Wait darn this actually has a decent solution.

Let $A = (x_1^2+1)\cdots(x_4^2+1)$. We claim that $A \geq 16$.

This bound is sharp when $x_1 = x_2 = x_3 = x_4 = 1$, so 16

is the minimum.

Compute $A = \cancel{(i+x_1)(i+x_2)(i+x_3)}(i+x_4)$

$\cancel{(-i+1}$

$$A = \prod_{k=1}^{4} (i-x_k)(-i-x_k) \cdot \cancel{(-1)} \quad (\text{here } i^2=-1)$$

$$= \cancel{(-1)}^4 P(i) \cdot P(-i)$$

$$= [1-b+d+i[c-a]][1-b+d-i[c-a]]$$

$$= \cancel{(b-d}$$

$$= (b-d-1)^2 + (a-c)^2$$

$$\geq (5-1)^2 + 0^2$$

$$= 16.$$

Answer: $f \equiv 0$, $f(x) \equiv x^2$, ~~and $f(x) = \begin{cases} 1 & \text{if } x = \pm 1 \\ 0 & \text{otherwise} \end{cases}$~~

It's easy to check $f \equiv 0$ works, and for $f(x) \equiv x^2$,

$$x(2y^2 - x)^2 + y^2(2x - y^2)^2 = x^3 + y^6$$
$$= \frac{(x^2)^2}{x} + (y \cdot y^2)^2$$

(where $x \neq 0$)

So $f(x) = x^2$ works too. ~~For the last function, see the end of the solution, page 4.~~

Now let's show these are all. Put $y = 0$ to obtain

$$x f(2f(0) - x) = \frac{f(x)^2}{x} + f(0). \quad (*)$$

Now we claim $\boxed{f(0) = 0}$. If not, pick a prime $p \nmid f(0)$ and choose $x = p \neq 0$. Then $\frac{f(x)^2}{x}$ is an integer, so $p \mid f(p)^2 \Rightarrow p \mid f(p) \Rightarrow p \mid \frac{f(p)^2}{p}$ and hence we obtain (oh also $p \mid pf(2f(0) - p)$) that $p \mid f(0)$, contradiction.

Hence $f(0) = 0$. So $(*)$ rewrites as.

$$x f(-x) = \frac{f(x)^2}{x} \quad \forall x \neq 0.$$

or

$$x^2 f(-x) = f(x)^2$$

which actually also holds at $x = 0$.

Next we show $\boxed{f \text{ is even.}}$ For $x \neq 0$, note

$$f(x)^2 = x^2 f(-x)$$
$$f(-x)^2 = x^2 f(x)$$
$$\Rightarrow \quad [f(x) - f(-x)][f(x) + f(-x) + x^2] = 0.$$

In the unlikely event that $f(x) + f(-x) = -x^2$, we find $f(x)^2 = x^2[-x^2 - f(x)]$

$$\Rightarrow \quad [f(x) + \tfrac{1}{2}x^2]^2 = -\tfrac{3}{4}x^4 < 0$$

which is absurd. (since $x \neq 0$)

So in fact, $\boxed{f(x)^2 = x^2 f(x).}$ The given is thus

$$x f(2f(y) - x) + y^2 f(2x - f(y))$$
$$= x f(x) + f(y f(y)) \quad \forall x \neq 0$$

Wait ~~also~~ I'm being dumb,* the boxed thing above is

$$f(x) \cdot [f(x) - x^2] = 0.$$

LOL. So $f(x) = 0$ or $f(x) = x^2$ for any particular x.

*Not that I'm not usually being dumb.

Now suppose there exists $t \in \mathbb{Z} - \{0\}$ such that $f(t) = 0$. We show f is zero everywhere.

Put $y = t$ in the given to obtain

$$t^2 \cdot f(2\cancel{x}) = 0 \quad \forall x \neq 0$$

Adding on $f(0) = 0$, $\Rightarrow$ $\boxed{f(2k) = 0 \text{ for all } k.}$

Now, select $x = 2k \neq 0$ in the given.. We find

$$2k \cdot f(2[f(y) - k]) + y^2 f(4k - f(y))$$
$$= \frac{f(2k)^2}{2k} + f(yf(y))$$
$$\Rightarrow \quad y^2 \cdot f(4k - f(y)) = f(yf(y)).$$

Assume for contradiction that $f(y) = y^2$ now. (Here $y \neq 0$).
Evidently
$$y^2 \cdot f(4k - y^2) = f(y^3)..$$

If $f(y^3) \neq 0$ then $f(4k - y^2) \neq 0$, in which case we get $y^4 = (4k - y^2)^2$ for all $k \neq 0$, impossible. Hence $f(4k - y^2) = f(y^2 - 4k) = f(y^3) = 0$ for all $k \neq 0$.

Since y is odd, $y^2 \equiv 1 \pmod 4$, and so

$f(n) = 0$ for all n other than $\pm y^2$. ~~In particular,~~

~~However f is even, so now $f(n)=0$ for all n except possibly~~ $f(-1)$ and $f(1)$.

But unless $y=\pm 1$, we can now take $n=y$, contradiction

Thus we either have $f(x)\equiv 0$ or $f(x)=\begin{cases}1 & x=\pm 1\\ 0 & \text{otherwise}\end{cases}$

We now check the last function. FAILS:

- If $x=\pm 1$ and $y=\pm 1$:

$$xf(2-x)+f(2x-1)\overset{?}{=}xf(x)+1$$

This holds for both $x=1$ and $x=-1$.

- If $x=\pm 1$ but $y\neq\pm 1 \Rightarrow f(y)=0$.

$$xf(x)+\underbrace{y^2f(2x)}_{=0}\overset{?}{=}xf(x). \checkmark$$

- If $x\neq\pm 1$, but $y=\pm 1$:

$$xf(2-x)+f(2x-1)=1$$

But this is wrong at $x=5$. LOL!
Darn I was pretty scared for a while.

Thus $f(x)\equiv 0$, $f(x)\equiv x^2$ are the only solutions to this functional equation. ∎

Let X, Y, Z denote Lincoln, NE, the North Pole, and any vertex of the Bermuda Triangle. (Those are noncollinear since they lie on the spherical Earth.) We use barycentric coordinates with respect to XYZ.* For every $n \in \mathbb{Z}$ let us select the point $\boxed{P_n = \left(1 : n - \frac{2014}{3}, \left(n - \frac{2014}{3}\right)^3\right)}$

The coordinate sum is

$$27\left(1 + \left(n - \tfrac{2014}{3}\right) + \left(n - \tfrac{2014}{3}\right)^3\right) \equiv 2014^3 \pmod 3$$
$$\not\equiv 0 \pmod 3$$

so this means it is nonzero, and P_n is a Euclidean point (and not a point at infinity).

Let $x = a - \frac{2014}{3}$, $y = b - \frac{2014}{3}$, $z = c - \frac{2014}{3}$.

Define

$$D(a,b,c) = D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

$$= \sum_{cyc} -x^3(y-z).$$

In that case the points P_a, P_b, P_c are collinear if and only if the determinant D is zero.

*Darn I just realized Cartesian $y = \left(x - \frac{2014}{3}\right)^3$ works too. Oops.

But observe that

$$(x-y)(y-z)(z-x)(x+y+z)$$

$$= \sum_{cyc} x\left(-x^2y + x^2z - y^2z + y^2x - z^2x + z^2y\right)$$

$$= \sum_{cyc} -x^3(y-z) - xy^2z + xyz^2 - zx^2 + y^2x^2$$

$$= \sum_{cyc} -x^3(y-z).$$

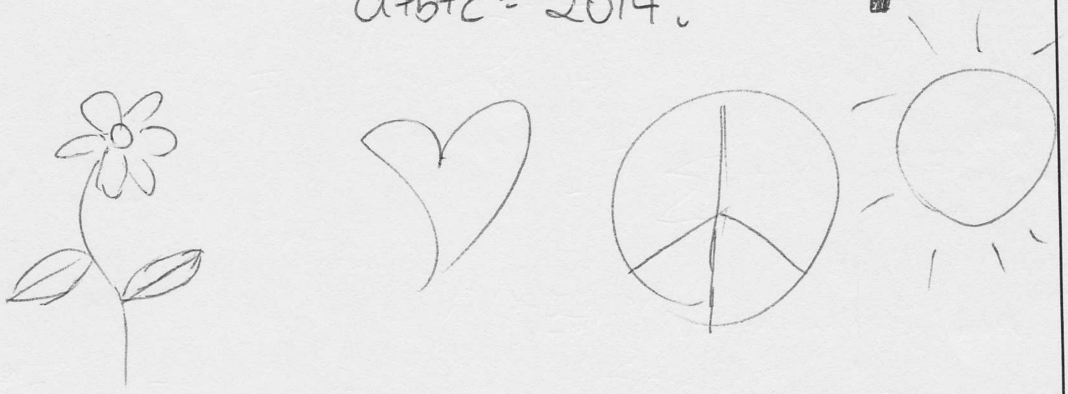
So then

$$D(a,b,c) =$$

$$(a-b)(b-c)(c-a)(a+b+c-2014).$$

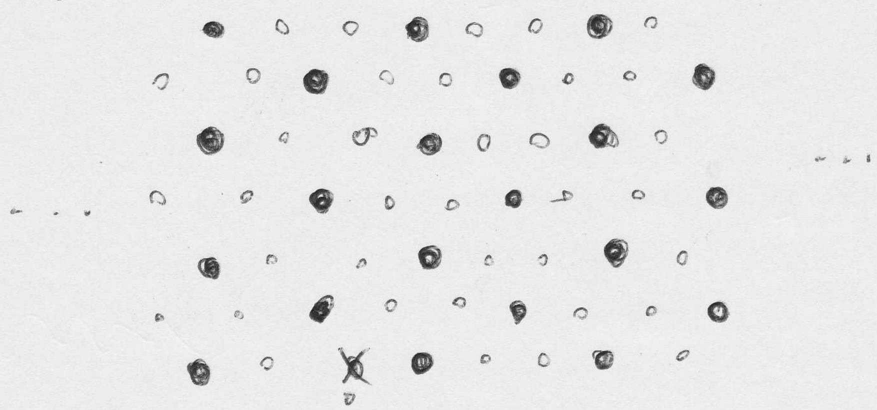
Hence P_a, P_b, P_c are collinear if and only if two of a,b,c coincide, or ~~a+b+c=2014~~.

$$a+b+c = 2014.$$



The answer is $k=6$. Call the players Ana & Banana, for A and B.

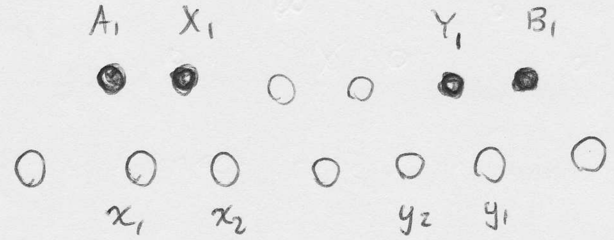
First, we prove that if $k \geq 6$, then Ana cannot always win. Consider the board below, where each space is a circle:



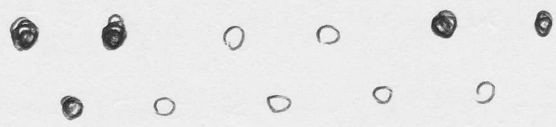
We have marked several spaces black; extend this pattern indefinitely. We claim that Banana can merely remove the blackened spots. This is always possible since no two such spots are adjacent. Now this implies that at the beginning of Ana's turn, any consecutive six grid cells have at least two spaces without a counter which are also nonadjacent (in fact, we always have exactly two blackened cells). This makes it impossible to win at $k=6$.

Next we construct a winning strategy for $k=5$. After Banana's first move, there is exactly one token left, so Ana can construct an "equilateral triangle", whence Banana leaves her with two adjacent counters.

Now, let Ana place two counters at a "gap" two away, as shown:



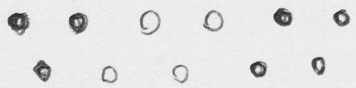
Note that this forces Banana to take either X_1 or Y_1. WLOG he takes X_1. ~~Now we co~~ Then Ana adds in X_1 and x_1.

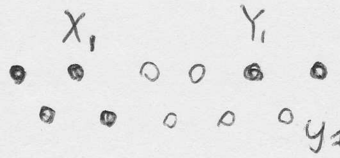


~~Now, we consider two cases.~~

We then repeat this procedure until either x_1 and x_2 are both present, or both y_1 & y_2 are.

At each step, Banana must take either X_1 or Y_1, and we simply add it back in, ~~until one~~ along with a lower-case letter, until one lowercase letter is full. So we have essentially two cases; ~~[illegible]~~ Banana to move.

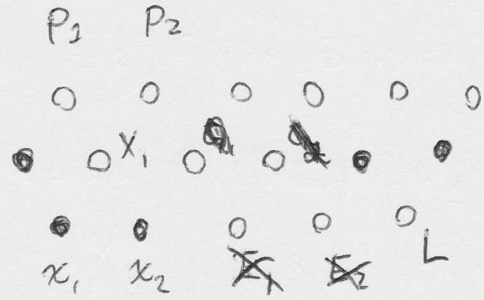




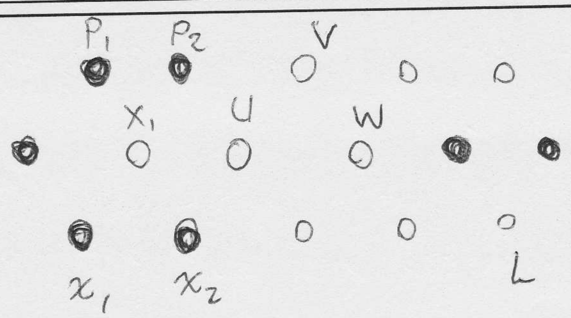
Case I ~~[illegible]~~ $(1+2)$ Case II. $(0+2)$

Let's start with Case II. If Y_1 is removed, then add in $\{y_1, Y_1\}$, reducing to Case I.

If X_1 is removed, then we get the following.



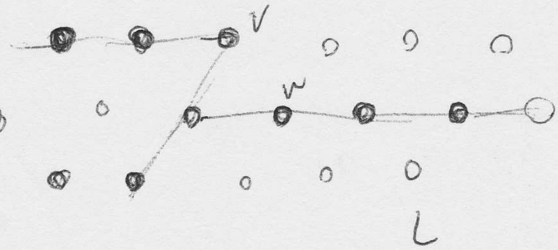
Adding on P_1 and and X_1 forces Banana to take X_1 (since $P_1 - X_1 - X_2$ is a threat, as is the same $A_1 - X_1 - Y_1 - B_1$). Similarly we can add P_2, X_1 and again force Banana to take X_1.



We get this

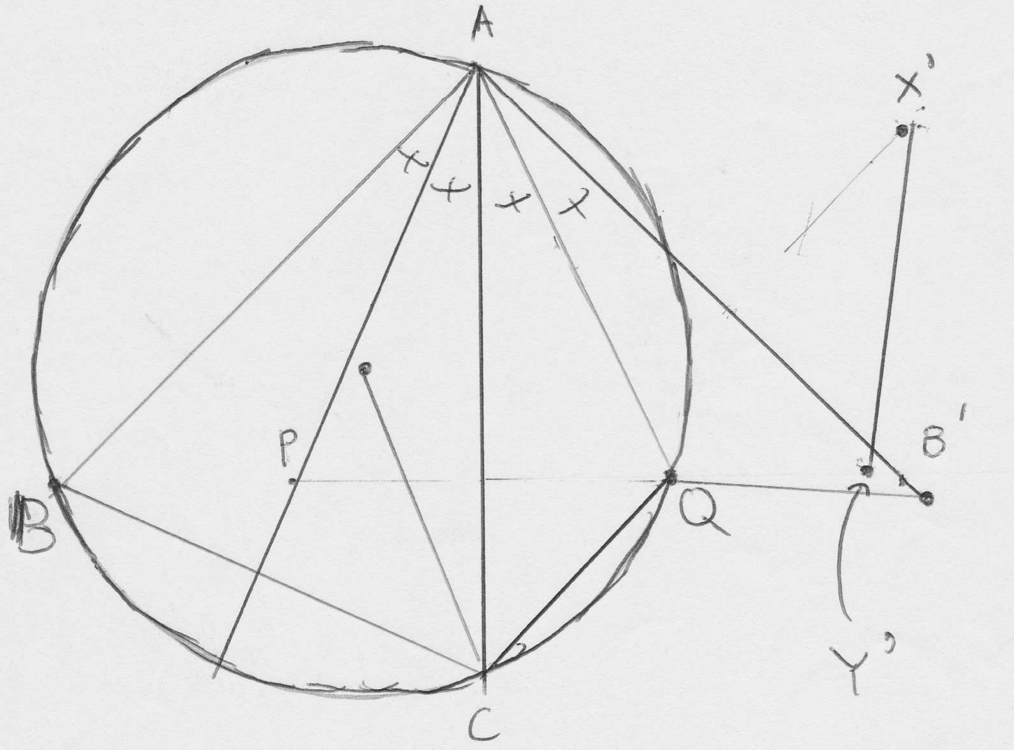
If we now add X_1 and U, then $P_1 - X_1 - x_2$ and $P_2 - X_1 - x_1$ forces Banana to take X_1.

Now adding in V, W leads to a clear win:



Case I is analogous, except the ~~square~~ cell marked L is this time not empty, which does not alter the strategy.

Hence Ana can force a win when $k=5$, as required.



Let Q be the reflection of P across $\overline{AC}$.

Then, let B', X', Y' denote the reflections of B, X, Y across $\overline{AC}$.

Directing angles,

$$\measuredangle AQC = -\measuredangle APC = -\measuredangle AHC = \measuredangle ABC$$

well-known lemma here: reflection of H across $\overline{AC}$ is on (ABC). This is just angle chasing.

Hence Q lies on (ABC).

Now complex bash with (ABC) the unit circle.

Because $\angle BAC = 2\angle CAQ$, we have

$$\frac{1}{q}\cdot b = \left(\frac{1}{q}\cdot c\right)^3$$

$$\Rightarrow b = c^3/q^2.$$

Also

$$b' = a + c - ac\bar{b}$$ [See Addendum 1]

$$= a + c - aq^2/c^2.$$

Moreover,

$$y' = a+q+c.$$

It remains to compute x'. Note that $x' - a$ is the circumcenter of the circle with vertices 0, $q-a$, $b'-a$; hence

$$x' - a = \frac{\begin{vmatrix} q-a & |q-a|^2 & 1 \\ b'-a & |b'-a|^2 & 1 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} q-a & \overline{q-a} & 1 \\ b'-a & \overline{b'-a} & 1 \\ 0 & 0 & 1 \end{vmatrix}}$$

[Circumcenter Formula]

[See Addendum 2]

$$= \frac{(q-a)(b'-a)\left[\overline{b'-a} - \overline{q-a}\right]}{(q-a)\overline{(b'-a)} - \overline{(q-a)}(b'-a)}$$

So now, $b' - a = c - aq^2/c^2$

$$\Rightarrow \overline{b'-a} = \frac{1}{c} - c^2/aq^2.$$

Hence

$$x' - a = \frac{(q-a)(c - \frac{aq^2}{c^2})(\frac{1}{c} - \frac{c^2}{aq^2} - \frac{1}{q} + \frac{1}{a})}{(q-a)(\frac{1}{c} - \frac{c^2}{aq^2}) - (\frac{1}{q} - \frac{1}{a})(c - \frac{aq^2}{c^2})}$$

$$= \frac{(q-a)(c^3 - aq^2)(\frac{aq^2 - c^3}{\cancel{A}\, aq^2c} - \frac{1}{q} + \frac{1}{a})}{(q-a)(c - \frac{c^4}{aq^2}) + \frac{q-a}{aq} \cdot (c^3 - aq^2)}$$

$$= \frac{aq(c^3 - aq^2)(\frac{1}{c} - \frac{c^2}{aq^2} - \frac{1}{q} + \frac{1}{a})}{aqc - \frac{c^4}{q} + c^3 - aq^2}$$

$$= \frac{aq(\frac{aq^2 - c^3}{aq^2c} - \frac{1}{q} + \frac{1}{a})(c^3 - aq^2)}{-\frac{c}{q}(c^3 - aq^2) + (c^3 - aq^2)}$$

$$= \frac{aq(\frac{aq^2 - c^3}{aq^2c} + \frac{q-a}{aq})}{1 - \frac{c}{q}}$$

$$= \frac{aq^2 - c^3 + qc(q-a)}{qc - c^2}$$

$$= \frac{aq(q-c) + c(q-c)(q+c)}{c(q-c)}$$

$$= \frac{aq}{c} + q + c.$$

Hence,

$$x' = a + q + c + \frac{aq}{c}$$

$$y' = a + q + c$$

$$\Rightarrow |x' - y'| = \left|\frac{aq}{c}\right| = \frac{1 \cdot 1}{1} = 1$$

as required.

Page 5 mentions addendums, which clarify the theorems I cited.

Addendum 1: It's well-known that if $\overline{AB}$ is a chord of the unit circle and $z \in \mathbb{C}$ then the foot of z on AB is

$$p = \frac{1}{2}(a+b+z-ab\bar{z})$$

Thus the reflection is $2p - z = a+b-ab\bar{z}$.

Addendum 2: The arbitrary circumcenter has formula

$$p = \frac{\begin{vmatrix} x & x\bar{x} & 1 \\ y & y\bar{y} & 1 \\ z & z\bar{z} & 1 \end{vmatrix}}{\begin{vmatrix} x & \bar{x} & 1 \\ y & \bar{y} & 1 \\ z & \bar{z} & 1 \end{vmatrix}}.$$

To see this note that if p is said center then

$$R^2 = |p-x|^2 = p\bar{p} - p\bar{x} - \bar{p}x + x\bar{x}$$

$$\Rightarrow \quad \bar{x}p + x\bar{p} = -R^2 + p\bar{p} + x\bar{x}.$$

and so on. Then Cramer's Rule on $p, \bar{p}$ is

$$p = \frac{\begin{vmatrix} x & -R^2+p\bar{p}+x\bar{x} & 1 \\ y & -R^2+p\bar{p}+y\bar{y} & 1 \\ z & -R^2+p\bar{p}+z\bar{z} & 1 \end{vmatrix}}{\begin{vmatrix} x & \bar{x} & 1 \\ y & \bar{y} & 1 \\ z & \bar{z} & 1 \end{vmatrix}} = \frac{\begin{vmatrix} x & x\bar{x} & 1 \\ y & y\bar{y} & 1 \\ z & z\bar{z} & 1 \end{vmatrix}}{\begin{vmatrix} x & \bar{x} & 1 \\ y & \bar{y} & 1 \\ z & \bar{z} & 1 \end{vmatrix}},$$

Let $N = n+1$. Construct an $N \times N$ table as below:

b \ a	0	1	2	3	…	n
0	(0,0)	(1,0)	(2,0)	…		
1	(0,1)	⋱				
2	⋮					
3						
⋮						
n						

(i,j)

Here $N=6$. Now in each square, write the smallest prime p dividing $\gcd(a+i, b+j)$.

We claim that for N large enough, at least $0.28N^2$ of the squares have a prime greater than ~~$0.001n$~~ $0.001n^2$ written in them. Indeed, let $p_1, p_2, \ldots, p_r$ denote the set of primes less than ~~$0.001n$~~. $0.001n^2$.

Now note that a prime p is written at most

$$\left\lceil \frac{N}{p} \right\rceil^2 \leq \left(\frac{N}{p}+1\right)^2$$

times. Thusly the number of "bad" squares is less than

$$\sum_{i=1}^{r} \left(\frac{N}{p_i}+1\right)^2 = N^2\left[\frac{1}{p_1^2}+\dots+\frac{1}{p_r^2}\right]$$

$$+ \quad 2N\left[\frac{1}{p_1}+\dots+\frac{1}{p_r}\right]$$

$$+ \quad r \qquad (*)$$

~~By the Prime Number Theorem, r grows asymptotically as N/ln N. So we have asymptotic bounds~~

$$\cancel{\frac{1}{p_1}+\dots+\frac{1}{p_r} < \frac{1}{1}+\dots+\frac{1}{r} \sim}$$

Now we have asymptotic bounds

$$\frac{1}{p_1}+\dots+\frac{1}{p_r} < \frac{1}{1}+\dots+\frac{1}{r}$$

$$\lesssim 10 \ln r \qquad \lesssim o(N)$$

$$\lesssim 20 \ln 0.001n \qquad \ll N^2$$

and of course $r \le 0.001n^2$. So the last two terms of (*) can be bounded by $0.002N^2$ for large N, say.

Moreover,

$$\frac{1}{p_1^2} + \dots + \frac{1}{p_r^2} < \frac{\pi^2}{6} - \frac{1}{1} - \frac{1}{16}$$

$$< \frac{3.2^2}{6} - 1 - \frac{1}{16}$$

$$< \cancel{0.7}\ \cancel{0.698}\ 0.708$$

So the entire sum (*) is bounded above by $0.71N^2$, establishing the claim.

Hence some ~~column~~ row has at least ~~29~~ 28% of its entries as primes exceeding $0.001n^2$. If N is large enough, then (addendum at end) $N < 0.001n^2$, so all these primes are distinct,* and moreover there are at least $\cancel{0.26n^2 < 0.28N}$ $0.26n < 0.28N$ such primes.

Consequently, ~~a+i is~~ for some $0 \le i \le n$,

$$a + i \geq \cancel{(0.26n^2)}$$

$$\Rightarrow \quad a \geq (0.001 n^2)^{0.26n} - i$$

$$\Rightarrow \quad a > c^n \cdot n^{\frac{1}{2}n}$$

for some constant $c > 0$. As a, b are symmetric, we must therefore have

$$\min\{a, b\} > c^n \cdot n^{\frac{1}{2}n}.$$

when n is big enough. Now we simply pick c even smaller so that $c^n \cdot n^{\frac{1}{2}n} < 1$ for the finitely many smaller n. We're done.

~~Bloody problem~~

ADDENDUM:

(*) as if p divides $a+i$, $a+j$ then $p \mid i-j$ ($i \neq j$ here) so $p \leq |i-j|$. Hence $p \leq N$.