A Brief Introduction to Olympiad Inequalities

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The goal of this document is to provide an easier introduction to olympiad inequalities than the standard exposition Olympiad Inequalities, by Thomas Mildorf. I was motivated to write it by feeling guilty for getting free 7’s on problems by simply regurgitating a few tricks I happened to know, while other students were unable to solve the problem.

Warning: These are notes, not a full handout. Lots of the exposition is very minimal, and many things are left to the reader.

In a problem with $n$ variables, these respectively mean to cycle through the $n$ variables, and to go through all $n!$ permutations. To provide an example, in a three-variable problem we might write

\[
\sum_{\text{cyc}} a^2 = a^2 + b^2 + c^2
\]
\[
\sum_{\text{cyc}} a^2b = a^2b + b^2c + c^2a
\]
\[
\sum_{\text{sym}} a^2 = a^2 + a^2 + b^2 + b^2 + c^2 + c^2
\]
\[
\sum_{\text{sym}} a^2b = a^2b + a^2c + b^2c + b^2a + c^2a + c^2b.
\]

§1 Polynomial Inequalities

§1.1 AM-GM and Muirhead

Consider the following theorem.

**Theorem 1.1 (AM-GM)**

For nonnegative reals $a_1, a_2, \ldots, a_n$ we have

\[
\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.
\]

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

For example, this implies

\[
a^2 + b^2 \geq 2ab, \quad a^3 + b^3 + c^3 \geq 3abc.
\]

Adding such inequalities can give us some basic propositions.
Example 1.2
Prove that \(a^2 + b^2 + c^2 \geq ab + bc + ca\) and \(a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab\).

Proof. By AM-GM,
\[
\frac{a^2 + b^2}{2} \geq ab \quad \text{and} \quad \frac{2a^4 + b^4 + c^4}{4} \geq a^2bc.
\]
Similarly,
\[
\frac{b^2 + c^2}{2} \geq bc \quad \text{and} \quad \frac{2b^4 + c^4 + a^4}{4} \geq b^2ca.
\]
\[
\frac{c^2 + a^2}{2} \geq ca \quad \text{and} \quad \frac{2c^4 + a^4 + b^4}{4} \geq c^2ab.
\]
Summing the above statements gives
\[
a^2 + b^2 + c^2 \geq ab + bc + ca \quad \text{and} \quad a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab.
\]

Exercise 1.3. Prove that \(a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a\).

Exercise 1.4. Prove that \(a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab \geq abc(ab + bc + ca)\).

The fundamental intuition is being able to decide which symmetric polynomials of a given degree are bigger. For example, for degree 3, the polynomial \(a^3 + b^3 + c^3\) is biggest and \(abc\) is the smallest. Roughly, the more “mixed” polynomials are the smaller. From this, for example, one can immediately see that the inequality
\[(a + b + c)^3 \geq a^3 + b^3 + c^3 + 2abc
\]
must be true, since upon expanding the LHS and cancelling \(a^3 + b^3 + c^3\), we find that the RHS contains only the piddling term \(24abc\). That means a straight AM-GM will suffice.

A useful formalization of this is Muirhead’s Inequality. Suppose we have two sequences \(x_1 \geq x_2 \geq \cdots \geq x_n\) and \(y_1 \geq y_2 \geq \cdots \geq y_n\) such that
\[x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n,
\]and for \(k = 1, 2, \ldots, n - 1\)
\[x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k.
\]Then we say that \((x_n)\) majorizes \((y_n)\), written \((x_n) \succ (y_n)\).
Using the above, we have the following theorem.

**Theorem 1.5 (Muirhead’s Inequality)**
If \(a_1, a_2, \ldots, a_n\) are positive reals, and \((x_n)\) majorizes \((y_n)\) then we have the inequality.
\[
\sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \cdots a_n^{y_n}.
\]
Example 1.6
Since $(5, 0, 0) \succ (3, 1, 1) \succ (2, 2, 1)$,
\[
a^5 + a^5 + b^5 + b^5 + c^5 + c^5 \geq a^3bc + a^3bc + b^3ca + b^3ca + c^3ab + c^3ab
\geq a^2b^2c + a^2b^2c + b^2c^2a + b^2c^2a + c^2a^2b + c^2a^2b.
\]
From this we derive \(a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab \geq abc(ab + bc + ca)\).

Notice that Muirhead is symmetric, not cyclic. For example, even though \((3, 0, 0) \succ (2, 1, 0)\), Muirhead’s inequality only gives that
\[2(a^3 + b^3 + c^3) \geq a^2b + a^2c + b^2c + b^2a + c^2a + c^2b\]
and in particular this does not imply that \(a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a\). These situations must still be resolved by AM-GM.

§1.2 Non-homogeneous inequalities

Consider the following example.

Example 1.7

Prove that if \(abc = 1\) then \(a^2 + b^2 + c^2 \geq a + b + c\).

Proof. AM-GM alone is hopeless here, because whenever we apply AM-GM, the left and right hand sides of the inequality all have the same degree. So we want to use the condition \(abc = 1\) to force the problem to have the same degree. The trick is to notice that the given inequality can be rewritten as
\[a^2 + b^2 + c^2 \geq a^{1/3}b^{1/3}c^{1/3} (a + b + c).
\]
Now the inequality is homogeneous. Observe that if we multiply \(a, b, c\) by any real number \(k > 0\), all that happens is that both sides of the inequality are multiplied by \(k^2\), which doesn’t change anything. That means the condition \(abc = 1\) can be ignored now. Since \((2, 0, 0) \succ (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})\), applying Muirhead’s Inequality solves the problem.

The importance of this problem is that it shows us how to eliminate a given condition by homogenizing the inequality; this is very important. (In fact, we will soon see that we can use this in reverse – we can impose an arbitrary condition on a homogeneous inequality.)

§1.3 Practice Problems

1. \(a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3\).
2. If \(a + b + c = 1\), then \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 + 2 \cdot \frac{(a^3 + b^3 + c^3)}{abc} \).
3. \(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c\).
4. If \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1\), then \((a + 1)(b + 1)(c + 1) \geq 64\).
5. (USA 2011) If \(a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4\), then
\[\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3\]
6. If \(abcd = 1\), then \(a^4b + b^4c + c^4d + d^4a \geq a + b + c + d\).
Let \( f : (u,v) \to \mathbb{R} \) be a function and let \( a_1, a_2, \ldots, a_n \in (u,v) \). Suppose that we fix \( \frac{a_1 + a_2 + \cdots + a_n}{n} = a \) (if the inequality is homogeneous, we will often insert such a condition) and we want to prove that
\[
f(a_1) + f(a_2) + \cdots + f(a_n)
\]
is at least (or at most) \( nf(a) \). In this section we will provide three methods for doing so.

We say that function \( f \) is **convex** if \( f''(x) \geq 0 \) for all \( x \); we say it is **concave** if \( f''(x) \leq 0 \) for all \( x \). Note that \( f \) is convex if and only if \( -f \) is concave.

### §2.1 Jensen / Karamata

#### Theorem 2.1 (Jensen’s Inequality)
If \( f \) is convex, then
\[
\frac{f(a_1) + \cdots + f(a_n)}{n} \geq f\left( \frac{a_1 + \cdots + a_n}{n} \right).
\]
The reverse inequality holds when \( f \) is concave.

#### Theorem 2.2 (Karamata’s Inequality)
If \( f \) is convex, and \((x_n) \) majorizes \((y_n) \) then
\[
f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).
\]
The reverse inequality holds when \( f \) is concave.

### Example 2.3 (Shortlist 2009)
Given \( a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \), prove that
\[
\frac{1}{(2a + b + c)^2} + \frac{1}{(a + 2b + c)^2} + \frac{1}{(a + b + 2c)^2} \leq \frac{3}{16}.
\]

**Proof.** First, we want to eliminate the condition. The original problem is equivalent to
\[
\frac{1}{(2a + b + c)^2} + \frac{1}{(a + 2b + c)^2} + \frac{1}{(a + b + 2c)^2} \leq \frac{3}{16} \cdot \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{a + b + c}.
\]
Now the inequality is homogeneous, so we can assume that \( a + b + c = 3 \). Now our original problem can be rewritten as
\[
\sum_{cyc} \frac{1}{16a} - \frac{1}{(a + 3)^2} \geq 0.
\]
Set \( f(x) = \frac{1}{16x} - \frac{1}{(x+3)^2} \). We can check that \( f \) over \((0,3)\) is convex so Jensen completes the problem. \( \square \)
Example 2.4

Prove that
\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 2 \left( \frac{1}{a+b+c} + \frac{1}{b+c+a} \right) \geq \frac{9}{a+b+c}. \]

Proof. The problem is equivalent to
\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{a+b+c} + \frac{1}{b+c+a} + \frac{1}{c+a+b}. \]
Assume WLOG that \( a \geq b \geq c \). Let \( f(x) = \frac{1}{x} \). Since \((a,b,c) \succ (a+b+c, a+b+c, a+b+c)\) the conclusion follows by Karamata.

Example 2.5 (APMO 1996)

If \( a, b, c \) are the three sides of a triangle, prove that
\[ \sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}. \]

Proof. Again assume WLOG that \( a \geq b \geq c \) and notice that \((a,b,c) \succ (b+c-a, c+a-b, a+b-c)\). Apply Karamata on \( f(x) = \sqrt{x} \).

§2.2 Tangent Line Trick

Again fix \( a = \frac{a_1 + \cdots + a_n}{n} \). If \( f \) is not convex, we can sometimes still prove the inequality
\[ f(x) \geq f(a) + f'(a) (x-a). \]
If this inequality manages to hold for all \( x \), then simply summing the inequality will give us the desired conclusion. This method is called the tangent line trick.

Example 2.6 (David Stoner)

If \( a + b + c = 3 \), prove that
\[ 18 \sum_{\text{cyc}} \frac{1}{(3-c)(4-c)} + 2(ab+bc+ca) \geq 15. \]

Proof. We can rewrite the given inequality as
\[ \sum_{\text{cyc}} \left( \frac{18}{(3-c)(4-c)} - c^2 \right) \geq 6. \]
Using the tangent line trick lets us obtain the magical inequality
\[ \frac{18}{(3-c)(4-c)} - c^2 \geq \frac{c+3}{2} \iff c(c-1)^2(2c-9) \leq 0 \]
and the conclusion follows by summing.
Example 2.7 (Japan)
Prove \( \sum_{\text{cyc}} \frac{(b+c-a)^2}{a^2+(b+c)^2} \geq \frac{3}{5} \).

Proof. Since the inequality is homogeneous, we may assume WLOG that \( a + b + c = 3 \). So the inequality we wish to prove is

\[
\sum_{\text{cyc}} \frac{(3-2a)^2}{a^2+(3-a)^2} \geq \frac{3}{5}.
\]

With some computation, the tangent line trick gives away the magical inequality:

\[
\frac{(3-2a)^2}{(3-a)^2+a^2} \geq \frac{1}{5} - \frac{18}{25} (a-1) \iff \frac{18}{25} (a-1)^2 \frac{2a+1}{2a^2-6a+9} \geq 0.
\]

§2.3 \( n-1 \) EV

The last such technique is \( n-1 \) EV. This is a brute force method involving much calculus, but it is nonetheless a useful weapon.

Theorem 2.8 \( (n-1 \text{ EV}) \)

Let \( a_1, a_2, \ldots, a_n \) be real numbers, and suppose \( a_1 + a_2 + \cdots + a_n \) is fixed. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function with exactly one inflection point. If

\[
f(a_1) + f(a_2) + \cdots + f(a_n)
\]

achieves a maximal or minimal value, then \( n-1 \) of the \( a_i \) are equal to each other.

Proof. See page 15 of *Olympiad Inequalities*, by Thomas Mildorf. The main idea is to use Karamata to “push” the \( a_i \) together.

Example 2.9 (IMO 2001 / APMOC 2014)

Let \( a, b, c \) be positive reals. Prove \( 1 \leq \sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}} < 2 \).

Proof. Set \( e^x = \frac{b}{a^2}, e^y = \frac{c}{a^2}, e^z = \frac{ab}{c^2} \). We have the condition \( x + y + z = 0 \) and want to prove

\[
1 \leq f(x) + f(y) + f(z) < 2
\]

where \( f(x) = \frac{1}{\sqrt{1+8e^x}} \). You can compute

\[
f''(x) = \frac{4e^x(4e^x-1)}{(8e^x+1)^2}
\]

so by \( n-1 \) EV, we only need to consider the case \( x = y \). Let \( t = e^x \); that means we want to show that

\[
1 \leq \frac{2}{\sqrt{1+8t}} + \frac{1}{\sqrt{1+8/t}} < 2.
\]

Since this a function of one variable, we can just use standard Calculus BC methods.
Example 2.10 (Vietnam 1998)
Let \(x_1, x_2, \ldots, x_n\) be positive reals satisfying \(\sum_{i=1}^{n} \frac{1}{1998+x_i} = \frac{1}{1998}\). Prove
\[
\sqrt[n]{x_1x_2\cdots x_n} \geq 1998.
\]

Proof. Let \(y_i = \frac{1998}{1998+x_i}\). Since \(y_1 + y_2 + \cdots + y_n = 1\), the problem becomes
\[
\prod_{i=1}^{n} \left( \frac{1}{y_i} - 1 \right) \geq (n-1)^n.
\]
Set \(f(x) = \ln \left( \frac{1}{x} - 1 \right)\), so the inequality becomes
\[
f(y_1) + \cdots + f(y_n) \geq nf \left( \frac{1}{n} \right).
\]
We can prove that \(f''(y) = 1 - \frac{2y}{(y^2-y)^2}\).
So \(f\) has one inflection point, we can assume WLOG that \(y_1 = y_2 = \ldots y_{n-1}\). Let this common value be \(t\); we only need to prove
\[
(n-1) \ln \left( \frac{1}{t} - 1 \right) + \ln \left( \frac{1}{1-(n-1)t} - 1 \right) \geq n \ln(n-1).
\]
Again, since this is a one-variable inequality, calculus methods suffice. \(\square\)

§2.4 Practice Problems
1. Use Jensen to prove AM-GM.
2. If \(a^2 + b^2 + c^2 = 1\) then \(\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq \frac{1}{6ab+ca} + \frac{1}{6bc+a^2} + \frac{1}{6ca+b^2}\).
3. If \(a+b+c = 3\) then
\[
\sum_{\text{cyc}} \frac{a}{2a^2 + a + 1} \leq \frac{3}{4}.
\]
4. (MOP 2012) If \(a+b+c+d = 4\), then \(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \geq a^2 + b^2 + c^2 + d^2\).

§3 Eliminating Radicals and Fractions
§3.1 Weighted Power Mean
AM-GM has the following natural generalization.

Theorem 3.1 (Weighted Power Mean)
Let \(a_1, a_2, \ldots, a_n\) and \(w_1, w_2, \ldots, w_n\) be positive reals with \(w_1 + w_2 + \cdots + w_n = 1\). For any real number \(r\), we define
\[
P(r) = \begin{cases} 
(w_1 a_1^r + w_2 a_2^r + \cdots + w_n a_n^r)^{1/r} & \text{if } r \neq 0 \\
w_1^{w_1} a_1^{w_2} \cdots a_n^{w_n} & \text{if } r = 0.
\end{cases}
\]
If \(r > s\), then \(P(r) \geq P(s)\) equality occurs if and only if \(a_1 = a_2 = \cdots = a_n\).
In particular, if \( w_1 = w_2 = \cdots = w_n = \frac{1}{n} \), the above \( \mathcal{P}(r) \) is just

\[
\mathcal{P}(r) = \begin{cases} 
\left( \frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{1/r} & r \neq 0 \\
\sqrt[\lambda]{a_1 a_2 \cdots a_n} & r = 0.
\end{cases}
\]

By setting \( r = 2, 1, 0, -1 \) we derive

\[
\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} \geq \sqrt[\lambda]{a_1 a_2 \cdots a_n} \geq \frac{1}{a_1} + \cdots + \frac{1}{a_n}
\]

which is QM-AM-GM-HM. Moreover, AM-GM lets us “add” roots, like

\[
\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3\sqrt{\frac{a+b+c}{3}}.
\]

**Example 3.2** (Taiwan TST Quiz)

Prove \( 3(a+b+c) \geq 8\sqrt{abc} + \frac{3}{9}(a^3+b^3+c^3) \).

**Proof.** By Power Mean with \( r = 1, s = \frac{1}{3}, w_1 = \frac{1}{9}, w_2 = \frac{8}{9}, \) we find that

\[
\left( \frac{1}{9} \sqrt[3]{a^3+b^3+c^3} + \frac{8}{9} \sqrt[3]{abc} \right)^3 \leq \frac{1}{9} \left( a^3+b^3+c^3 \right) + \frac{8}{9} (abc).
\]

so we want to prove \( a^3+b^3+c^3+24abc \leq (a+b+c)^3 \), which is clear. \( \square \)

§3.2 Cauchy and Hölder

**Theorem 3.3** (Hölder’s Inequality)

Let \( \lambda_1, \lambda_2, \ldots, \lambda_z \) be positive reals with \( \lambda_1 + \lambda_2 + \cdots + \lambda_z = 1 \). Let \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m, \ldots, z_1, z_2, \ldots, z_n \) be positive reals. Then

\[
(a_1 + \cdots + a_n)^{\lambda_a} (b_1 + \cdots + b_m)^{\lambda_b} \cdots (z_1 + \cdots + z_n)^{\lambda_z} \geq \sum_{i=1}^{n} a_1^{\lambda_{a_1}} b_1^{\lambda_{b_1}} \cdots z_i^{\lambda_i}.
\]

Equality holds if \( a_1 : a_2 : \cdots : a_n \equiv b_1 : b_2 : \cdots : b_m \equiv \cdots \equiv z_1 : z_2 : \cdots : z_n \).

**Proof.** WLOG \( a_1 + \cdots + a_n = b_1 + \cdots + b_m = \cdots = 1 \) (note that the degree of the \( a_i \) on either side is \( \lambda_a \)). In that case, the LHS of the inequality is 1, and we just note

\[
\sum_{i=1}^{n} a_i^{\lambda_{a}} b_1^{\lambda_{b}} \cdots z_i^{\lambda_z} \leq \sum_{i=1}^{n} (\lambda_a a_i + \lambda_b b_i + \ldots) = 1.
\]

If we set \( \lambda_a = \lambda_b = \frac{1}{2} \), we derive what is called the Cauchy-Schwarz inequality.

\[
(a_1 + a_2 + \cdots + a_n) (b_1 + b_2 + \cdots + b_m) \geq \left( \sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n} \right)^2.
\]
Cauchy can be rewritten as

\[
\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + \cdots + y_n}.
\]

This form is often called Titu’s Lemma in the United States.

Cauchy and Hölder have at least two uses:

1. eliminating radicals,
2. eliminating fractions.

Let us look at some examples.

**Example 3.4** (IMO 2001)

Prove

\[
\sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \geq 1.
\]

**Proof.** By Hölder

\[
\left( \sum_{\text{cyc}} a(a^2 + 8bc) \right)^{\frac{1}{3}} \left( \sum_{\text{cyc}} \frac{a}{\sqrt{a^2 + 8bc}} \right)^{\frac{2}{3}} \geq (a + b + c)
\]

So it suffices to prove \((a + b + c)^3 \geq \sum_{\text{cyc}} a(a^2 + 8bc) = a^3 + b^3 + c^3 + 24abc\). Does this look familiar?

In this problem, we used Hölder to clear the square roots in the denominator.

**Example 3.5** (Balkan)

Prove

\[
\frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \geq \frac{27}{2(a+b+c)^2}.
\]

**Proof.** Again by Hölder,

\[
\left( \sum_{\text{cyc}} a \right)^{\frac{1}{3}} \left( \sum_{\text{cyc}} b + c \right)^{\frac{1}{3}} \left( \sum_{\text{cyc}} \frac{1}{a(b+c)} \right)^{\frac{1}{3}} \geq 1 + 1 + 1 = 3.
\]

**Example 3.6** (JMO 2012)

Prove

\[
\sum_{\text{cyc}} \frac{a^3 + 5b^3}{3a+b} \geq \frac{3}{2} \left( a^2 + b^2 + c^2 \right).
\]

**Proof.** We use Cauchy (Titu) to obtain

\[
\sum_{\text{cyc}} \frac{a^3}{3a+b} = \sum_{\text{cyc}} \frac{(a^2)^2}{3a^2 + ab} \geq \frac{(a^2 + b^2 + c^2)^2}{\sum_{\text{cyc}} 3a^2 + ab}.
\]

We can easily prove this is at least \(\frac{1}{4}(a^2 + b^2 + c^2)\) (recall \(a^2 + b^2 + c^2\) is the “biggest” sum, so we knew in advance this method would work)). Similarly \(\sum_{\text{cyc}} \frac{5b^3}{3a+b} \geq \frac{5}{4}(a^2 + b^2 + c^2)\). 


Example 3.7 (USA TST 2010)
If \(abc = 1\), prove
\[
\frac{1}{a^3(b+2c)} + \frac{1}{b^3(c+2a)} + \frac{1}{c^3(a+2b)} \geq \frac{1}{3}.
\]

Proof. We can use Hölder to eliminate the square roots in the denominator:
\[
\left( \sum \text{cyc} ab + 2ac \right)^2 \left( \sum \text{cyc} \frac{1}{a^3(b+2c)^2} \right) \geq \left( \sum \frac{1}{a} \right)^3 \geq 3(ab + bc + ca)^2.
\]

§3.3 Practice Problems
1. If \(a + b + c = 1\), then
\[
\sqrt{ab} + \sqrt{bc} + \sqrt{ca} + b \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}.
\]
2. If \(a^2 + b^2 + c^2 = 12\), then \(a \cdot \sqrt{b^2 + c^2} + b \cdot \sqrt{c^2 + a^2} + c \cdot \sqrt{a^2 + b^2} \leq 12\).
3. (ISL 2004) If \(ab + bc + ca = 1\), prove
\[
\sqrt{1 + 3b} + \sqrt{1 + 3c} \leq 3(1 + 3b + 3c).
\]
4. (MOP 2011) \(\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2} + 9\sqrt{abc} \leq 4(a + b + c)\).
5. (Evan Chen) If \(a^3 + b^3 + c^3 + abc = 4\), prove
\[
(5a^2 + 2b)^2 \frac{(a + b)(a + c)}{(a + b)^2 + (b + c)(b + a)} + (5b^2 + 2c)^2 \frac{(b + c)(b + a)}{(b + c)^2 + (c + a)(c + b)} + (5c^2 + 2a)^2 \frac{(c + a)(c + b)}{(c + a)^2 + (a + b)(a + c)} \geq \frac{(10 - abc)^2}{a + b + c}.
\]

When does equality hold?

§4 Problems
1. (MOP 2013) If \(a + b + c = 3\), then
\[
\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \geq \sqrt{3}.
\]
2. (IMO 1995) If \(abc = 1\), then \(\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}\).
3. (USA 2003) Prove \(\sum \text{cyc} \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \leq 8\).
4. (Romania) Let \(x_1, x_2, \ldots, x_n\) be positive reals with \(x_1x_2\ldots x_n = 1\). Prove that \(\sum_{i=1}^{n} \frac{1}{x_i} \leq 1\).
5. (USA 2004) Let \(a, b, c\) be positive reals. Prove that
\[
(a^5 - a^2 + 3) (b^5 - b^2 + 3) (c^5 - c^2 + 3) \geq (a + b + c)^3.
\]
6. (Evan Chen) Let \(a, b, c\) be positive reals satisfying \(a + b + c = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}\). Prove \(a^3b^3c^3 \geq 1\).