assumption that $H(f, t)$ and $h(\tau, t)$, considering f and τ as parameters, can be adequately modeled as sample functions of stationary ergodic random processes. Obviously, for channels such as chaff channels, where the scattering particles fall to the ground in a number of hours, the stationarity assumption cannot be strictly valid. However, over the period of a measurement it is perfectly reasonable to assume stationarity. As Brennan[5] has pointed out, the assumption of ergodicity is no problem when one is constructing a random process model from a sample function.

The first averages that one might consider are $\overline{h(\tau, t)}$ and $\overline{H(f, t)}$, where we use overhead bars to represent averages. It is not hard to see that, for communication by reflection from a collection of scatterers in motion relative to each other, both $\overline{h(\tau, t)}$ and $\overline{H(f, t)}$ will be zero. To achieve this result, it is sufficient for the phase of the return from each scatterer to be uniformly distributed between 0 and 2π. Naturally, on channels for which $\overline{h(\tau, t)}$ and $\overline{H(f, t)}$ are not zero, these quantities can be easily measured and their effect considered as a specular component of channel return. We shall simplify our notation, however, by assuming these quantities to be 0.

A. Two-Frequency Correlation Function

The next average we shall consider is a quantity called the two-frequency correlation function $\mathcal{R}(\Delta f, \Delta t)$ first discussed by Hagfors.[6] This is defined as

$$\mathcal{R}(\Delta f, \Delta t) = \overline{2H^*(f_o - \frac{\Delta f}{2}, t)\, H(f_o + \frac{\Delta f}{2}, t + \Delta t)} \quad . \tag{33}$$

Using the complex low-pass representation for H from Eq. (30), this becomes

$$\mathcal{R}(\Delta f, \Delta t) = \frac{1}{2}\, \overline{\underline{H}^*(-\frac{\Delta f}{2}, t)\, \underline{H}(\frac{\Delta f}{2}, t + \Delta t)} \quad . \tag{34}$$

The right side of Eq. (33) is not a function of t because of the stationarity assumption. However, it does appear to be a function of the carrier frequency f_o. We shall justify later the fact that, for most time- and frequency-spread channels of interest, $\mathcal{R}(\Delta f, \Delta t)$ is relatively independent of f_o over a very broad range of f_o.

In order to interpret $\mathcal{R}(\Delta f, \Delta t)$, let us first consider the special case $\mathcal{R}(0, \Delta t)$. From Eq. (33), this is simply the autocorrelation of the channel response to a sinusoid of frequency f_o. We shall define the coherence time of the channel T_c as the interval in Δt over which $\mathcal{R}(0, \Delta t)$ is essentially nonzero. We are deliberately vague about this definition since we wish to use T_c in an order of magnitude sense only. It can be seen that T_c is a measure of the time over which coherent integration can be performed on the channel output and also of the duration of fades on the channel.

It is to be seen that a large T_c is of questionable value, since the large coherent integration time is offset by the presence of long fades. It is sometimes convenient to think of a coherent channel as the limit as T_c approaches infinity of a time- and frequency-spread channel, but the above relationship between coherent integration time and fading duration suggests that considerable caution should be exercised here.

Next let us consider $\mathcal{R}(\Delta f, 0)$. If frequency diversity to transmit information is used, it is $\mathcal{R}(\Delta f, 0)$ that tells us how far apart in frequency the channels must be to achieve essentially uncorrelated returns on each. Even more important, if we transmit a signal of bandwidth W, where W is small enough so that $\mathcal{R}(W, 0) \approx \mathcal{R}(0, 0)$, $H(f, t)$ will be approximately independent of f over the bandwidth used. Thus the received signal will be the same as the transmitted signal except for an over-all amplitude and phase that change over a period of time which is of the order of magnitude T_c. On the other hand, if the input bandwidth is so large that $\mathcal{R}(W, 0) \approx 0$, the amplitude and phase of different frequency components of the input signal will be changed relative to each other, and the received waveform will no longer bear any simple resemblance to the transmitted waveform. We vaguely define the coherence bandwidth of the channel F_c as the frequency at which $\mathcal{R}(F_c, 0)$ has dropped significantly toward 0 from $\mathcal{R}(0, 0)$.

It is somewhat tempting to try to relate the coherence time T_c to the over-all impulse response duration of the channel and to relate F_c to the over-all doppler spreading. Such a relationship does not exist and these quantities arise in physically unrelated ways. We shall show later, however, that if $R(\Delta f, \Delta t)$ is unimodal, then T_c is roughly the reciprocal of the doppler spreading, and F_c is roughly the reciprocal of the impulse response duration. For the example of orbital dipole channels previously discussed, $T_c \approx 1$ msec, and $F_c \approx 10^4$ cps.

B. Tap-Gain Correlation Function

The tap-gain correlation function $\mathfrak{R}(\tau, \Delta t)$ is defined as the inverse Fourier transform on Δf of $\mathfrak{R}(\Delta f, \Delta t)$.

$$R(\tau, \Delta t) \triangleq \int_{-\infty}^{\infty} \mathfrak{R}(\Delta f, \Delta t) \exp[j2\pi\Delta f\tau]\, d\Delta f \tag{35}$$

$$\mathfrak{R}(\Delta f, \Delta t) = \int_{-\infty}^{\infty} R(\tau, \Delta t) \exp[-j2\pi\Delta f\tau]\, d\tau \quad . \tag{36}$$

We can obtain a more convenient expression for $R(\tau, \Delta t)$ by applying Eqs. (34) and (31) to $\mathfrak{R}(\Delta f, \Delta t)$.

$$\mathfrak{R}(\Delta f, \Delta t) = \frac{1}{2} \overline{\underline{H}^*(-\frac{\Delta f}{2}, t)\, \underline{H}(\frac{\Delta f}{2}, t + \Delta t)}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau' d\tau'' \overline{\underline{h}^*(\tau', t) \exp[-j2\pi\tau'\Delta f/2]\, \underline{h}(\tau'', t + \Delta t) \exp[-j2\pi\tau''\Delta f/2]} \; .$$

Changing the variables of integration by letting $\delta = \tau'' - \tau'$, and then $\tau = \tau' + (\delta/2)$, we find that

$$\mathfrak{R}(\Delta f, \Delta t) = \int_{-\infty}^{\infty} d\tau \exp[-j2\pi\Delta f\tau] \left[\frac{1}{2} \int_{-\infty}^{\infty} d\delta\, \overline{\underline{h}^*(\tau - \frac{\delta}{2}, t)\, \underline{h}(\tau + \frac{\delta}{2}, t + \Delta t)}\right] \quad . \tag{37}$$

Comparing Eq. (37) with (36), we see that

$$R(\tau, \Delta t) = \frac{1}{2} \int_{-\infty}^{\infty} d\delta\, \overline{\underline{h}^*(\tau - \frac{\delta}{2}, t)\, \underline{h}(\tau + \frac{\delta}{2}, t + \Delta t)} \quad . \tag{38}$$

If $\mathfrak{R}(\Delta f, \Delta t)$ depends upon f_o, $R(\tau, \Delta t)$ will also depend upon f_o. This can be seen in Eq. (38) because the low-pass representation $\underline{h}$ is defined from h using the frequency f_o.

Let us investigate Eq. (38) for the case of a channel composed of a collection of scatterers, and let us suppose that each scatterer has an impulse response duration of, at most, ϵ.† For chaff particles and orbital dipole scatterers, $\epsilon \approx 10^{-9}$. The quantity ϵ is not to be confused with the over-all impulse response duration of the cloud. The latter quantity is many orders of magnitude larger than ϵ because the scatterers are spread out, and each one responds with a different over-all delay. We now wish to establish that

$$\overline{\underline{h}^*(\tau - \frac{\delta}{2}, t)\, \underline{h}(\tau + \frac{\delta}{2}, t + \Delta t)} = 0 \quad \text{for } |\delta| > \epsilon \quad . \tag{39}$$

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† Although it is possible for the impulse response of a scatterer to be strictly limited to a time duration ϵ, it is not possible to strictly time limit the complex low-pass representation of the impulse response since its Fourier transform is strictly zero for $f \leqslant -f_o$. It is not hard to convince oneself, however, that the errors due to this approximation can be ignored.

Unfortunately, to prove this mathematically, we would first have to set up a mathematical model for the statistics of the scatterer locations. Any model that we could handle mathematically would be far too restrictive for the physical cases we wish to consider. Thus we must be satisfied with an intuitive argument. Let $\underline{h}_i(\tau, t)$ be the response from the i^{th} of a collection of scatterers. Then, since $\underline{h}$ is the sum of the responses from the individual scatterers, Eq. (39) becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\underline{h}_i^*(\tau - \frac{\delta}{2}, t)\, \underline{h}_j(\tau + \frac{\delta}{2}, t + \Delta t)} = 0 \quad \text{for } |\delta| > \epsilon \quad . \tag{40}$$

To establish Eq. (40), it is certainly sufficient to show that†

$$\overline{\underline{h}_i^*(\tau - \frac{\delta}{2}, t)\, \underline{h}_j(\tau + \frac{\delta}{2}, t + \Delta t)} = 0 \quad \text{for } i \neq j \quad . \tag{41}$$

Equation (41) will follow, however, if the phase angles of both $\underline{h}_i^*[\tau - (\delta/2), t]$ and $\underline{h}_j[\tau + (\delta/2), t + \Delta t]$ are independent and uniformly distributed. This will occur if the probability density of the path length of each scattering particle is flat over an RF wavelength independent of the other particles; this appears to be eminently reasonable for any collection of scatterers. Using this assumption, Eq. (38) can be written

$$R(\tau, \Delta t) = \frac{1}{2} \int_{-\epsilon}^{\epsilon} d\delta\, \overline{\underline{h}^*(\tau - \frac{\delta}{2}, t)\, \underline{h}(\tau + \frac{\delta}{2}, t + \Delta t)} \quad . \tag{42}$$

Assuming also that $R(\tau, \Delta t)$ is relatively constant over an interval ϵ in τ, this becomes

$$R(\tau, \Delta t) = \frac{1}{2} \int_{\tau-\epsilon}^{\tau+\epsilon} d\tau'\, \overline{\underline{h}^*(\tau, t)\, \underline{h}(\tau', t + \Delta t)} \quad . \quad \approx \epsilon\, \overline{\underline{h}^*(\tau, t)\, h(\tau, t+\Delta t)} \tag{43}$$

In order to interpret $R(\tau, \Delta t)$, suppose that the inputs to the channel are restricted to a bandwidth W around the carrier frequency f_o. Then, as shown by Kailath, the channel, aside from additive noise, can be represented by a tapped delay line with taps spaced $1/W$ apart. To accomplish this, the input $\underline{x}(t)$ can be represented by the sampling theorem as

$$\underline{x}(t - \tau) = \sum_{k=-\infty}^{\infty} \underline{x}(t - \frac{k}{W}) \frac{\sin \pi W(\tau - \frac{k}{W})}{\pi W(\tau - \frac{k}{W})} \quad . \tag{44}$$

Applying this to Eq. (32), the channel output can be expressed as

$$\underline{y}(t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \underline{x}(t - \frac{k}{W}) \int_{-\infty}^{\infty} d\tau\, \frac{\sin \pi W(\tau - \frac{k}{W})}{\pi W(\tau - \frac{k}{W})}\, \underline{h}(\tau, t) \quad . \tag{45}$$

Defining

$$\underline{h}_W(\frac{k}{W}, t) \triangleq \int_{-\infty}^{\infty} d\tau\, \frac{\sin \pi W(\tau - \frac{k}{W})}{\pi W(\tau - \frac{k}{W})}\, \underline{h}(\tau, t) \tag{46}$$

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† This follows from the fact that $\underline{h}_i^*[\tau - (\delta/2), t]\, h_i[\tau + (\delta/2), t]$ must be 0 for $\delta > \epsilon$, since $\underline{h}_i$ is nonzero only for a duration ϵ in τ. If particle i has a deterministic velocity component v, $\underline{h}_i^*[\tau - (\delta/2), t]\, h_i[\tau + (\delta/2), t + \Delta t]$ might be nonzero for $\delta < \epsilon + (\Delta t v/c)$, but we shall neglect this effect for simplicity since it does not appear to have any important effects.

we have

$$\underline{y}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2}\, \underline{x}\left(t - \frac{k}{W}\right) \underline{h}_W\left(\frac{k}{W}, t\right) \quad . \tag{47}$$

Then the channel is represented as a delay line with taps $1/W$ seconds apart and gain $\underline{h}_W(k/W, t)$ on the k^{th} tap.

We shall next find the autocorrelation of the tap gain $\underline{h}_W(k/W, t)$ under the restriction that $(1/W) >> \epsilon$ and also that $1/W$ is much less than the time in τ over which $R(\tau, \Delta t)$ changes appreciably.

$$\frac{1}{2}\, \overline{\underline{h}_W^*\left(\frac{k}{W}, t\right) \underline{h}_W\left(\frac{k}{W}, t + \Delta t\right)} = \frac{1}{2} \iint d\tau\, d\tau' \left[\overline{\underline{h}^*(\tau, t)\, \underline{h}(\tau', t + \Delta t)} \times \frac{\sin \pi W\left(\tau - \frac{k}{W}\right)}{\pi W\left(\tau - \frac{k}{W}\right)} \frac{\sin \pi W\left(\tau' - \frac{k}{W}\right)}{\pi W\left(\tau' - \frac{k}{W}\right)} \right] \tag{48}$$

$$\doteq \int d\tau\, R(\tau, \Delta t) \left[\frac{\sin \pi W\left(\tau - \frac{k}{W}\right)}{\pi W\left(\tau - \frac{k}{W}\right)} \right]^2 \tag{49}$$

$$\doteq \frac{R\left(\frac{k}{W}, \Delta t\right)}{W} \quad . \tag{50}$$

Equation (48) follows from Eq. (46), Eq. (49) from Eq. (43) and the assumption that $W << (1/\epsilon)$, and Eq. (50) from the assumption that $1/W$ is much less than the time over which $R(\tau, \Delta t)$ changes appreciably. Using the same conditions, it follows in the same way that successive tap gains are uncorrelated. For orbital scatter channels, the fact that the tap gains are uncorrelated agrees with an earlier result by Bello.[7] For a discussion of channels in which the tap gains are correlated, see Spilker[8] and Kailath.[9]

The quantity on the left side of Eq. (48) is the tap-gain correlation function used by Kailath. We have shown that under some very plausible conditions it is proportional to the tap-gain correlation defined here. If the conditions are not satisfied, one must give up Eq. (50) if one uses the definition here, and one must give up the Fourier relationships between $\mathfrak{R}(\Delta f, \Delta t)$, $R(\tau, \Delta t)$, and the scattering function if one uses Kailath's definition.

The difficulty here is not due to the use of sampling. If one wishes to maintain Fourier relationships between $\mathfrak{R}(\Delta f, \Delta t)$, $R(\tau, \Delta t)$, and the scattering function, the definition of $R(\tau, \Delta t)$ must involve some smoothing over τ [see Eqs. (38) and (43)]. If $W << 1/\epsilon$, the taps on the delay-line model are far enough apart for this smoothing to be automatically accomplished. If $W > 1/\epsilon$, there is no reason to expect the two definitions to be equivalent.

The quantity $R(\tau, \Delta t)$ is sometimes complex; when this is so, $\text{Re}[h_W(\tau, t)]$ and $\text{Im}[\underline{h}_W(\tau, t + \Delta t)]$ are correlated. This condition is physically meaningful and, as we shall see later, allows us to distinguish, for example, between an expanding and a contracting cloud of scatterers.

Sometimes it is desirable to separate the tap gain $\underline{h}_W(k/W, t)$ into a real and an imaginary part and find the autocorrelations and cross correlation of these two quantities. Equation (50) is not sufficient to find these quantities, but if we use the assumption of scatterers with independent uniformly distributed phases, we immediately see that the real and imaginary parts of

$\underline{h}_W(k/W, t)$ have the same autocorrelation. Using Eq. (50) for the sum of these quantities,

$$\overline{\mathrm{Re}[\underline{h}_W(\tfrac{k}{W}, t)]\,\mathrm{Re}[\underline{h}_W(\tfrac{k}{W}, t + \Delta t)]} = \frac{\mathrm{Re}[R(\frac{k}{W}, \Delta t)]}{W} \tag{51}$$

$$\overline{\mathrm{Im}[\underline{h}_W(\tfrac{k}{W}, t)]\,\mathrm{Im}[\underline{h}_W(\tfrac{k}{W}, t + \Delta t)]} = \frac{\mathrm{Re}[R(\frac{k}{W}, \Delta t)]}{W} \quad . \tag{52}$$

To find some physical interpretation for $R(\tau, \Delta t)$, we can first look at $R(\tau, 0)$ as given by Eq. (43). This is proportional to the average returned power at delay τ, or in other words, is the return due to scatterers for which the path length yields a delay τ. Similarly, $R(\tau, \Delta t)$ is the correlation function of the return due to scatterers at path lengths of delay τ. Clearly, we cannot try to resolve τ too closely in this picture, because of the individual response time ϵ and because our statistical model will break down if a large number of scatterers are not in each resolution zone.

We shall define the multipath smear L as the time duration in τ over which $R(\tau, 0)$ is effectively nonzero. This is clearly the same as the effective duration in τ of the impulse response $\underline{h}(\tau, t)$. Since $R(\tau, \Delta t)$ is the Fourier transform of $\mathcal{R}(\Delta f, \Delta t)$, L will have the order of magnitude of $1/F_c$, if $R(\tau, \Delta t)$ and $\mathcal{R}(\Delta f, \Delta t)$ are well-behaved unimodal functions of τ and Δf. The time coherence of the channel T_c previously defined as the interval in Δt over which $\mathcal{R}(\Delta f, \Delta t)$ is effectively nonzero is also the interval in Δt over which $R(\tau, \Delta t)$ is effectively nonzero.

In order to interpret the significance of L and T_c, consider the use of phase-shift keying for transmitting digital data. The duration of a symbol in such a system should be considerably greater than L to avoid inter-symbol interference. On the other hand, it should be less than T_c to allow for coherent integration of the received signal. These conditions cannot be met if $T_c < L$; under these circumstances phase-shift keying is clearly a very poor data transmission method. Channels for which $T_c < L$ are called overspread channels and will be discussed later.

C. Scattering Function

The scattering function $\sigma(\tau, f)$ of a time- and frequency-spread channel is defined as

$$\sigma(\tau, f) \triangleq \int_{-\infty}^{\infty} R(\tau, \Delta t) \exp[-j2\pi f \Delta t]\, d\Delta t \quad . \tag{53}$$

As with $\mathcal{R}(\Delta f, \Delta t)$ and $R(\tau, \Delta t)$, $\sigma(\tau, f)$ is implicitly a function of the carrier frequency f_o, but as before, we assume that $\sigma(\tau, f)$ is relatively independent of f_o over a wide band of frequencies.

The function $\sigma(\tau, f)$, unlike $R(\tau, \Delta t)$ and $\mathcal{R}(\Delta f, \Delta t)$, is necessarily real and non-negative. To demonstrate this, it is sufficient to show that $R(\tau, \Delta t) = R^*(\tau, -\Delta t)$. Using Eq. (38) we have

$$R^*(\tau, -\Delta t) = \frac{1}{2} \int_{-\infty}^{\infty} d\delta\, \overline{\underline{h}(\tau - \tfrac{\delta}{2}, t)\, \underline{h}^*(\tau + \tfrac{\delta}{2}, t - \Delta t)} \tag{54}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} d\delta\, \overline{\underline{h}^*(\tau + \tfrac{\delta}{2}, t)\, \underline{h}(\tau - \tfrac{\delta}{2}, t + \Delta t)} \tag{55}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} d\delta\, \overline{\underline{h}^*(\tau - \tfrac{\delta}{2}, t)\, \underline{h}(\tau + \tfrac{\delta}{2}, t + \Delta t)} = R(\tau, \Delta t) \quad . \tag{56}$$

Equation (55) follows from using stationarity to replace t with $t + \Delta t$. Equation (56) follows from using $-\delta$ for δ as the variable of integration.

The scattering function has a much more direct physical interpretation than either $\mathfrak{R}(\Delta f, \Delta t)$ or $R(\tau, \Delta t)$. Going back to the tapped delay-line model for the channel, we see that $\sigma(k/W, f)/W$ gives the power density spectrum of the k^{th} tap gain. Thus $\sigma(k/W, f)/W$ is a measure of the power returned at delay k/W with doppler shift f, and we can think of $\sigma(\tau, f)$ as being a measure of the power returned by scatterers at delay τ with doppler shift f. This is obscured somewhat because frequency spreading is caused not only by the doppler shift produced by motion of scatterers in the direction of changing path length, but also by the modulation introduced by the rotation of the scatterers. When the cloud of scatterers is actually an astronomical body such as the moon, rotation of individual pieces of the body is no problem, and $\sigma(\tau, f)$ resolves the return from the body according to path length and velocity in the direction of changing path length. This was originally pointed out by Green,[10] and Pettengill's moon-mapping experiments[11] were based on this principle. For chaff clouds and orbital dipole belts, rotation of particles is not negligible, and therefore $\sigma(\tau, f)$ cannot quite be interpreted as the return from particles with a given path length and path length velocity component. Even in these cases, however, $\sigma(\tau, f)$ is closely related to the actual physical mechanisms in the cloud.

We can now justify our contention that $\sigma(\tau, f)$, and hence also $R(\tau, \Delta t)$ and $\mathfrak{R}(\Delta f, \Delta t)$, are relatively insensitive to changes of the carrier frequency f_o. The relationship between doppler shift and velocity depends linearly upon f_o. Thus for $f_o = 10$ kMc, a change in carrier frequency of 1 kMc will change the doppler shift and hence the scale of $\sigma(\tau, f)$ by 10 percent. Similarly, for short dipole scatterers a 10-percent change in carrier frequency should not materially affect the individual scatterer response. Although 1 kMc is not large relative to f_o, it is certainly large relative to conventional communication bandwidths, and this is the rationale in ignoring f_o.

The most important over-all characteristics of $\sigma(\tau, f)$ are: L, the previously defined time duration in τ; and B, the frequency interval in f outside of which $\sigma(\tau, f)$ is effectively zero. The quantity B is commonly called the doppler spread of the channel. If $\sigma(\tau, f)$ and $R(\tau, \Delta t)$ are both well behaved and unimodal in f and Δt, B has the order of magnitude of $1/T_c$.

V. MEASUREMENT OF TIME- AND FREQUENCY-SPREAD CHANNELS

In Sec. IV we showed that the two-frequency correlation function, the tap-gain correlation function, and the scattering function are of fundamental importance in characterizing a time- and frequency-spread channel. In this section we shall analyze several techniques for measuring these quantities. We shall find both the mean and variance of the measurements discussed, and hopefully, make it relatively easy to decide what type of measurement to make in a given situation. (For a different approach to the analysis of channel measurements, see Levin.[12])

A. Measurement by Pseudo-Random Inputs

First we shall discuss the measurement of $R(\tau, \Delta t)$ or $\sigma(\tau, f)$ by using a pseudo-random input waveform on the channel. Green[10] and Kailath[9] have discussed, from very different standpoints, what is operationally this same measurement. We shall discuss the measurement technique itself, show how it fits in with the work of Kailath and Green, and resolve an anomaly about overspread channels arising between the results obtained by these men.

Let $\underline{x}(t)$ and $\underline{y}(t)$ be the complex low-pass representations of the input and output from the channel. We shall estimate $R(\tau, \Delta t)$ from the equation:

$$\text{Est}\, R(\tau, \Delta t) = \alpha \int_{-\infty}^{\infty} dt \cdot \frac{1}{2} \left[\frac{1}{2}\, \underline{x}(t-\tau)\, \underline{y}^*(t)\right] \left[\frac{1}{2}\, \underline{x}^*(t+\Delta t-\tau)\, \underline{y}(t+\Delta t)\right] \tag{57}$$

where α is a constant to be evaluated later.† Some restrictions obviously must be put on $\underline{x}(t)$, but these restrictions will be imposed later.

Physically, this measurement is obtained by transmitting $x(t)$ over the channel and multiplying the channel output by a frequency offset version of $x(t)$ delayed by τ seconds. The complex low-pass equivalent of the resulting waveform is $\frac{1}{2}\, \underline{x}^*(t-\tau)\, \underline{y}(t)$. The complex low-pass equivalent of the autocorrelation of this waveform is then proportional to $\text{Est}\, R(\tau, \Delta t)$.

If we neglect the additive noise for the present, we can use Eq. (32) to express $\underline{y}$ in terms of $\underline{x}$ and $\underline{h}$.

$$\text{Est}\, R(\tau, \Delta t) = \frac{\alpha}{32} \iiint dt\, d\tau'\, d\tau'' \left[\underline{x}(t-\tau)\, \underline{x}^*(t-\tau')\, \underline{x}^*(t+\Delta t-\tau)\, \underline{x}(t+\Delta t-\tau'')\right.$$
$$\left. \times\, \underline{h}^*(\tau', t)\, \underline{h}(\tau'', t+\Delta t)\right] \quad . \tag{58}$$

Next we find the average value of $\text{Est}\, R(\tau, \Delta t)$ averaged over the channel fluctuation. If we assume that $\underline{x}(t)$ is essentially constant over the interval ϵ discussed prior to Eq. (39), we can use Eq. (42) to integrate over τ''.

$$\overline{\text{Est}\, R(\tau, \Delta t)} = \frac{\alpha}{16} \iint dt\, d\tau' \left[\underline{x}(t-\tau)\, \underline{x}^*(t-\tau')\, \underline{x}^*(t+\Delta t-\tau)\, \underline{x}(t+\Delta t-\tau')\right.$$
$$\left. \times\, R(\tau', \Delta t)\right] \quad . \tag{59}$$

Now we assume that $\text{Re}[x(t)]$ and $\text{Im}[x(t)]$ are each independent sample functions of duration $0 < t < T$ chosen from a stationary Gaussian random process of autocorrelation $R_{\underline{x}}(\tau)$. Then

$$\frac{1}{2}\, \overline{\underline{x}^*(t)\, \underline{x}(t+\tau)} = R_{\underline{x}}(\tau) \quad \text{for } 0 \leqslant t \leqslant T \quad ; \quad 0 \leqslant t+\tau \leqslant T \tag{60}$$

$$\frac{1}{2}\, \overline{\underline{x}(t)\, \underline{x}(t+\tau)} = 0 \quad . \tag{61}$$

There is a slight logical contradiction here, since a complex low-pass equivalent can have no frequency components below $-f_0$ and therefore cannot be strictly time limited. This is obviously unimportant if the bandwidth of $\underline{x}$ is much less than f_0. We now average Eq. (59) over the input ensemble by using Eqs. (60) and (61). Recall that any higher order moment of a Gaussian random process can be expressed as the sum of products of second-order moments.‡ Using a double bar to indicate an average over both the channel and the input, we have

† The factors of 1/2 are inserted here to be consistent with the rules for convolution on low-pass complex equivalents. The reader need not concern himself with them, however, since they will eventually be absorbed into the constant α.

‡ See Ref. 13, pp. 93 and 95. The extension to complex random processes with independent, identically distributed, real and imaginary parts is easy.

$$\overline{\overline{\text{Est}\, R(\tau, \Delta t)}} = \frac{\alpha}{4} \int_{\tau'=-\infty}^{\infty} \int_t d\tau'\, dt\, [R_{\underline{x}}^2(\tau - \tau') + R_{\underline{x}}^2(\Delta t)]\ R(\tau', \Delta t) \quad . \tag{62}$$

The limits on the integral on t of Eq. (62) are such that they keep $t - \tau$, $t - \tau'$, $t + \Delta t - \tau$, and $t + \Delta t - \tau'$ all between 0 and T. If we assume $T >> L$, where L is the duration in τ' of $R(\tau', \Delta t)$, we can take the limits on t to be τ and $T - \Delta T + \tau$. If we also assume that the duration of $R_{\underline{x}}(\tau)$ is much shorter than the time in τ' over which $R(\tau', \Delta t)$ changes, we can integrate Eq. (62) to find that

$$\overline{\overline{\text{Est}\, R(\tau, \Delta t)}} = \frac{\alpha}{4} (T - \Delta t) \left[R(\tau, \Delta t) \int_{\tau'} R_{\underline{x}}^2(\tau')\, d\tau' + R_{\underline{x}}^2(\Delta t) \int_{\tau'} R(\tau', \Delta t)\, d\tau' \right] \quad . \tag{63}$$

Finally, setting

$$\alpha = \frac{4}{(T - \Delta t) \int_{\tau'} R_{\underline{x}}^2(\tau')\, d\tau'} \tag{64}$$

we have

$$\overline{\overline{\text{Est}\, R(\tau, \Delta t)}} = R(\tau, \Delta t) + \frac{R_{\underline{x}}^2(\Delta t)}{\int R_{\underline{x}}^2(t)\, dt} \int_{\tau'} R(\tau', \Delta t)\, d\tau' \quad . \tag{65}$$

Equation (65) was derived on the assumptions that the measurement interval T is much greater than the multipath spread L and that the duration of $R_{\underline{x}}(\tau)$ is much greater than ϵ and much less than the time in τ over which $R(\tau, \Delta t)$ changes. For the common time- and frequency-spread channels, meeting this restriction is no problem. If, for some peculiar channel, ϵ is comparable to the time in τ over which $R(\tau, \Delta t)$ changes, $R(\tau, \Delta t)$ will not characterize the channel very well. The quantity $\overline{\underline{h}^*(\tau', t)\, \underline{h}(\tau'', t + \Delta t)}$ could still be estimated in this case, however.

Equation (66) shows that the average of the proposed estimate of $R(\tau, \Delta t)$ is the true value plus an error term; however, the error term is only significant for Δt so small that $R_{\underline{x}}(\Delta t)$ is nonzero. If we increase the bandwidth of the input process, we can make $R_{\underline{x}}(\Delta t)$ be nonzero over as small an interval as we wish down to ϵ, but it can be seen from Eq. (65) that the magnitude of the disturbance at $\Delta t = 0$ grows with the bandwidth. For a large input bandwidth, we can consider the error term to be an impulse in Δt of magnitude $\int_{\tau'} R(\tau', \Delta t)\, d\tau'$. Figure 1 shows $\overline{\overline{\text{Est}\, R(\tau, \Delta t)}}$ as a function of Δt.

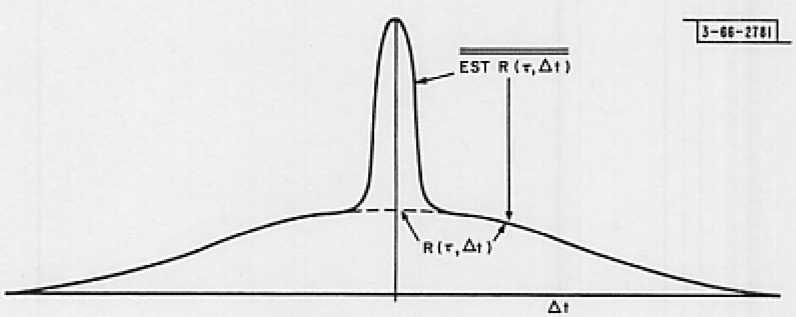

Fig. 1. $\overline{\overline{\text{Est}\, R(\tau, \Delta t)}}$ as a function of Δt.

The impulsive behavior of this error term does not appear to be a consequence of the assumption that the input was a sample function of a Gaussian random process. The same behavior can be demonstrated when the input is a carrier that is phase-shift keyed by a random binary sequence, and all the evidence indicates that any input waveform of long duration and large bandwidth will give a similar behavior.

The error term in Eq. (65) is of course no problem practically. We simply pick the input bandwidth large enough so that $R(\tau, \Delta t)$ is essentially constant in Δt over the duration of $R_{\underline{x}}(\tau)$. We then extrapolate $R(\tau, \Delta t)$ at $\Delta t = 0$ from nearby points.

The error term in Eq. (65) is stressed because it is the key to the anomaly between the results of Kailath and Green. Kailath's analysis was formally analogous to what has been done here, except that he used white Gaussian noise as the input and measured $\overline{h(\tau', t)\, h(\tau'', t + \Delta t)}$. He noted an impulse in his measurement corresponding to the error term found here but did not discuss its effect on a practical measurement. The use of white Gaussian noise as an input simplifies the formal manipulations but precludes any analysis of the noise or the variance of the estimate and makes interpretation of some of the results quite difficult.

B. Green's Estimate of $\sigma(\tau, f)$

In Green's analysis, $\sigma(\tau, f)$ was estimated by passing the received waveform through a filter matched to the input but offset in time by τ and in frequency by f. This amounts to finding the energy at frequency f in the waveform $\underline{y}(t)\, \underline{x}(t - \tau)$, and thus qualitatively it appears that Green's estimate of $\sigma(\tau, f)$ should be the Fourier transform of Est $R(\tau, \Delta t)$. Analytically, Green[10] shows that the average value of his offset matched filter estimate is given by

$$\overline{\text{Est}\,\sigma(\tau, f)} = \frac{\alpha}{2} \iint d\tau'\, df' \cdot \frac{1}{2} \psi^2(\tau' - \tau, f - f')\, \sigma(\tau', f') \tag{66}$$

where ψ^2 is the ambiguity function of the input waveform $\psi^2(\tau, f) = \psi(\tau, f) \cdot \psi^*(\tau, f)$,

$$\psi(\tau, f) = \int dt\, \frac{1}{2}\, \underline{x}^*(t)\, \underline{x}(t + \tau)\, \exp[j2\pi f t] \quad . \tag{67}$$

We now show that the inverse Fourier transform of Eq. (66) is indeed Eq. (59). Taking the inverse Fourier transform of Eq. (66) on f and recognizing that the right-hand side is a convolution on f, we can use the convolution theorem to get

$$F^{-1}\, \overline{\text{Est}\,\sigma(\tau, f)} = \frac{\alpha}{4} \int_{\tau'} d\tau'\, F^{-1}[\psi^2(\tau' - \tau, f)] \cdot F^{-1}[\sigma(\tau', f)] \tag{68}$$

where the notation F^{-1} denotes inverse Fourier transform.

From the definition of $\sigma(\tau, f)$ in Eq. (53), we have $F^{-1}[\sigma(\tau', f)] = R(\tau', \Delta t)$.

Also, from Eq. (67),

$$F^{-1}\psi(\tau' - \tau, f) = \frac{1}{2}\, \underline{x}(t)\, \underline{x}^*(t + \tau' - \tau) \quad .$$

Thus $\psi^2(\tau' - \tau, f)$ is the energy density spectrum of the function $\frac{1}{2}\, \underline{x}(t)\, \underline{x}^*(t + \tau' - \tau)$ considered as a function of t. Consequently, the inverse transform of ψ^2 is the autocorrelation of $\frac{1}{2}\, \underline{x}(t)\, \underline{x}^*(t + \tau' - \tau)$ or $\frac{1}{4} \int \underline{x}^*(t)\, \underline{x}(t + \tau' - \tau)\, \underline{x}(t + \Delta t)\, \underline{x}^*(t + \Delta t + \tau' - \tau)\, dt$.

Finally, substituting $t - \tau'$ for t, we have

$$F^{-1}\,[\psi^2(\tau' - \tau, f)] = \int_{t=-\infty}^{\infty} \frac{1}{4}\, \underline{x}^*(t - \tau')\, \underline{x}(t - \tau)\, \underline{x}(t + \Delta t - \tau')\, \underline{x}^*(t + \Delta t - \tau)\, dt \quad . \tag{69}$$

Substituting Eq. (69) into (68), we have

$$F^{-1}\, \overline{\text{Est}\,\sigma(\tau, f)} = \frac{\alpha}{16} \int_{\tau'} R(\tau', \Delta t) \int_t \underline{x}(t - \tau)\, \underline{x}^*(t - \tau')\, \underline{x}^*(t + \Delta t - \tau) \times \underline{x}(t + \Delta t - \tau')\, dt\, d\tau' \quad . \tag{70}$$

Comparing (70) with (59), we see that the expressions are identical. Thus

$$\overline{\text{Est}\,\sigma(\tau, f)} = \int \overline{\text{Est}\,R(\tau, \Delta t)}\, \exp[-j2\pi f \Delta t]\, d\Delta t \quad . \tag{71}$$

This result is not really surprising; both techniques perform the same operation. In both cases we multiply the received signal by a delayed version of the transmitted signal and either correlate the resulting waveform or find its spectrum.

Let us now use Eq. (71) to calculate $\overline{\overline{\text{Est}\,\sigma(\tau, f)}}$ when $\underline{x}(t)$ is a sample of a complex, stationary, Gaussian, random process. Taking the Fourier transform of Eq. (65) and assuming that $R_{\underline{x}}(\Delta t)$ is nonzero only when $R(\tau', \Delta t) \approx R(\tau', 0)$,

$$\overline{\overline{\text{Est}\,\sigma(\tau, f)}} = \sigma(\tau, f) + \int_{\tau'} R(\tau', 0)\, d\tau' \tag{72}$$

$$\overline{\overline{\text{Est}\,\sigma(\tau, f)}} = \sigma(\tau, f) + \int_{\tau'} \int_{f'} \sigma(\tau', f')\, d\tau'\, df' \quad . \tag{73}$$

The term on the right side of Eq. (73) is usually called self-noise, but this name is somewhat misleading since the term is a fixed deterministic quantity independent of τ and f. To get an idea of the relative magnitude of the two terms in Eq. (73), assume that $\sigma(\tau, f)$ is constant over the region $-L/2 < \tau < L/2$, $-B/2 < f < B/2$, and zero outside this region. Then the ratio of the error term to $\sigma(\tau, f)$ is simply BL, the spreading constant of the channel. Green[10] also observed this result and pointed out that if the channel were overspread, i.e., if $BL > 1$, then the estimate of $\sigma(\tau, f)$ would not be close to the true value. The new contribution here is to point out that when this error term is averaged over the channel and the input ensemble, it is independent of τ and f and therefore can be subtracted out.

The preceding results have established the connection between Green's offset match-filter measuring technique and Kailath's correlation technique. We have seen that neither the self-noise spike in $\overline{\overline{\text{Est}\,R(\tau, \Delta t)}}$ at $\tau = 0$, $\Delta t = 0$, nor the corresponding plateau in $\overline{\overline{\text{Est}\,\sigma(\tau, f)}}$ causes any insurmountable measurement problems. These results, however, concern only the average value of our estimates. In order to show that any particular measurement will yield a good estimate, we shall estimate the variance of our estimate of $R(\tau, \Delta t)$ and show that it goes to zero with increasing measurement time. The estimate of $\sigma(\tau, f)$ approaches $\sigma(\tau, f)$ in a somewhat more complicated way and will not be treated in detail here. Since Est $\sigma(\tau, f)$ is the power spectrum of a finite sample of the random process $\underline{y}(t)\, \underline{x}(t - \tau)$, some smoothing must be done on Est $\sigma(\tau, f)$ before the variance goes to zero with increasing measurement time. The problem of finding the power spectrum of a process from a sample function is discussed in detail by Blackman and Tukey.[14]

C. Variance of Est $R(\tau, \Delta t)$ with Pseudo-Random Inputs

In order to find the variance of Est $R(\tau, \Delta t)$, we must discuss in more detail how the received signal is filtered to reduce the additive noise. In practice, this would involve a circuit such as that in Fig. 2.[15]

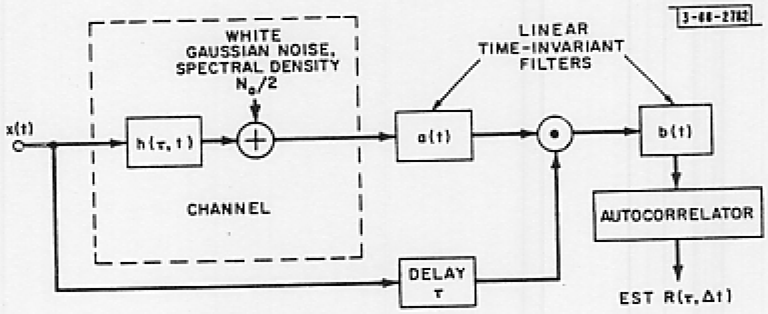

Fig. 2. Block diagram for estimating $R(\tau, \Delta t)$.

If the bandwidth of the input process is W_x, then presumably the bandwidth of a(t) should be about $W_x + B$, and the bandwidth of b(t) should be about B. In order to simplify the analysis, we shall assume that the input process has an autocorrelation $R_x(\tau) = S \exp[-\pi W_x^2 \tau^2]$, corresponding to an input power S and nominal bandwidth W_x. We also take the low-pass equivalent of the impulse of filters a(t) and b(t) to be:

$$\underline{a}(t) = 2\sqrt{2}\, W_a \exp[-2\pi W_a^2 t^2]$$

$$\underline{b}(t) = 2\sqrt{2}\, W_b \exp[-2\pi W_b^2 t^2] \quad .$$

This particular form of filter is chosen only for analytical convenience. The fact that a(t) and b(t) are unrealizable is of no consequence since they can be realized as closely as desired by adding delay. By taking the Fourier transform of these filters and moving them up to bandpass by Eq. (9), we see that each filter has unity gain in the center of the band. Even with these simplifications, the general analysis of such a system is exceptionally tedious. Two special cases have been analyzed however, and these are sufficient for our purposes. In Case A, we assume $W_b >> W_a > W_x$ so that the filter b(t) can be ignored. In Case B, we assume that $W_a > W_x >> W_b$ so that the filter a(t) can be ignored. Case A is analyzed in Appendix A, and Case B although slightly more tedious, can be analyzed in the same way. In both cases, it is assumed that $1/W_x >> \epsilon$ (ϵ is the impulse response duration of an individual scatterer), and that $1/W_x$ is much less than the time interval in τ and Δt over which $R(\tau, \Delta t)$ changes significantly. It is also assumed that T, the duration of the input signal, is long enough to neglect end effects in the integrations. Finally, it is assumed that there are many scattering particles in an interval $1/W_x$ of path delay, and that $1/W_x < \Delta t$.

The variance of Est $R(\tau, \Delta t)$ is broken up into three terms: $\overline{N_C^2}$, the variance due to channel and input variations; $\overline{N_A^2}$, the variance due to additive noise; and $\overline{N_{CA}^2}$, the variance due to cross products. For Case A, the following approximate expressions and also some more exact expressions are derived in Appendix A.

$$\overline{N_C^2} = \frac{1}{T} \int |R(\tau, t)|^2 \, dt + \frac{W_x}{T} \left[\int R(\tau', 0) \, d\tau' \right]^2 \tag{74}$$

$$N_A^2 = \frac{4\sqrt{2}\, W_x W_a^2 N_o^2}{TS^2} \left[1 + \left(\frac{W_a}{W_x} \right)^2 \right]^{-1/2} \tag{75}$$

$$\overline{N_{CA}^2} = \frac{4W_x}{T} \frac{W_a N_o}{S} \frac{\int R(\tau', 0) \, d\tau'}{\sqrt{\frac{3}{4} + \frac{1}{4}\left(\frac{W_a}{W_x}\right)^2}} \quad . \tag{76}$$

For Case B, the analogous expressions are:

$$\overline{N_C^2} = \frac{1}{T} \int |R(\tau, t)|^2 \, dt + \frac{W_b}{\sqrt{2}T} \left[\int R(\tau', 0) \, d\tau' \right]^2 \tag{77}$$

$$\overline{N_A^2} = \frac{4\sqrt{2}\, W_x^2 W_b N_o^2}{TS^2} \tag{78}$$

$$\overline{N_{CA}^2} = \frac{4W_x}{T} \frac{W_b N_o}{S} \int R(\tau', 0) \, d\tau' \quad . \tag{79}$$

As expected, these expressions indicate that the variance of Est $R(\tau, \Delta t)$ decreases as T and S increase. The behavior with respect to the input and filter bandwidths is less obvious. The results clearly indicate that all these bandwidths should be made as small as possible. One might argue that by making W_x large, one could resolve Est $R(\tau, \Delta t)$ more closely, and then by smoothing over τ, one could reduce the variance. The argument is valid if the additive noise is unimportant, but in most measurements, $\overline{N_A^2}$ is the dominant term; then since Case B is valid for large W_x, we see that even after smoothing over τ, the variance increases with W_x. The reason for the increase in variance with an increase in W_x can be seen somewhat from Eqs. (63) and (64). As W_x increases, the scaling constant α increases linearly corresponding to a decrease in the level of the autocorrelation of $\underline{x}(t - \tau)\, \underline{y}^*(t)$. This constant α appears squared in the variance of Est $R(\tau, \Delta t)$, giving rise to the W_x^2 term in Eq. (78).

When W_x, W_a, and W_b are all reduced as much as possible without violating the restrictions put on them, W_b will be limited by B, the doppler spread of the channel, and W_x will be limited by the reciprocal of the time in τ over which $R(\tau, \Delta t)$ changes. For underspread channels, i.e., channels in which $LB \ll 1$, we will have $W_x \gg W_b$ and the results for Case B will be valid. For other channels, W_x, W_a, and W_b will all be about the same and Case A and Case B will give the same order of magnitude of results. These results will be discussed in more detail after describing another technique for measuring spread channels.

D. Measurement by a Chirp Input Signal

Although the measurement technique described in the last section yields a perfectly acceptable way to measure $R(\tau, \Delta t)$, the required instrumentation is elaborate and expensive. In this section we shall describe another measurement technique that appears both to be simpler to instrument and to have a smaller variance than the pseudo-random input technique.

We shall use a chirp signal of power S and duration T as an input to the channel:

$$x(t) = \sqrt{2S}\cos 2\pi\left(f_o t + \frac{\alpha t^2}{2}\right) \quad 0 \leqslant t \leqslant T \quad . \tag{80}$$

The instantaneous frequency of this signal is $f_o + \alpha t$ and is thus linearly increasing (or decreasing for $\alpha < 0$) with time. We shall take the complex low-pass representation of x(t) to be†

$$\underline{x}(t) = \sqrt{2S}\exp[j2\pi\alpha t^2/2] \quad 0 \leqslant t \leqslant T \quad . \tag{81}$$

Neglecting additive noise, the received, complex, low-pass representation $\underline{y}(t)$ is given by Eq. (32) as

$$\underline{y}(t) = \frac{1}{2}\int_\tau \underline{h}(\tau, t)\,\underline{x}(t-\tau)\,d\tau \tag{82}$$

$$= \sqrt{\frac{S}{2}}\int_{\tau=0}^{t} \underline{h}(\tau, t)\exp[j\pi\alpha(t^2 - 2\tau t + \tau^2)]\,d\tau \quad . \tag{83}$$

If we bring the received signal down to an intermediate frequency by multiplying it by another chirp signal of the same α, we have

$$\underline{y}_{IF}(t) = \sqrt{\frac{S}{2}}\int_{\tau=0}^{t} \underline{h}(\tau, t)\exp[j\pi\alpha(-\tau t + \tau^2)]\,d\tau \quad . \tag{84}$$

We shall show that the autocorrelation of $\underline{y}_{IF}(t)$ can be used as an estimate of the two-frequency correlation function $\mathfrak{R}(\Delta f, \Delta t)$.

$$\frac{1}{2}\underline{y}^*_{IF}(t)\,\underline{y}_{IF}(t+\Delta t) = \frac{S}{4}\int_{\tau=0}^{t}\int_{\tau'=0}^{t+\Delta t} d\tau\,d\tau'\,\underline{h}^*(\tau, t)\,\underline{h}(\tau', t+\Delta t)$$

$$\exp\left\{j\pi\alpha\left[2\tau t - \tau^2 - 2\tau'(t+\Delta t) + (\tau')^2\right]\right\} \quad . \tag{85}$$

We now average both sides of Eq. (85) and integrate over τ'. Recall that $\overline{\underline{h}^*(\tau, t)\,\underline{h}(\tau', t+\Delta t)}$ is effectively nonzero only for $|\tau - \tau'| < \epsilon$, where ϵ is the response duration of a single scattering particle. If $\pi\alpha(2t + 2\Delta t - \tau')$ is much less than $1/\epsilon$, then the exponential term in Eq. (85) will remain essentially constant as τ' changes by ϵ, and Eq. (85) can be integrated to give

$$\overline{\frac{1}{2}y^*_{IF}(t)\,y_{IF}(t+\Delta t)} = \frac{S}{2}\int_{\tau=0}^{t} R(\tau, \Delta t)\exp[j\pi\alpha(-2\tau\Delta t)]\,. \tag{86}$$

Applying Eq. (36), and assuming $R(\tau, \Delta t)$ nonzero only for $0 \leqslant \tau \leqslant t$, we get

$$\overline{\frac{1}{2}y^*_{IF}(t)\,y_{IF}(t+\Delta t)} = \frac{S}{2}\mathfrak{R}(\alpha\Delta t, \Delta t) \quad . \tag{87}$$

In order to establish the validity of Eq. (87), we must justify the assumption that $2\pi\alpha(t + \Delta t - \tau'/2) \ll 1/\epsilon$. We can interpret $\alpha(t + \Delta t - \tau'/2)$ as the change in instantaneous frequency between time 0 and time $t + \Delta t - \tau'/2$. Also, since ϵ is the impulse-response duration of a scatterer, $1/\epsilon$ can be crudely interpreted as the over-all bandwidth of the channel. Thus Eq. (87) will be

† Strictly, $\underline{x}(t)$ must persist for all time since its Fourier transform must be 0 for $f < -f_o$. It can be shown, however, that the fraction of energy at frequencies below $-f_o$ in $\underline{x}(t)$, as given by Eq. (81), is less than $2/\pi^2 f_o T$ for $\alpha > 0$, and $2/[\pi^2(f_o + \alpha T)T]$ for $\alpha < 0$.

valid if the frequency sweep of the chirp signal stays well within the over-all channel bandwidth. For chaff or orbital scatter, this restricts us to a sweep of less than 1 kMc which is hardly serious. From Eq. (87) we see that it is reasonable to estimate $\mathfrak{R}(\Delta f, \Delta t)$ as follows:

$$\text{Est } \mathfrak{R}(\alpha\Delta t, \Delta t) = \frac{1}{S(T - L - \Delta t)} \int_{t=L+\Delta t}^{T} y^*_{IF}(t)\, \underline{y}_{IF}(t + \Delta t)\, dt \quad . \tag{88}$$

The limits on the integral in Eq. (88) are simply the limits for which Eq. (87) is valid. We cannot be quite so cavalier in neglecting Δt and L relative to T as we were in the last section since T is limited by $\alpha T \ll 1/\epsilon$.

We see from Eq. (88) that with a chirp signal of given α we can measure $\mathfrak{R}(\Delta f, \Delta t)$ along a straight line of slope α passing through the origin in the $\Delta t, \Delta f$ plane. Thus to measure $\mathfrak{R}(\Delta f, \Delta t)$ everywhere, we must use a variety of different values of α.

In Appendix A we calculate the variance of Est $\mathfrak{R}(\alpha\Delta t, \Delta t)$. As before, this variance is separated into three parts: $\overline{N_C^2}$, the variance due to channel fluctuations; $\overline{N_A^2}$, the variance due to additive noise; and $\overline{N_{CA}^2}$, the variance due to cross terms between the channel and the additive noise. We assume that the noise added to $\underline{y}_{IF}(t)$ is Gaussian and is filtered so that it has an autocorrelation

$$R_{\underline{n}}(\tau) = N_o W_r \exp[-\pi W_r^2 \tau^2] \quad . \tag{89}$$

We assume that the bandwidth of this filter W_r is large enough to pass $y_{IF}(t)$ with negligible distortion. We also assume that T is much larger than the duration in Δt over which $\mathfrak{R}(\alpha\Delta t, \Delta t)$ is nonzero. Subject to these restrictions, the following relationships are derived in Appendix A:

$$\overline{N_C^2} = \frac{1}{T - L - \Delta t} \int_{\Delta t=-\infty}^{\infty} |\mathfrak{R}(\alpha\Delta t, \Delta t)|^2\, d\Delta t \tag{90}$$

$$\overline{N_A^2} = \frac{2\sqrt{2}}{(T - L - \Delta t)\, W_r} \left(\frac{N_o W_r}{S}\right)^2 \tag{91}$$

$$\overline{N_{CA}^2} = \frac{4\,\mathfrak{R}(0,0)}{(T - L - \Delta t)\, W_r} \left(\frac{N_o W_r}{S}\right) \quad . \tag{92}$$

Along with these variance terms, it is possible for the additive noise to change the mean value of Est $\mathfrak{R}(\alpha\Delta t, \Delta t)$ if Δt is not much larger than $1/W_r$. This additional term is real and positive and has the value $2N_o W_r/S \exp[-\pi W_r^2 (\Delta t)^2]$. Since the term can be subtracted from the estimate, we shall ignore it in the following discussion.

E. Comparison of Measurement Techniques

We have discussed two measurement techniques, one measuring $R(\tau, \Delta t)$ by a pseudorandom input, and the other measuring $\mathfrak{R}(\Delta f, \Delta t)$ by a chirp-signal input. In this section we wish to compare the measurements on the basis of the time required to measure the channel. It appears to this author that the chirp-signal method has definite advantages, but a clear and convincing argument is difficult since the techniques measure different quantities, and the expressions for variance are not simple and depend on a number of assumptions.

We can give a reasonably convincing argument, however, by considering one example. Suppose the channel has a two-frequency correlation function given by

$$\mathfrak{R}(\Delta f, \Delta t) = \mathfrak{R}(0,0)\ \exp\{-\pi[(\Delta f)^2/F_c^2 + (\Delta t)^2/T_c^2]\} \quad . \tag{93}$$

Using Eqs. (35) and (53), the tap-gain correlation function and the scattering function are given by

$$R(\tau, \Delta t) = F_c \mathfrak{R}(0,0)\ \exp\{-\pi[F_c^2\tau^2 + (\Delta t)^2/T_c^2]\} \tag{94}$$

$$\sigma(\tau, f) = F_c T_c \mathfrak{R}(0,0)\ \exp\{-\pi[F_c^2\tau^2 + T_c^2 f^2]\} \quad . \tag{95}$$

The quantities F_c and T_c represent the frequency and time coherences of the channel in the sense that about 60 percent of the volume of $\mathfrak{R}(\Delta f, \Delta t)$ is contained within the region where $|\Delta f| \leqslant F_c/2$, $|\Delta t| \leqslant T_c/2$. In a similar sense, $L = 1/F_c$ and $B = 1/T_c$ can be taken as the multipath smear and doppler spread of the channel. We assume that the signal power available for measuring the channel is so limited that N_A^2 is the only important term in the variance of Est $R(\tau, \Delta t)$ or Est $\mathfrak{R}(\Delta f, \Delta t)$.

Consider a measurement of $R(\tau, \Delta t)$ by the pseudo-random input technique in which we attempt to resolve $R(\tau, \Delta t)$ to intervals of L/m in τ, and T_c/m in Δt, where m is a given positive integer. If we assume the channel to be underspread so that $L < T_c$, Eq. (78) is valid for the variance of Est $R(\tau, \Delta t)$. In order to resolve τ to intervals of L/m, we must take W_x to be at least m/L, and in order to resolve Δt to intervals of T_c/m, we must take W_b to be at least m/T_c. Substituting these values into Eq. (78), we have

$$\overline{N_A^2} = \frac{4\sqrt{2}\, m^3 N_o^2}{T L^2 T_c S^2} \quad . \tag{96}$$

If we normalize $\overline{N_A^2}$ by dividing by $R^2(0,0)$, which is proportional to the square of the mean of the estimate, we have, using Eq. (94),

$$\frac{\overline{N_A^2}}{R^2(0,0)} = \frac{4\sqrt{2}\, m^3 N_o^2}{T T_c S^2 \mathfrak{R}^2(0,0)} \tag{97}$$

Suppose we use the chirp-signal technique to measure the same channel. Since we are now measuring $\mathfrak{R}(\Delta f, \Delta t)$, we will attempt to resolve Δf in increments of F_c/m and Δt in increments of T_c/m. Since $y_{IF}(t)$ is filtered before it is autocorrelated to give $\mathfrak{R}(\alpha\Delta t, \Delta t)$, the filter bandwidth W_r must be at least m/T_c in order to resolve Δt in increments of T_c/m. Using $W_r = m/T_c$ in Eq. (91) and neglecting L and Δt relative to T, we get a variance of

$$\overline{N_A^2} = \frac{2\sqrt{2}\, N_o^2 m}{T T_c S^2} \quad . \tag{98}$$

In order to resolve Δf to F_c/m, we must make about 2m measurements with different values of α. If we let $T_t = 2mT$ be the total time for all 2m measurements and normalize Eq. (98) by dividing by $\mathfrak{R}^2(0,0)$, which is proportional to the mean of Est $\mathfrak{R}(\alpha\Delta t, \Delta t)$, we have

$$\frac{\overline{N_A^2}}{\mathfrak{R}^2(0,0)} = \frac{4\sqrt{2}\, N_o^2 m^2}{T_t T_c S^2\, \mathfrak{R}^2(0,0)} \quad . \tag{99}$$

Comparing Eqs. (97) and (99), we see that the normalized variance for the chirp case is less than that for the pseudo-random case by a factor of m. Thus, if the measurement is limited by additive noise, the total time required for a chirp measurement will be m times less than the equivalent pseudo-random noise measurement.

We can rationalize these results in the following way. The assumption that the additive noise is the major limitation in the measurement indicates that the major problem is that of building up a sufficient signal-to-noise ratio at the receiver. Price and Green[16] have shown that this signal-to-noise ratio is proportional to the number of degrees of freedom in the output signal, times the square of the energy-to-noise ratio per degree of freedom when this energy-to-noise ratio is small. To achieve a good signal-to-noise ratio, therefore, one must reduce the number of degrees of freedom in the output signal.

In the comparison just considered, the output bandwidth, and therefore the number of degrees of freedom per second, was the same in both the chirp and pseudo-random cases. However, for each sample measured in the chirp case, we used a measurement interval of $T_t/2m$, whereas for each sample in the pseudo-random case, we used the total interval T_t. Thus the number of degrees of freedom entering into the measurement of each sample in the chirp case was smaller than in the pseudo-random case by a factor of 2m. It is not surprising that the variance in a chirp measurement sample is about $1/m$ of the variance in a pseudo-random measurement sample.

When the additive noise is not the limiting factor in the measurement, there appears to be no simple comparison between chirp and pseudo-random inputs. Qualitatively, we can see that the chirp input has less interference between the measurement of different sample points than the pseudo-random input. On the other hand, it is clear that the chirp input does not minimize the time required to measure the channel. If signal power is not a consideration, one can use several chirp inputs together, for example, to decrease the time required for a measurement.

While certain advantages for chirp inputs over pseudo-random inputs have been demonstrated in the preceding paragraphs, the question of the "optimum" input to use in measuring a channel still remains. It is the author's opinion, after analyzing these two cases in detail and several other types of input in somewhat less detail, that the similarities between the performance of different inputs is much more striking than the differences. It appears that almost any reasonable input signal, if properly processed at the receiver, will produce a measurement whose variance goes down as the reciprocal of the measurement time, and that this result is true for all values of B times L. The major factor influencing the choice of an input signal in a very noisy environment appears to be the desire to measure the minimum number of quantities at a time. The reason for this is not so much to reduce interference between samples as to decrease the number of degrees of freedom entering into each sample measurement.

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APPENDIX A
VARIANCE OF ESTIMATES

I. VARIANCE OF EST $R(\tau, \Delta t)$ FOR PSEUDO-RANDOM INPUT

In this section we shall calculate the variance of Est $R(\tau,\Delta t)$ for the circuit given in Fig. 2 for the case where $W_x \gg W_a \gg W_b$. We shall first calculate this variance in the absence of additive noise, in which case both filters may be neglected. We shall find the mean square value of Est $R(\tau,\Delta t)$. From Eq. (58) we have

$$\text{Est}\,R(\tau,\Delta t) = \frac{\alpha}{32}\iiint dt\,d\tau'\,d\tau''\,[\underline{x}(t-\tau)\,\underline{x}^*(t-\tau')\,\underline{x}^*(t+\Delta t-\tau)$$
$$\times\,\underline{x}(t+\Delta t-\tau'')\,\underline{h}^*(\tau',t)\,\underline{h}(\tau'',t+\Delta t)] \tag{A-1}$$

$$|\text{Est}\,R(\tau,\Delta t)|^2 = \frac{\alpha^2}{2^{10}}\iiiint\!\!\iint dt\,dt'\,d\tau'\,d\tau''\,d\tau'''\,d\tau''''\,[\underline{x}(t-\tau)\,\underline{x}^*(t-\tau')$$
$$\times\,\underline{x}^*(t+\Delta t-\tau)\,\underline{x}(t+\Delta t-\tau'')\,\underline{x}^*(t'-\tau)\,\underline{x}(t'-\tau''')$$
$$\times\,\underline{x}(t'+\Delta t-\tau)\,\underline{x}^*(t'+\Delta t-\tau'''')\,\underline{h}^*(\tau',t)\,\underline{h}(\tau'',t+\Delta t)$$
$$\times\,\underline{h}(\tau''',t')\,\underline{h}^*(\tau'''',t'+\Delta t)]\quad . \tag{A-2}$$

We wish to find the average value of Eq. (A-2). We assume that: the input is statistically independent of $h(\tau,t)$; the input satisfies Eqs. (60) and (61); and Δt is large enough so that $R_{\underline{x}}(\Delta t) \approx 0$. We further assume that $R(\tau,\Delta t)$ is essentially constant in both τ and Δt over the interval in which $R_{\underline{x}}(\tau)$ is nonzero, and that $R_{\underline{x}}(\tau)$ is essentially constant over the interval ϵ so that we can approximate the fourth moment of $\underline{h}$ by (see Appendix B)

$$\overline{\underline{h}^*(\tau',t)\,\underline{h}(\tau'',t+\Delta t)\,\underline{h}(\tau''',t')\,\underline{h}^*(\tau'''',t'+\Delta t)}$$
$$= 4R(\tau',\Delta t)\,R^*(\tau''',\Delta t)\,\delta(\tau''-\tau')\,\delta(\tau''''-\tau''')$$
$$+\,4R(\tau',t'-t)\,R^*(\tau'',t'-t)\,\delta(\tau'''-\tau')\,\delta(\tau''''-\tau'')$$
$$+\,4\,\frac{KL}{n}\,|R(\tau',t'-t)|^2\,\delta(\tau''-\tau')\,\delta(\tau'''-\tau')\,\delta(\tau''''-\tau') \tag{A-3}$$

where δ is a unit impulse, L is the multipath smear, n is the number of scatterers, and K is a factor of the order of magnitude of 1.

We now average Eq. (A-2) over both the channel fluctuations and the input. After substituting Eq. (A-3) into Eq. (A-2) and integrating out the delta functions, we are left with

$$\overline{|\text{Est}\,R(\tau,\Delta t)|^2} = \frac{\alpha^2}{2^8}\iiiint dt\,dt'\,d\tau'\,d\tau'''\,[R(\tau',\Delta t)\,R^*(\tau''',\Delta t)\,\overline{\underline{x}(t-\tau)}$$
$$\overline{\times\,\underline{x}^*(t-\tau')\,\underline{x}^*(t+\Delta t-\tau)\,\underline{x}(t+\Delta t-\tau')\,\underline{x}^*(t'-\tau)\,\underline{x}(t'-\tau''')}$$
$$\overline{\times\,\underline{x}(t'+\Delta t-\tau)\,\underline{x}^*(t'+\Delta t-\tau''')}] + \frac{\alpha^2}{2^8}\iiiint dt\,dt'\,d\tau'\,d\tau''$$
$$\times\,[R(\tau',t'-t)\,R^*(\tau'',t'-t)\,\overline{\underline{x}(t-\tau)\,\underline{x}^*(t-\tau')\,\underline{x}^*(t+\Delta t-\tau)}$$
$$\overline{\times\,\underline{x}(t+\Delta t-\tau'')\,\underline{x}^*(t'-\tau)\,\underline{x}(t'-\tau')\,\underline{x}(t'+\Delta t-\tau)\,\underline{x}^*(t'+\Delta t-\tau'')}]$$

$$+ \frac{\alpha^2}{2^8} \frac{KL}{n} \iiint dt\,dt'\,d\tau' \; [|R(\tau', t'-t)|^2 \; \overline{\underline{x}(t-\tau)\,\underline{x}^*(t-\tau')}$$

$$\overline{\times\, \underline{x}^*(t+\Delta t-\tau)\,\underline{x}(t+\Delta t-\tau')\,\underline{x}^*(t'-\tau)\,\underline{x}(t'-\tau')}$$

$$\overline{\times\, \underline{x}(t'+\Delta t-\tau)\,\underline{x}^*(t'+\Delta t-\tau')}] \quad . \qquad \text{(A-4)}$$

Each of the eighth-order moments in $\underline{x}$ in Eq. (A-4) can now be broken up into a sum of products of second-order moments since $\underline{x}$ is a Gaussian random process. When we do this, using Eqs. (60) and (61) and the assumption that $R_{\underline{x}}(\tau) \approx 0$ for $|\tau| > |\Delta t|$, we find that the first term breaks into eight, the second into fourteen, and the third into eight terms. It may be illuminating to write explicitly the first two of the thirty terms. We derive the first by coupling terms 1 + 2, 3 + 4, 5 + 6, and 7 + 8 in the first eighth-order moment; and the second from coupling 1 + 2, 3 + 6, 4 + 5, and 7 + 8. This gives us

$$\overline{\overline{|\mathrm{Est}\, R(\tau, \Delta t)|^2}} = \frac{\alpha^2}{16} \iiiint dt\,dt'\,d\tau'\,d\tau''' \; [R(\tau', \Delta t)\, R^*(\tau''', \Delta t)\, R_{\underline{x}}^2(\tau - \tau')\, R_{\underline{x}}^2(\tau - \tau''')$$

$$+ R(\tau', \Delta t)\, R^*(\tau''', \Delta t)\, R_{\underline{x}}(\tau - \tau')\, R_{\underline{x}}(t + \Delta t - \tau - t' + \tau''')$$

$$\times R_{\underline{x}}(t + \Delta t - \tau' - t' + \tau''')\, R_{\underline{x}}(\tau - \tau''')]$$

$$+ \text{28 other terms} \quad . \qquad \text{(A-5)}$$

Equation (A-5), if we had written out all the terms, would not provide a very illuminating solution. However, if we assume a particularly simple form for $R_{\underline{x}}(\tau)$, these integrals can all be evaluated. We therefore assume

$$R_{\underline{x}}(\tau) = S \exp[-\pi W_x^2 \tau^2] \quad . \qquad \text{(A-6)}$$

Thus $\underline{x}(t)$ is a low-pass representation of a Gaussian random process of power S with a Gaussian shaped autocorrelation, a Gaussian shaped spectrum, and a nominal bandwidth W_x. We assume that W_x is much larger than $1/\Delta t$ and much larger than the reciprocal of the time over which $R(\tau, \Delta t)$ changes appreciably. We also assume that T, the duration of the input, is so large that end effects on the integrals can be neglected.

Integrating Eq. (A-5), we have

$$\overline{\overline{|\mathrm{Est}\, R(\tau, \Delta t)|^2}} = \frac{\alpha^2}{16} \left[\frac{S^4 T^2}{2W_x^2} |R(\tau, \Delta t)|^2 + \frac{S^4 T}{2W_x^3} |R(\tau, \Delta t)|^2\right]$$

$$+ \text{28 other terms} \quad . \qquad \text{(A-7)}$$

Substituting Eq. (A-6) into (64), and again assuming that Δt is negligible with respect to T,

$$\alpha = \frac{4\sqrt{2}\, W_x}{TS^2} \quad . \qquad \text{(A-8)}$$

Substituting Eq. (A-8) into (A-7),

$$\overline{|\mathrm{Est}\,R(\tau,\Delta t)|^2} = |R(\tau,\Delta t)|^2 + \frac{1}{TW_x}\,|R(\tau,\Delta t)|^2$$

$$+ \text{ 28 other terms } . \qquad \text{(A-9)}$$

We now note that the first term in Eq. (A-9) is the square of the mean of Est $R(\tau, \Delta t)$. Thus the remaining terms give the variance of Est $R(\tau, \Delta t)$. Since $|R(\tau, \Delta t)|^2$ is so much larger than any of the other terms, we might be concerned about the approximations used in evaluating it. This is no problem, however, since the term in Eq. (A-5), from which $|R(\tau, \Delta t)|^2$ came, is precisely the square of $\overline{\mathrm{Est}\,R(\tau, \Delta t)}$ as given in Eq. (62). and thus the other terms in Eq. (A-5) do constitute the variance. We call this variance $\overline{N_C^2}$ since it is due to channel and input variations and does not involve the additive noise. Evaluating and adding together the remaining twenty-nine terms, we have

$$\overline{N_C^2} = \frac{1}{TW_x}\,[6\,|R(\tau,\Delta t)|^2 + |R(\tau-\Delta t,\Delta t)|^2 + |R(\tau+\Delta t,\Delta t)|^2 + 2R(\tau,0)$$

$$\times R(\tau-\Delta t,0) + 2R(\tau,0)\,R(\tau+\Delta t,0) + \tfrac{1}{2}\,R(\tau-\Delta t,0)\,R(\tau+\Delta t,0)]$$

$$+ \frac{1}{T}\left\{\int_{\tau'} |R(\tau',\Delta t)|^2\,d\tau' + \int_t |R(\tau,t)|^2\,dt + \int R(\tau',0)\,R(\tau'+\Delta t,0)\,d\tau'\right.$$

$$\left. + [2R(\tau,0) + \frac{1}{\sqrt{2}}\,R(\tau-\Delta t,0) + \frac{1}{\sqrt{2}}\,R(\tau+\Delta t,0)]\int_{\tau'} R(\tau',0)\,d\tau'\right\}$$

$$+ \frac{W_x}{T}\left[\int R(\tau',0)\,d\tau'\right]^2 + \frac{1}{T}\left(\frac{W_xKL}{n}\right)\left[\int |R(\tau,t)|^2\,dt + \int |R(\tau',0)|^2\,d\tau'\right]$$

$$+ \frac{1}{W_xT}\left(\frac{W_xKL}{n}\right)\left[2\,|R(\tau,0)|^2 + \frac{1}{\sqrt{2}}\,|R(\tau,\Delta t)|^2 + \frac{1}{\sqrt{2}}\,|R(\tau-\Delta t)|^2\right.$$

$$\left. + \frac{1}{\sqrt{2}}\,|R(\tau+\Delta t,0)|^2 + \frac{1}{\sqrt{2}}\,|R(\tau-\Delta t,0)|^2\right] \quad . \qquad \text{(A-10)}$$

Equation (A-10) is still too complicated to be of much use without some further analysis. Those terms involving W_xKL/n come from the third part of Eq. (A-3) and can be neglected relative to the other terms if $(W_xKL/n) \ll 1$. If we interpret $1/W_x$ as the desired resolution in τ of $R(\tau, \Delta t)$, corresponding to the assumption that $1/W_x$ is less than the interval over which $R(\tau, \Delta t)$ changes, then n/W_xL is the average number of scatterers in a resolution interval of path delays. Therefore, the term involving W_xKL/n can be neglected if there are many scatterers in each resolution interval. If these terms cannot be neglected, then our results can only be considered as order of magnitude results due to the factor K. We shall refer to these terms as shot noise because, if they are important, the impulse response of the channel, filtered to bandwidth W_x, will look like shot noise.

For the other terms in Eq. (A-10), we shall first assume a particular form for R; namely,

$$R(\tau,\Delta t) = R(0,0) \begin{cases} -\frac{L}{2} \leqslant \tau \leqslant \frac{L}{2} \\ -\frac{T_c}{2} \leqslant \Delta t \leqslant \frac{T_c}{2} \end{cases}$$

$$= 0 \qquad \text{otherwise} \quad .$$

Then for τ and Δt both close to zero, Eq. (A-10) reduces to

$$\overline{N_C^2} \approx \frac{R^2(0,0)}{T} \left(\frac{12.5}{W_x} + T_c + 5.4L + W_x L^2\right) \quad . \tag{A-11}$$

Our assumptions already imply that $L >> (1/W_x)$; thus only the second and fourth terms are important, and finding the terms in Eq. (A-10) from which these come, we have

$$\overline{N_C^2} \approx \frac{1}{T}\int_t |R(\tau,t)|^2\, dt + \frac{W_x}{T}\left[\int_{\tau'} R(\tau', 0)\, d\tau'\right]^2 \quad . \tag{A-12}$$

Next we consider the effect of additive noise on the variance of Est $R(\tau, \Delta t)$. We have been using $\underline{y}(t)$ as the channel output in the absence of noise, so we let $\underline{y}(t) + \underline{n}(t)$ be the actual channel output.

$$\text{Est } R(\tau, \Delta t) = \frac{\alpha}{8}\int_{t=0}^{T} dt\; \underline{x}(t-\tau)\,[\underline{y}^*(t) + \underline{n}^*(t)]\; \underline{x}^*(t+\Delta t-\tau)\,[\underline{y}(t+\Delta t) + \underline{n}(t+\Delta t)] \quad . \tag{A-13}$$

When we square the magnitude of Eq. (A-13) and expand the sums, we find a term involving $\underline{y}$ to the fourth order, a term involving $\underline{n}$ to the fourth order, and various cross terms. In the preceding section we found the mean of the term involving $\underline{y}$ to the fourth order, and in this section we shall find the mean of the term involving $\underline{n}$ to the fourth order, $\overline{N_A^2}$.

$$\overline{N_A^2} = \frac{\alpha^2}{64}\int_t\int_{t'} dt\, dt'\; \overline{[\underline{x}(t-\tau)\, \underline{x}^*(t+\Delta t-\tau)\, \underline{x}^*(t'-\tau)\, \underline{x}(t'+\Delta t-\tau)}$$

$$\overline{\times\, \underline{n}^*(t)\, \underline{n}(t+\Delta t)\, \underline{n}(t')\, \underline{n}(t'+\Delta t)]} \quad . \tag{A-14}$$

We recall that the received signal is passed through the filter a(t). We have assumed that this does not affect $\underline{y}(t)$ since $W_a >> W_x$. On the other hand, the noise will be affected. We assume that the noise before filtering is white Gaussian noise of spectral density $N_o/2$, corresponding to N_o per unit positive bandwidth. After passing through the filter a(t) with a low-pass equivalent impulse response, $\underline{a}(t) = 2\sqrt{2}\, W_a \exp[-\pi 2 W_a^2 t^2]$, it follows after some calculation that the low-pass equivalent autocorrelation of the filtered noise is given by

$$R_{\underline{n}}(\tau) = \frac{1}{2}\, \overline{\underline{n}^*(t)\, \underline{n}(t+\tau)} = N_o W_a \exp[-\pi W_a^2 \tau^2] \quad . \tag{A-15}$$

Assuming the same input process as before, we can write Eq. (A-14) as

$$\overline{N_A^2} = \frac{\alpha^2}{4}\int_o^T\int_o^T dt\, dt'\; [R_{\underline{x}}^2(\Delta t) + R_{\underline{x}}^2(t-t')]\, [R_{\underline{n}}^2(\Delta t) + R_{\underline{n}}^2(t-t')] \quad . \tag{A-16}$$

Assuming, as before, that $R_{\underline{x}}(\Delta t) = R_{\underline{n}}(\Delta t) = 0$, we can substitute Eqs. (60) and (A-15) into Eq. (A-16), and integrate to arrive at

$$\overline{N_A^2} = \frac{\alpha^2 S^2 N_o^2 W_a^2 T}{4\sqrt{2(W_x^2 + W_a^2)}} \quad . \tag{A-17}$$

Substituting Eq. (A-8) for α, this becomes

$$\overline{N_A^2} = \frac{4\sqrt{2}\, W_x}{T\sqrt{1 + \left(\frac{W_a}{W_x}\right)^2}} \left(\frac{W_a N_o}{S}\right)^2 \quad . \tag{A-18}$$

We observe that $N_o W_a$ is the received noise power N_r. The received signal power is given by $S_r = \frac{1}{2} S\Re(0, 0)$, or

$$S_r = \frac{1}{2} S \int_{\tau'=-\infty}^{\infty} R(\tau', 0)\, d\tau' \quad . \tag{A-19}$$

Substituting this into Eq. (A-18), we have

$$\overline{N_A^2} = \frac{\sqrt{2}\, W_x}{T\sqrt{1 + \left(\frac{W_a}{W_x}\right)^2}} \left(\frac{N_r}{S_r}\right)^2 \left[\int_{\tau'} R(\tau', 0)\, d\tau'\right]^2 \quad . \tag{A-20}$$

Comparing Eq. (A-20) with the second term of Eq. (A-12), we see that the additive noise is important when N_r/S_r is less than 1.

We can now return to Eq. (A-13) and compute the variance due to the cross terms between noise and channel and input. The calculation is similar to those already given, and the answer is

$$\overline{N_{CA}^2} = 4\, \frac{W_x W_a N_o}{TS} \int R(\tau', 0)\, d\tau' \left[\frac{1}{\sqrt{\frac{3}{4} + \frac{1}{4}\left(\frac{W_a}{W_x}\right)^2}} + \text{negligible terms}\right] \quad . \tag{A-21}$$

It can easily be shown that $\overline{N_{CA}^2} \leqslant \overline{N_A^2} + \overline{N_C^2}$; consequently, this term is not of great importance.

II. VARIANCE OF EST $\Re(\alpha\Delta t, \Delta t)$ FOR CHIRP INPUT

In this section we find the variance of Est $\Re(\alpha\Delta t, \Delta t)$ as given by Eq. (88). The analysis is similar to that in the last section but somewhat simpler. First, we find $\overline{|\text{Est}\, \Re(\alpha\Delta t, \Delta t)|^2}$ in the absence of additive noise.

$$\text{Est}\, \Re(\alpha\Delta t, \Delta t) = \frac{1}{S(T - L - \Delta t)} \int_{t=L+\Delta t}^{T} \underline{y}_{IF}^*(t)\, \underline{y}_{IF}(t + \Delta t)\, dt \quad . \tag{A-22}$$

From Eq. (85), this is

$$\text{Est}\, \Re(\alpha\Delta t, \Delta t) = \frac{1}{2(T - L - \Delta t)} \int_{t=L+\Delta t}^{T} \int_{\tau=0}^{t} \int_{\tau'=0}^{t+\Delta t} dt\, d\tau\, d\tau'\, \underline{h}^*(\tau, t)\, \underline{h}(\tau', t + \Delta t)$$

$$\times \exp\{j\pi\alpha\, [2\tau t - \tau^2 - 2\tau'(t + \Delta t) + (\tau')^2]\} \quad . \tag{A-23}$$

Squaring Eq. (A-23) in magnitude, we have

$$\left|\text{Est } \mathfrak{R}(\alpha\Delta t, \Delta t)\right|^2 = \frac{1}{4(T - L - \Delta t)^2} \int_{t=L+\Delta t}^{T} dt \int_{t'=L+\Delta t}^{T} dt' \int_{\tau=0}^{t} d\tau \int_{\tau'=0}^{t+\Delta t} d\tau'$$

$$\times \int_{\tau''=0}^{t} d\tau'' \int_{\tau'''=0}^{t+\Delta t} d\tau''' \; \underline{h}^*(\tau, t)\, \underline{h}(\tau', t + \Delta t)\, \underline{h}(\tau'', t')\, \underline{h}^*(\tau''', t' + \Delta t)$$

$$\times \exp\{j\pi\alpha\, [2\tau t - \tau^2 - 2\tau'(t + \Delta t) + (\tau')^2 - 2\tau'' t'$$

$$+ (\tau'')^2 + 2\tau'''(t' + \Delta t) - (\tau''')^2]\} \quad . \tag{A-24}$$

Averaging this over $\underline{h}$, using Eq. (A-3) for the fourth moment of $\underline{h}$, and integrating over the δ functions,

$$\overline{\left|\text{Est } \mathfrak{R}(\alpha\Delta t, \Delta t)\right|^2} = \frac{1}{(T - L - \Delta t)^2} \iiiint dt\, dt'\, d\tau\, d\tau''\; R(\tau, \Delta t)\, R^*(\tau'', \Delta t)$$

$$\times \exp[j\pi\alpha(-2\tau\Delta t + 2\tau'''\Delta t)]$$

$$+ \frac{1}{(T - L - \Delta t)^2} \iiiint dt\, dt'\, d\tau\, d\tau'\; R(\tau, t' - t)\, R^*(\tau', t' - t)$$

$$\times \exp\{j\pi\alpha\, [-2(\tau - \tau')\, (t' - t]\}$$

$$+ \frac{KL}{n(T - L - \Delta t)^2} \iiint dt\, dt'\, d\tau\, \left|R(\tau, t' - t)\right|^2 \quad . \tag{A-25}$$

The first term in Eq. (A-25) is the magnitude squared of $\overline{\text{Est } \mathfrak{R}(\alpha\Delta t, \Delta t)}$, and therefore, the variance of the estimate which we call $\overline{N_C^2}$ is given by the second and third terms. Using Eq. (36) we can integrate the second term

$$\overline{N_C^2} = \frac{1}{(T - L - \Delta t)^2} \int_{t=L+\Delta t}^{T} \int_{t'=L+\Delta t}^{T} dt\, dt' \left\{ \left|\mathfrak{R}\; [\alpha(t' - t),\, t' - t]\right|^2 \right.$$

$$\left. + \frac{KL}{n} \int_{\tau} \left|R(\tau, t' - t)\right|^2 d\tau \right\} \quad . \tag{A-26}$$

The second term in Eq. (A-26) can be neglected relative to the first term if the number of scattering particles is large. If we then substitute Δt for $t' - t$, and integrate over t (neglecting end effects), we get

$$\overline{N_C^2} \approx \frac{1}{T - L - \Delta t} \int_{\Delta t} \left|\mathfrak{R}\,(\alpha\Delta t, \Delta t)\right|^2 d\Delta t \quad . \tag{A-27}$$

Next, we consider the effect of additive noise on the variance of Est $\mathfrak{R}(\alpha\Delta t, \Delta t)$. Letting $\underline{y}_{IF}(t) + \underline{n}(t)$ be the complex representation of the noisy received IF waveform, we have

$$\text{Est } \mathfrak{R}(\alpha\Delta t, \Delta t) = \frac{1}{S(T - L - \Delta t)} \int_{t=L+\Delta t}^{T} [\underline{y}_{IF}^*(t) + \underline{n}^*(t)]\, [\underline{y}_{IF}(t + \Delta t) + \underline{n}(t + \Delta t)]\, dt \quad . \tag{A-28}$$

We assume as before that $\underline{n}(t)$ represents filtered Gaussian noise and that its autocorrelation $R_{\underline{n}}(\tau)$ is

$$R_{\underline{n}}(\tau) = \frac{1}{2} \underline{n}^*(t)\, \underline{n}(t+\tau) = N_o W_r \exp[-\pi W_r^2 \tau^2] \quad . \tag{A-29}$$

We shall determine an appropriate value for the bandwidth W_r of the filter later.

When we square the magnitude of Eq. (A-28) and expand the sums, we find a term involving $\underline{y}$ to the fourth order, a term involving $\underline{n}$ to the fourth order, and various cross terms. In the last section we found the mean of the term involving $\underline{y}$ to the fourth order. Here we find the mean of the term involving $\underline{n}$ to the fourth order $\overline{N_A^2}$. The mean of the cross term $\overline{N_{CA}^2}$ is found the same way, and only the result will be given.

$$\overline{N_A^2} = \frac{1}{S^2(T-L-\Delta t)^2} \int_{t=L+\Delta t}^{T} dt \int_{t'=L+\Delta t}^{T} dt' \; \overline{\underline{n}^*(t)\, \underline{n}(t+\Delta t)\, \underline{n}(t')\, \underline{n}^*(t'+\Delta t)} \quad . \tag{A-30}$$

Breaking up the fourth moment of $\underline{n}$ in second moments,

$$\begin{aligned} \overline{N_A^2} &= \frac{4}{S^2(T-L-\Delta t)^2} \iint dt\, dt' \; [R_n^2(\Delta t) + R_{\underline{n}}^2(t'-t)] \\ &= \frac{4(N_o W_r)^2}{S^2} \exp[-\pi W_r^2 (\Delta t)^2] + \frac{2\sqrt{2}(N_o W_r)^2}{S^2 W_r (T-L-\Delta t)} \quad . \end{aligned} \tag{A-31}$$

The first term in Eq. (A-31) is not part of the variance of Est $\Re(\alpha\Delta t, \Delta t)$ but is simply part of the square of the mean as can be seen from Eq. (A-28). It can be neglected for $\Delta t >> 1/W_r$. The analogous term in Eq. (A-16) for pseudo-random inputs was completely neglected. The second term in Eq. (A-31) is the actual variance due to additive noise.

$$\overline{N_A^2} = \frac{2\sqrt{2}}{(T-L-\Delta t)\, W_r} \left(\frac{N_o W_r}{S}\right)^2 \quad . \tag{A-32}$$

A similar analysis of the variance due to cross terms between noise and channel gives us

$$\overline{N_{CA}^2} = \frac{4\,\Re(0,0)}{(T-L-\Delta t)\, W_r} \cdot \frac{N_o W_r}{S} \quad . \tag{A-33}$$

If we rewrite both these equations using $N_r = W_r N_o$ and $S_r = \frac{1}{2} S \Re(0,0)$, we have

$$\overline{N_A^2} = \frac{\Re^2(0,0)}{\sqrt{2}(T-L-\Delta t)\, W_r} \left(\frac{N_r}{S_r}\right)^2 \tag{A-34}$$

$$\overline{N_{CA}^2} = \frac{2\,\Re^2(0,0)}{(T-L-\Delta t)\, W_r} \cdot \left(\frac{N_r}{S_r}\right) \quad . \tag{A-35}$$

From Eq. (A-32), we see that W_r should be made as small as possible to reduce the additive-noise interference. On the other hand, W_r must be large enough to pass $\underline{y}_{IF}(t)$ with negligible change. The power spectrum of $\underline{y}_{IF}(t)$ is the Fourier transform of the autocorrelation of $\underline{y}_{IF}(t)$ which was shown to be $\Re(\alpha\Delta t, \Delta t)$. Thus if α is small, most of the power of $\underline{y}_{IF}(t)$ will be in a bandwidth B, the doppler spread of the channel. If α is large, most of the power will be in a bandwidth αL. Thus W_r should be taken somewhat larger than both B and αL.

APPENDIX B
FOURTH MOMENT OF CHANNEL IMPULSE RESPONSE

In this section we shall analyze the fourth moment:

$$\underline{h}_4 = \overline{\underline{h}^*(\tau', t)\, \underline{h}(\tau'', t + \Delta t)\, \underline{h}(\tau''', t')\, \underline{h}^*(\tau'''', t + \Delta t)} \tag{B-1}$$

of a channel made up of a large number n of scatterers. We assume that each scatterer has an impulse response $\underline{h}_i(\tau, t)$ limited to a time duration ϵ. Assume for simplicity that the impulse responses are square waves of duration ϵ starting at times independently and uniformly distributed between 0 and L. The amplitude of each square wave ξ is a zero-mean, complex, random variable with identically distributed, independent, real and imaginary parts.

We can then rewrite Eq. (B-1) as

$$\underline{h}_4 = \sum_{i, j, k, \ell = 1}^{n} \overline{\underline{h}_i^*(\tau', t)\, \underline{h}_j(\tau'', t + \Delta t)\, \underline{h}_k(\tau''', t')\, \underline{h}_\ell^*(\tau'''', t' + \Delta t)} \quad . \tag{B-2}$$

Due to the independence of the scatterers and the zero-mean amplitudes, only the terms in Eq. (B-2), with all four indices the same or with pairs of indices the same, will be nonzero.

$$\begin{aligned} \underline{h}_4 = {} & \sum_{i=1}^{n} \overline{\underline{h}_i^*(\tau', t)\, \underline{h}_i(\tau'', t + \Delta t)\, \underline{h}_i(\tau''', t')\, \underline{h}_i^*(\tau'''', t' + \Delta t)} \\ & + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{\underline{h}_i^*(\tau', t)\, \underline{h}_i(\tau'', t + \Delta t)\, \underline{h}_j(\tau''', t')\, \underline{h}_j^*(\tau'''', t' + \Delta t)} \\ & + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{\underline{h}_i^*(\tau', t)\, \underline{h}_j(\tau'', t + \Delta t)\, \underline{h}_i(\tau''', t')\, \underline{h}_j^*(\tau'''', t' + \Delta t)} \\ & + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{\underline{h}_i^*(\tau', t)\, \underline{h}_j(\tau'', t + \Delta t)\, \underline{h}_j(\tau''', t')\, \underline{h}_i^*(\tau'''', t' + \Delta t)} \quad . \end{aligned} \tag{B-3}$$

First assume for simplicity that $t' = t$ and $\Delta t = 0$. The first term in Eq. (B-3) is nonzero only when τ', τ'', τ''', and τ'''' are all within ϵ of each other. If the equivalent low-pass input to the channel is essentially constant over a period ϵ, as we have assumed throughout this report, we can represent this term as a function of τ' with unit impulse functions for τ'', τ''', and τ''''. The probability that a particular scatterer will be giving a return at time τ' is ϵ/L. Given a return at τ', we can integrate over τ'', τ''', and τ'''', getting the first term in Eq. (B-3) to be $\sum_{k=1}^{n} \epsilon/L \cdot \epsilon^3 \overline{\underline{\xi}^4}\, \delta(\tau'' - \tau')\, \delta(\tau''' - \tau')\, \delta(\tau'''' - \tau')$. The second, third, and fourth terms can be handled similarly to give, for $t' = t$ and $\Delta t = 0$,

$$\underline{h}_4 = \frac{n\epsilon^4}{L}\overline{\underline{\xi}^4}\,\delta(\tau'' - \tau')\,\delta(\tau''' - \tau')\,\delta(\tau'''' - \tau') + \frac{n^2\epsilon^4}{L^2}\overline{\underline{\xi}^2}^2$$

$$\times [\delta(\tau'' - \tau)\,\delta(\tau'''' - \tau''') + \delta(\tau''' - \tau')\,\delta(\tau'''' - \tau'')] \quad . \tag{B-4}$$

Using the same argument for $R(\tau, \Delta t)$, we find that for $\Delta t = 0$,

$$R(\tau, 0) = \frac{n\epsilon^2}{2L}\overline{\underline{\xi}^2} \quad . \tag{B-5}$$

Now we define:

$$K = \frac{\overline{\underline{\xi}^4}}{\overline{\underline{\xi}^2}^2} \quad . \tag{B-6}$$

For any reasonable type of scatterer, K will have the order of magnitude of 1. Substituting Eqs. (B-6) and B-5) into Eq. (B-4), we have

$$h_4 = \frac{4KL}{n} R^2(\tau', 0)\,\delta(\tau'' - \tau')\,\delta(\tau''' - \tau')\,\delta(\tau'''' - \tau') + 4R(\tau', 0)\,R^*(\tau''', 0)$$

$$\times\,\delta(\tau'' - \tau')\,\delta(\tau'''' - \tau''') + 4R(\tau', 0)\,R^*(\tau'', 0)\,\delta(\tau''' - \tau')\,\delta(\tau'''' - \tau'') \quad . \tag{B-7}$$

If $\Delta t \neq 0$ and $t' \neq t$, we could still use the same argument. The only difference is that the movement of the particles will spread out the difference in times τ', τ'', τ''', and τ'''' over which the first term of Eq. (B-3) is zero. We assume that this time spread is small enough so that Eq. (B-4) can still be used. Finally, if the impulse response is arbitrary, the argument is still valid, except that the factor K defined in Eq. (B-6) will be more difficult to estimate.