Generating series and nonlinear systems:
Analytic aspects, local realizability, and i/o representations[1]

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ABSTRACT

This paper studies fundamental analytic properties of generating series for nonlinear control systems, and of the operators they define. It then applies the results obtained to the extension of facts, which relate realizability and algebraic input/output equations, to local realizability and analytic equations.

1 Introduction

State space models tend to play a central role in nonlinear control theory. However, other descriptions involving directly the input/output behavior, such as Volterra series, are also appropriate and often useful. Indeed, these are closer to the transfer functions and transfer matrices that appear in the development of linear control theory; there, i/o representations were used first in the analysis of linear systems and state-space approaches were only later introduced. In particular, *generating series* have been a popular choice for representing input/output operators, starting with the work of Fliess in the late 1970s (see for instance the references [2], [4]).

This paper has two main goals. First, we study a number of properties of generating series and the operators they define. While many of these properties have long been known and are used often by previous authors, it is hard to find complete proofs of them in the literature. In particular, the results proved here provide a mathematical foundation for many of the properties cited in our paper [15]. A second objective of this paper is to relate the rich theory that exists for differential-geometric nonlinear realization (for which see for instance [6], [8], [9], and [13]) to the results in [15]. The purely algebraic material in that reference shows the equivalence between on the one hand the existence of algebraic differential equations relating inputs and outputs, and on the other hand realizability by rational systems. In order to relate to the differential geometric framework, one must generalize [15] to analytic realizations and i/o equations. This is done here, using analogous ideas but quite different mathematical techniques. There are major differences with the algebraic case not only in the tools used but in the fact that, in contrast to that case, it is generally not true that every state space system gives rise to a global i/o equation, even under analyticity assumptions; only the converse, going from i/o equations to realizability, holds.

Outline of this work: We start by giving the basic terminology regarding series and convergence. Then for operators defined by the evaluation of these series, we study smoothness properties of their output functions, including the fact, needed in later results, that analytic

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inputs give rise to analytic outputs. Later, our results from [15] relating algebraic i/o equations to internal realizability are complemented here with a result relating analytic i/o equations and local internal realizability. To do this, we first construct a "meromorphic" realization by studying the properties of meromorphically finitely generated field extensions, and we impose similar properties on the observation fields already introduced in the former paper. Finally, by a perturbation approach, together with the Lie rank condition for realizability (cf. [3] and [12]), we show that around each point there is local analytic realization. (This could also be done by using the rank condition studied in [7].) In the last section, we compare the results of this paper with those in [15].

2 Analytic Aspects of Generating Series

We develop in this section the basic facts about operators defined by generating series.

2.1 Definitions

We call a *power series in the noncommutative variables* $\eta_0, \eta_1, \ldots, \eta_m$, where m is a fixed integer, a formal power series

$$c = \sum_{\iota \in I^*} \langle c,\ \eta_\iota \rangle \eta_\iota, \tag{1}$$

where we use the notation $\eta_\iota = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_l}$ for each multiindex $\iota = i_1 i_2 \cdots i_l$. The coefficients $\langle c,\ \eta_\iota\rangle$ are assumed to be real. The set of all power series (over a fixed but arbitrary alphabet P) forms a vector space with "+" defined coefficientwise.

We shall say that the power series c is *convergent* if there exist $K,\ M \geq 0$ such that

$$|\langle c,\ \eta_\iota\rangle| \leq K M^k k! \quad \text{for each } \iota \in I^k, \text{ and each } k \geq 0. \tag{2}$$

For any fixed real number $T > 0$, let $\mathcal{U}_T$ be the set of all essentially bounded measurable functions

$$u:\ [0,\ T] \to \mathbb{R}^m$$

endowed with the L^1 norm. We write $\|u\|_1$ for $\max\{\|u_i\|_1\ :,\ 1 \leq i \leq m\}$ and $\|u\|_\infty$ for $\max\{\|u_i\|_\infty\ :,\ 1 \leq i \leq m\}$ where u_i is the i-th component of u, and $\|u_i\|_1$ is the L^1 norm of u_i, $\|u_i\|_\infty$ is the L^∞ norm of u_i. For each $u \in \mathcal{U}_T$ and each $\iota \in I^l$, we define inductively the functions

$$V_\iota = V_\iota[u] \ \in\ \mathcal{C}[0,\ T]$$

by

$$V_{i_1\cdots i_{l+1}}[u](t) = \int_0^t u_{i_1}(s) V_{i_2\cdots i_{l+1}}(s)\, ds, \tag{3}$$

where $V_\phi = 1$ and u_i is the i-th coordinate of $u(t)$ for $i = 1, 2, \ldots, m$ and $u_0(t) \equiv 1$.

For each formal power series c in $\eta_0, \eta_1, \ldots, \eta_m$, we define a formal operator on $\mathcal{U}_T$ in the following way:

$$F_c[u](t) = \sum \langle c,\ \eta_\iota\rangle V_\iota[u](t). \tag{4}$$

It has been known that for any $T < (Mm + M)^{-1}$, the series (4) converges uniformly and absolutely for all $t \in [0,\ T]$ and all those $u \in \mathcal{U}_T$ such that $\|u\|_\infty \leq 1$ (cf [6]). In fact, for any L_1 input u, there exists some $\delta > 0$ such that (4) converges uniformly and absolutely on $[0,\ \delta)$.

For each $T > 0$, we define

$$\mathcal{V}_T = \{u \in \mathcal{U}_T : \ \|u\|_\infty < 1\}, \tag{5}$$

and we shall say that T is *admissible for* c if $T < (M(m+1))^{-1}$ for some M such that (2) holds. Then F_c is always well defined on $\mathcal{V}_T$ if T is admissible for c. We shall call F_c an *input/output operator* defined on $\mathcal{V}_T$ if T is admissible for c. Hence, every convergent power series defines an i/o map, or more precisely, one such map on each $\mathcal{V}_T$ for which T is admissible. (We often identify any two such operators, when there is no danger of confusion, dealing in effect with "germs" of such operators.)

To each monomial $z = \eta_\kappa$, we associate a "shift" operator $c \mapsto z^{-1}c$ defined by

$$\langle z^{-1}c, \ \eta_\iota\rangle = \langle c, \ z\eta_\iota\rangle \quad \text{for} \quad \eta_\iota \in P^*.$$

It has been shown in [11] that if c is a convergent series and T is admissible for c, then T is admissible for $z^{-1}c$ for any $z \in P^*$.

2.2 Properties of I/O Operators

Assume c is a convergent series and pick up a T admissible for c. We shall first show that $F_c : \mathcal{V}_T \to \mathcal{C}[0, T]$ is a continuous operator with respect to the L^1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0, T]$. For this purpose, we need to establish the following lemma:

Lemma 2.1 For every multiindex $\iota \in I^*$, the map

$$V_\iota : \ \mathcal{V}_T \to \mathcal{C}[0, T], \quad u \mapsto V_\iota[u]$$

is continuous with respect to the L^1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0, T]$.

Proof. We use induction on the length of ι. For $\iota = i \in I^1$, we have

$$V_\iota[u](t) = \int_0^t u_i(s)\, ds.$$

It follows that for any $u, v \in \mathcal{V}_T$,

$$\|V_\iota[u] - V_\iota[v]\|_\infty \le \|u - v\|_1,$$

where "$\|\cdot\|_\infty$" denotes the $\mathcal{C}^0$ norm in $[0, T]$. Thus V_ι is continuous for any $\iota \in I^1$.

Suppose the conclusion is true for all ι with $|\iota| \le n$. Then for $\kappa = i\iota \in I^{n+1}$ and $u, v \in \mathcal{U}_T$, we have

$$\begin{aligned}
|V_\kappa[v](t) - V_\kappa[u](t)| &= \left| \int_0^t v_i(s) V_\iota[v](s)\, ds \; - \int_0^t u_i(s) V_\iota[u](s)\, ds \right| \\
&\le \left| \int_0^t (v_i(s) - u_i(s)) V_\iota[v](s)\, ds \right| + \left| \int_0^t u_i(s)(V_\iota[v](s) - V_\iota[u](s))\, ds \right| \\
&\le \|u - v\|_1 \, \|V_\iota[v]\|_\infty + \|u\|_1 \, \|V_\iota[v] - V_\iota[u]\|_\infty.
\end{aligned}$$

Notice V_ι is continuous, thus for any $\varepsilon > 0$ given, there exists some $\tau > 0$ such that

$$\|V_\iota[v]\|_\infty \le \|V_\iota[u]\|_\infty + 1 \quad \text{and} \quad \|u\|_1 \left(\|V_\iota[v] - V_\iota[u]\|_\infty\right) < \varepsilon/2,$$

for all $v \in \mathcal{B}_\tau(u)$, where $\mathcal{B}_\tau(u)$ is the ball of radius τ centered at u in $\mathcal{U}_T$. Now let

$$\delta \;=\; \min\left\{\tau, \;\; \frac{\varepsilon}{2(1+\|V_\iota[u]\|)}\right\}.$$

Then for any $v \in \mathcal{B}_\delta(u)$,

$$|V_\kappa[v](t) - V_\kappa[u](t)| \;<\; \varepsilon$$

for all $t \in [0,\, T]$, which implies that $\|V_\kappa[v] - V_\kappa[u]\|_\infty \;<\; \varepsilon$. This shows that V_κ is continuous, completing the induction step. ∎

Now let c be a convergent series, and pick any T admissible for c. Then for $u,\; v \in \mathcal{V}_T$ and $0 \le t \le T$,

$$|F_c[u](t) \;-\; F_c[v](t)| = \left|\sum_\iota \langle c,\; \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right|$$

$$\le \left|\sum_{|\iota|\le s} \langle c,\; \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right| \;+\; \left|\sum_{|\iota|> s} \langle c,\; \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right|$$

$$\le \left|\sum_{|\iota|\le s} \langle c,\; \eta_\iota\rangle \left(V_\iota[u](t) - V_\iota[v](t)\right)\right| \;+\; 2\sum_{i\ge s} \left(M(m+1)T\right)^i$$

for any $s \ge 0$. Since $V_\iota :\, \mathcal{V}_T \;\to \mathcal{C}[0,\, T]$ is continuous and

$$\sum_{i\ge s} \left(M(m+1)T\right)^i \;\longrightarrow\; 0 \;\text{ as }\; s \;\to\; \infty,$$

it follows that for any $\varepsilon > 0$ given, there exists some $\delta > 0$ such that

$$|F_c[u](t) \;-\; F_c[v](t)| \;<\; \varepsilon$$

for any $v \in \mathcal{V}_T$ satisfying $\|u - v\|_1 < \delta$. Thus, we get the following conclusion:

Lemma 2.2 Assume that c is a convergent power series and T is admissible for c. Then the operator

$$F_c :\; \mathcal{V}_T \;\to\; \mathcal{C}[0,\, T]$$

is continuous with respect to the L^1 norm in $\mathcal{V}_T$ and the $\mathcal{C}^0$ norm in $\mathcal{C}[0,\, T]$. □

We now turn to considering the smoothness properties of $F_c[u](t)$ as a function of time t. Notice that, for every multiindex ι, $V_\iota[u](t)$ is absolutely continuous as a function of t. It follows immediately from the fact that $F_c[u]$ defined by (4) converges uniformly that $F_c[u](t)$ is continuous on $[0,\, T]$. If fact, it was shown in [14] that for any $u \in \mathcal{V}_T$, $F_c[u](\cdot)$ is absolutely continuous.

Differentiability of these operators has been studied in [5], and the following formula, which follows from the definition (3), was provided in [5]:

$$\frac{d}{dt} F_c[u](t) \;=\; F_{\eta_0^{-1}c}[u](t) \;+\; \sum_{j=1}^{m} u_j(t)\, F_{\eta_j^{-1}c}[u](t) \tag{6}$$

for all $t \in [0, T]$, and each continuous $u \in \mathcal{V}_T$. In fact, (6) also holds for any $u \in \mathcal{V}_T$ for almost all $t \in [0, T]$. By (6), one can use induction to prove that $F_c[u]$ is of class $\mathcal{C}^{k+1}$ if u is of class $\mathcal{C}^k$. The following Lemma, however, is a less trivial one.

Lemma 2.3 Suppose c is a convergent series and T is admissible to c. Then $F_c[u]$ is analytic if $u \in \mathcal{V}_T$ is analytic.

To prove Lemma 2.3, we need the following fact: For $u \in \mathcal{V}_\tau$ $(\tau < T)$ and $v \in \mathcal{V}_{T-\tau}$, we use $u\#_\tau v$ to denote the concatenated control:

$$(u\#_\tau v)(t) = \begin{cases} u(t) & \text{if } 0 \le t \le \tau, \\ v(t-\tau) & \text{if } \tau < t \le T. \end{cases}$$

Lemma 2.4 Suppose c is a convergent series which satisfies (2) and T is admissible for c. For any $0 \le \tau < T$, $u \in \mathcal{V}_\tau$, let d be the series defined by

$$\langle d,\ \eta_\iota\rangle = F_{\eta_l^{-1}c}[u](\tau). \tag{7}$$

Then d is also a convergent power series and $T - \tau$ is admissible for d. Furthermore, for each $v \in \mathcal{V}_{T-\tau}$,

$$F_c[u\#_\tau v](t+\tau) = F_d[v](t). \tag{8}$$

Proof. It follows from (2) that

$$|\langle \eta_\iota^{-1}c,\ \eta_\kappa\rangle| \le KM^{l+k}(l+k)! \quad \text{for} \quad \iota \in I^l,\ \kappa \in I^k.$$

Thus for any $0 \le \tau \le T$ and any $u \in \mathcal{V}_\tau$,

$$\begin{aligned} |F_{\eta_\iota^{-1}c}[u](\tau)| &\le \left|\sum_{\kappa \in I^*} \langle c,\ \eta_\iota\eta_\kappa\rangle V_\iota[u](\tau)\right| \\ &\le \sum_{k=0}^{\infty} KM^{l+k}(l+k)!(m+1)^k \frac{\tau^k}{k!} \\ &= KM^l \sum_{k=0}^{\infty} \frac{s^k}{k!}(l+k)!\ , \end{aligned} \tag{9}$$

where $s = M(m+1)\tau$. For power series (9), we have

$$\sum_{k=0}^{\infty} \frac{s^k}{k!}(l+k)! = \frac{d^l}{ds^l} \sum_{k=0}^{\infty} \frac{s^k}{k!} = \frac{d^l}{ds^l} \frac{s^l}{1-s} = \frac{l!}{(1-s)^{l+1}}\ ,$$

for $|s| < 1$. Therefore,

$$|F_{\eta_\iota^{-1}c}[u](\tau)| \le \frac{KM^l l!}{(1 - M(m+1)\tau)^{l+1}}\ ,$$

i.e.,

$$|\langle d,\ \eta_\iota\rangle| \le K_\tau M_\tau^l l! \quad \text{for} \quad \iota \in I^l,\ l \ge 0\ , \tag{10}$$

where

$$K_\tau = \frac{K}{1 - M(m+1)\tau}\ ,$$

$$M_\tau \;=\; \frac{M}{1-M(m+1)\tau}\ ,$$

and the constants M and K are as in (2). Since

$$\frac{1}{M_\tau(m+1)} \;=\; \frac{1}{M(m+1)} - \tau \;>\; T-\tau,$$

it follows that $T-\tau$ is admissible for d.

Formula (8) will follow from the following formula:

$$V_\rho[u\#_\tau v](t+\tau) = \sum_{\iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau), \tag{11}$$

for any $\rho \in I^*$, since if we assume (11) holds, then

$$\begin{aligned}
F_c[u\#_\tau v](t+\tau) \;&=\; \sum_\rho \langle c,\ \eta_\rho\rangle \sum_{\iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau) && (12)\\
&=\; \sum_\iota \sum_\kappa \langle c,\ \eta_\iota\eta_\kappa\rangle V_\iota[v](\tau)V_\kappa[u](t) && (13)\\
&=\; \sum_\iota \sum_\kappa \langle \eta_\iota^{-1}c,\ \eta_\kappa\rangle V_\kappa[u](\tau)V_\iota[v](t) &&\\
&=\; \sum_\iota F_{\eta_\iota^{-1}c}[u](\tau)V_\iota[v](t) &&\\
&=\; F_d[v](t). && (14)
\end{aligned}$$

Note here that we can rearrange the terms in (12) to get (13) because the series of functions in (12) is absolutely convergent for $0 \le \tau + t \le T$.

We now return to prove (11) by induction on the length of ρ. Equation (11) is true when $\rho = j \in I^1$ because

$$\begin{aligned}
&V_\rho[u\#_\tau v](t+\tau) \;=\; \int_0^{t+\tau} (u_j\#_\tau v_j)(s)\,ds\\
&= \int_0^\tau u_j(s)\,ds + \int_0^t v_j(s)\,ds = V_\rho[u](\tau) + V_\rho[v](t).
\end{aligned}$$

Now assume that (11) holds for $\rho \in I^n$. For any

$$\rho = i_1 i_2 \cdots i_{n+1} \in I^{n+1},$$

we have

$$\begin{aligned}
&V_\rho[u\#_\tau](t+\tau) = \int_0^{t+\tau} (u_{i_1}\#_\tau v_{i_1})(s)V_{i_2i_3\cdots i_{n+1}}[u\#_\tau v](s)\,ds\\
&= \int_0^\tau u_{i_1}(s)V_{i_2i_3\cdots i_{n+1}}[u\#_\tau v](s) + \int_0^t v_{i_1}(s)V_{i_2i_3\cdots i_{n+1}}[u\#_\tau v](\tau+s)\,ds\\
&= V_\rho[u](\tau) + \sum_{i_1\iota\kappa=\rho} \int_0^t v_{i_1}(s)V_\iota[v](s)V_\kappa(\tau)\,ds\\
&= V_\rho[u](\tau) + \sum_{|\iota|\ge 1,\iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau)
\end{aligned}$$

$$= \sum_{\iota\kappa=\rho} V_\iota[v](t)V_\kappa[u](\tau).$$

We completed the proof of (11) by induction. ∎

We now return to prove Lemma 2.3.

Proof. Take an analytic control u in $\mathcal{V}_T$. First notice that if T is admissible for c, then there exists some $\varepsilon > 0$ such that $T+\varepsilon$ is also admissible for c. Assume ε is so small that u is analytic in $(-\varepsilon,\, T+\varepsilon)$. Let $T_1 = T+\varepsilon$ and

$$\tilde{u}(z) = (\tilde{u}_1(z), \ldots, \tilde{u}_m(z))$$

be the complex analytic function whose restriction to the real interval $(-\varepsilon, T_1)$ is u. One would like to say that the output is the restriction to real t of the complex output corresponding to $\tilde{u}$, from which analyticity would follow by the above differentiability (extended to the complexes). However, it is not necessary that $\tilde{u}$ be bounded by 1 for $|z| \le T_1$, so a local analysis is needed.

For any $0 \le t_0 \le T$, there exist some $\delta > 0$ and $\sigma > 0$ such that

$$|\tilde{u}(z)|M(m+1)T_1 \;\le\; 1-\sigma \quad \text{if} \quad z \in B_\delta(t_0)$$

since

$$|\tilde{u}(t_0)|M(m+1)T_1 \;<\; 1$$

where $B_\delta(t_0)$ is the ball of radius δ centered at t_0 and M is as in (2). Assume here that $\delta < \varepsilon$. By Lemma 2.4,

$$F_c[u](t) = F_d[v](t-t_0)$$

where d is defined as in (7) and $v(t) = u(t+t_0)$. Let $\tilde{v}(z) = \tilde{u}(z+t_0)$. For any complex vector function $w(z) = (w_1(z), \ldots, w_m(z))$ of dimension m which is defined and analytic for all z in a ball around the origin, we define

$$V_{i_1\cdots i_{l+1}}[w](z) = \int_0^z w_{i_1}(s)V_{i_2\cdots i_{l+1}}(s)\,ds,$$

inductively, where $V_\phi = 1$ and $w_0(z) :\equiv 1$. By induction, the integrand is analytic, so the integral is independent of the path and the result in analytic too. Then

$$\left|\sum_l \langle d,\ \eta_\iota\rangle V_l[\tilde{v}](z-t_0)\right| \;\le\; \sum_{l=0}^{\infty} K_{t_0}M_{t_0}^l(m+1)^l N^l|z-t_0|^l \tag{15}$$

where M_{t_0}, K_{t_0} are defined as in the proof of Lemma 2.4 and

$$N = \max_{z\in B_\delta(t_0)} \tilde{u}(z).$$

Notice that the series

$$\sum_{l=0}^{\infty} M_{t_0}^l(m+1)^l N^l|z-t_0|^l$$

converges uniformly for

$$|z-t_0| \;\le\; \frac{1-\sigma}{NM_{t_0}(m+1)}\ .$$

Now let

$$\tilde{\delta} = \min\left\{\delta, \ \frac{1-\sigma}{NM_{t_0}(m+1)}\right\}.$$

Then the series of complex functions

$$\sum \langle d, \ \eta_\iota\rangle V_l[\tilde{v}](z-t_0)$$

defines an analytic function in $B_{\tilde{\delta}}(t_0)$ since it converges uniformly, (cf: Theorem 5.1 in [1]). For t real,

$$F_d[v](t-t_0) = \sum \langle d, \ \eta_\iota\rangle V_\iota[v](t-t_0) = \sum \langle d, \ \eta_\iota\rangle V_\iota[\tilde{v}](t-t_0).$$

Thus, $F_d[v](t-t_0)$, i.e. $F_c[u](t)$, is analytic for $|t-t_0| < \tilde{\delta}$. Since t_0 can be chosen arbitrarily in $[0, \ T]$, we get the desired conclusion. ∎

Observe here that we have not claimed the following stronger statement: if u has a single convergent power series representation on $[0, \ T]$ then $F_c[u]$ also does. We only proved that $F_c[u]$ is analytic, that is, it has a local power series expansion around each point. The following example shows that the above stronger statement is not true in general.

Example 2.5 Consider the series

$$c = 1 + \eta_1 + 2\eta_1^{(2)} + 3!\eta_1^{(3)} + \cdots + k!\,\eta_1^{(k)} + \cdots \ .$$

It is not hard to see that any $T < 1$ is admissible for c, and

$$\begin{aligned} F_c[u](t) &= \sum_{k=0}^{\infty} k! \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} u(s_1)\cdots u(s_k)\ ds_k \cdots ds_1 \\ &= \sum_{k=0}^{\infty} \left(\int_0^t u(s)\,ds\right)^k = \frac{1}{1-\int_0^t u(s)\,ds}. \end{aligned}$$

Let $u = -\sin \pi t$. Then $\displaystyle\int_0^t u(s)\,ds = \frac{2}{\pi}\sin^2\frac{\pi}{2}t$. Hence

$$F_c[u](t) = \frac{\frac{\pi}{2}}{\frac{\pi}{2} + \sin^2\frac{\pi}{2}t} \ .$$

Consider the equation

$$\frac{\pi}{2} + \sin^2\frac{\pi}{2}\theta = 0 \tag{16}$$

on the complex plane. When $\theta = bj$ where $j = \sqrt{-1}$, equation (16) becomes

$$\frac{e^{b\pi/2} - e^{-b\pi/2}}{2} = \pm\sqrt{\frac{\pi}{2}} \ .$$

Let $f(b) = \frac{e^{b\pi/2} - e^{-b\pi/2}}{2}$. Then $f(0) = 0$ and

$$f(1) = \sum_{k=0}^{\infty} \frac{(\frac{\pi}{2})^{2k+1}}{(2k+1)!} \geq \frac{\pi}{2} \geq \sqrt{\frac{\pi}{2}} \ .$$

Therefore there exists some $b \in (0,\, 1)$ such that $f(b) \,=\, \sqrt{\frac{\pi}{2}}$ which implies that there exists some $\theta \in \mathbb{C}$ with $\|\theta\| < 1$ such that (16) holds. Therefore, the complex function

$$g(z) \;=\; \frac{\frac{\pi}{2}}{\frac{\pi}{2} \,+\, \sin^2 \frac{\pi}{2} z}$$

has at least one singularity inside the unit disc. It then follows that $F_c[u](t)$ cannot have a global convergent power series representation on $[0,\, T]$ if $0 \,<\, 1 - T \,<\, \delta$ for δ small enough, even though u has a global convergent power series representation. $\square$

3 Realizability by Analytic Systems

For any given convergent series c, we say that F_c is *realizable by an analytic system*

$$\Sigma = (\mathcal{M},\, (g_0,\, \ldots,\, g_m), x_0,\, h) \tag{17}$$

if there exist some analytic manifold $\mathcal{M}$, some $x_0 \in \mathcal{M}$, $(m+1)$ analytic vector fields

$$g_0,\, g_1,\, \ldots, g_m$$

on $\mathcal{M}$ and an analytic function

$$h:\; \mathcal{M} \;\to\; \mathbb{R}$$

such that for each $u \in \mathcal{V}_T$ with T admissible for c, there exists a solution $x(\cdot)$ of the equation

$$\begin{aligned} x' \;&=\; g_0(x) \,+\, \sum_{j=1}^{m} g_j(x)\, u_j \quad , \\ x(0) \;&=\; x_0 \end{aligned}$$

defined on all of $[0,\, T]$, and

$$F_c[u](t)(t) \;=\; h(x(t))\,, \;\; t \in [0,\, T]\,. \tag{18}$$

We shall say that F_c is *locally* realizable by an analytic system (17) if the solution $x(\cdot)$ of (17) is only defined for, and (18) only holds for, t small enough.

For any given power series c, we define the *observation space* $\mathcal{F}_1(c)$ as the $\mathbb{R}$-space spanned by all the series $\alpha^{-1}c$, the *observation algebra* $\mathcal{A}_1(c)$ is the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_1(c)$, under the shuffle product (see [15]), and the *observation field* $\mathcal{Q}_1(c)$ is the quotient field of $\mathcal{A}_1(c)$. Note that $\mathcal{Q}_1(c)$ is always defined since $\mathcal{A}_1(c)$ is an integral domain (cf [15]).

For any given convergent power series c, we say that $\mathcal{A}_1(c)$ is an *analytically finitely generated* $\mathbb{R}$-algebra if there exist an integer n and n elements $c_1,\, c_2,\, \ldots,\, c_n$ of $\mathcal{A}_1(c)$ such that for every element d in $\mathcal{A}_1(c)$, there exists some analytic function φ defined on $\mathbb{R}^n$ such that

$$F_d[u](t) \;=\; \varphi\,(F_{c_1}[u](t),\, \ldots,\, F_{c_n}[u](t))$$

for all $u \in \mathcal{V}_T,\; t \in [0,\, T]$ and for any T admissible for c.

We say that the observation field $\mathcal{Q}_1(c)$ is a *meromorphically finitely generated field extension* *of* $\mathbb{R}$ if there exists an integer n and

$$c_1,\, c_2,\, \ldots,\, c_n \,\in \mathcal{A}_1(c)$$

such that for each element d in $\mathcal{Q}_1(c)$, there exist some analytic functions φ_0 and φ_1 defined on $\mathbb{R}^n$ such that

$$\varphi_0\left(F_{c_1}[u](t),\, \ldots,\, F_{c_n}[u](t)\right)\, F_d[u](t) \;=\; \varphi_1\left(F_{c_1}[u](t),\, \ldots,\, F_{c_n}[u](t)\right)$$

for all $u \in \mathcal{V}_T,\ t \in [0,\, T]$ and for any T admissible for c, and,

$$\varphi_0\left(F_{c_1}[u],\, \ldots,\, F_{c_n}[u]\right) \,\neq\, 0$$

for some $u \in \mathcal{V}_T$, and some T admissible for c. If this is the case, we call $c_1,\, \ldots, c_n$ the generators of the field, or, we say that the field is generated by $c_1,\, \ldots,\, c_n$. Informally speaking, then, a meromorphically finitely generated field extension of $\mathbb{R}$ is one for which every element can be expressed as a meromorphic function of a finite set of generators.

The following Theorem shows that certain finiteness properties imply realizability.

Theorem 1 *Assume that* c *is a convergent series. Then*

(a) F_c *is realizable by an analytic system if* $\mathcal{A}_1(c)$ *is analytically finitely generated.*

(b) F_c *is locally realizable by an analytic system if* $\mathcal{Q}_1(c)$ *is a meromorphically finitely generated field extension of* $\mathbb{R}$.

Proof. (a): Assume that $\mathcal{A}_1(c)$ is generated by $c_1,\, c_2,\, \ldots,\, c_n$, for some integer n. It follows that there exist an analytic function g_{ij} such that

$$F_{\eta_j^{-1} c_i} \;=\; g_{ij}\left(F_{c_1},\, \ldots,\, F_{c_n}\right) \tag{19}$$

for any $i = 1, \ldots, n$, each $j = 0, 1, \ldots, m$, and an analytic function h such that

$$F_c \;=\; h\left(F_{c_1},\, F_{c_2},\, \ldots,\, F_{c_n}\right)\ .$$

Take $\mathcal{M}$ to be the Euclidean space $\mathbb{R}^n$ and let

$$x_0 \;=\; \left(\langle c_1,\, \phi\rangle,\, \langle c_2,\, \phi\rangle\, \ldots,\, \langle c_n,\, \phi\rangle\right)'\ . \tag{20}$$

It follows from (19) and formula (6) that the function

$$x(t) \;=\; \left(F_{c_1}[u](t),\, F_{c_2}[u](t),\, \ldots,\, F_{c_n}[u](t)\right)'$$

satisfies the equations

$$\begin{aligned} x' \;&=\; g_0(x) \,+\, \sum_{j=1}^{m} g_j(x)\, u_j\ \ , \\ x(0) \;&=\; x_0\ \ , \end{aligned}$$

for any $u \in \mathcal{V}_T$, and (20) implies that

$$y(t) \;=\; F_c[u](t) \;=\; h(x(t))$$

for all $t \in [0, T]$. We proved that F_c is realizable by an analytic system.

(b): Assume that $\mathcal{Q}_1(c)$ is meromorphically generated by $c_1, c_2, \ldots, c_n$. Without loss of generality, we may assume that $c_1 = c$. Similarly to part (a), one knows that for each $i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, m$, there exist analytic functions g_{ij} and q_{ij} such that

$$q_{ij}\left(F_{c_1}, \ldots, F_{c_n}\right) F_{\eta_j^{-1} c} = g_{ij}\left(F_{c_1}, \ldots, F_{c_n}\right) \tag{21}$$

and $q_{ij}\left(F_{c_1}, \ldots, F_{c_n}\right) \neq 0$. (These are all equations among operators.) Without loss of generality, we may assume that $q_{ij} = q$ for all i and j and

$$q\left(F_{c_1}[u], \ldots, F_{c_n}[u]\right) \neq 0\,. \tag{22}$$

It is not hard to see that for any $u \in \mathcal{V}_T$, the function

$$x(t) = \left(F_{c_1}[u](t), F_{c_2}[u](t), \ldots, F_{c_n}[u](t)\right)' \tag{23}$$

satisfies the equation

$$q(x(t))\, x'(t) = g_0(x(t)) + \sum_{j=0}^{m} g_j(x(t))\, u_j(t) \tag{24}$$

$$x(0) = x_0 = \left(\langle c_1, \phi\rangle, \langle c_2, \phi\rangle \ldots, \langle c_n, \phi\rangle\right)'\,, \tag{25}$$

and,

$$y(t) = F_c[u](t) = x_1(t).$$

It is clear that *if*

$$q(x_0) \neq 0\,, \tag{26}$$

then $q(x) \neq 0$ in a neighborhood $\mathcal{N}$ of x_0, thus, F_c is locally realized by the analytic system

$$(\mathcal{N}, \tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_m, x_0, h)\,,$$

where $\tilde{g}_i = \dfrac{g_i}{q}$ and $h(x) = x_1$.

We now assume $q(x_0) = 0$. Note that the function $x(\cdot)$ defined by (23) is of class $\mathcal{C}^k$ if u is of class $\mathcal{C}^{k-1}$ for any $k \geq 1$, and $x(\cdot)$ is analytic if u is analytic. Since analytic controls is dense in $\mathcal{V}_T$ (with respect to the L^1 topology), (22) implies that there is at least one analytic input u_0 for which the function

$$q\left(F_{c_1}[u_0], \ldots, F_{c_n}[u_0]\right)$$

is not identically zero. Fix such an u_0. Then the analyticity of $F_{c_i}[u](t)$ and the analyticity of q imply that there exists some $\delta > 0$ such that

$$q\left(F_{c_1}[u_0](t), \ldots, F_{c_n}[u_0](t)\right) \neq 0 \tag{27}$$

for all $t \in (0, \delta)$. For each $\lambda \in (0, \delta)$, we define a series c^λ by

$$\langle c^\lambda, \eta_\iota\rangle = F_{\eta_\iota^{-1} c}[u_0](\lambda).$$

By Lemma 2.4 c^λ is a convergent series and

$$F_{c^\lambda}[u](t) \;=\; F_c[u_0 \#_\lambda u](\lambda + t) \tag{28}$$

for any $u \in \mathcal{V}_{T-\lambda}$ and any $\lambda \in (0,\,\delta)$, which implies that, for each $u \in \mathcal{V}_{T-\lambda}$, the function

$$x^\lambda(t) \;=\; \Big(F_{c_1^\lambda}[u](t),\;\; F_{c_2^\lambda}[u](t),\;\; \ldots,\;\; F_{c_n^\lambda}[u](t)\Big)'$$

also satisfies equation (24) with the initial state

$$x^\lambda(0) \;=\; (F_{c_1}[u_0](\lambda),\; F_{c_2}[u_0](\lambda),\; \ldots,\; F_{c_n}[u_0](\lambda))'$$

and

$$F_c^\lambda[u](t) \;=\; x_1^\lambda(t)\;.$$

Notice that (28) means that

$$q(x_0^\lambda) \;\neq\; 0$$

for each $\lambda \in (0,\,\delta)$. thus for each fixed λ, there exists a neighborhood $\mathcal{N}_\lambda$ of x^λ such that

$$q(x) \;\neq\; 0, \;\;\text{for all } x \in \mathcal{N}_\lambda.$$

Hence, each of these "perturbed" operators F_{c^λ} is locally realized by the analytic system

$$\Big(\mathcal{N}_\lambda,\; (\tilde{g}_0,\, \tilde{g}_1,\, \ldots,\, \tilde{g}_m),\; x^\lambda,\; h\Big)\,.$$

To show that F_{c^λ} is still realizable when $\lambda = 0$, we need to introduce more notations. Let $\mathcal{P}$ be the set of polynomials in $\eta_0, \eta_1, \ldots, \eta_m$. We now define the *Lie bracket* $[\cdot\;,\;\cdot]$ on $\mathcal{P}$ as follows:

$$[P_1,\; P_2] \;=\; P_1 \cdot P_2 \;-\; P_2 \cdot P_1$$

where "$\cdot$" denotes the standard product defined for polynomials. With $[\cdot\;,\;\cdot]$ defined as above, $\mathcal{P}$ forms a Lie algebra. Let $\mathcal{L}$ be the subalgebra of $\mathcal{P}$ generated by $\eta_0, \eta_2, \ldots, \eta_m$. The elements of $\mathcal{L}$ will be called *Lie polynomials*.

We now define

$$\psi_c(w) \;=\; \sum \langle w,\; \eta_\kappa\rangle\, \eta_\kappa^{-1} c$$

for any polynomial $w = \sum \langle w,\; \eta_\kappa\rangle$. Now for a given series c, we define the *Lie rank* $\rho(c)$ of c to be the dimension of the $\mathbb{R}$-space spanned by $\psi_c(w)$, over all Lie polynomials $w \in \mathcal{L}$, i.e.,

$$\rho(c) \;=\; \dim\,(\mathrm{span}_{\,\mathbb{R}}\,\{\psi_c(w) :\; w \in \mathcal{L}\})\;\;.$$

It is well-known that the i/o operator F_c is locally realizable by an analytic system if and only if the Lie rank $\rho(c)$ is finite, and, if F_c is realizable by a system of dimension n, then the Lie rank $\rho(c)$ is less than or equal to n (cf [3], [6], [12]).

It follows from the second part of the above statement that the Lie rank $\rho(c^\lambda)$ of c^λ is bounded by n for any $\lambda \in (0,\,\delta)$.

Claim: $\;\rho(c) \;\leq\; n.$

Suppose $\rho(c) > n$. Then there exist $w_1, \ldots, w_{n+1}$ such that the $n+1$ series

$$\psi_c(w_1),\; \ldots,\; \psi_c(w_{n+1})$$

are linearly independent.

We now enumerate the elements of P^*, the set of all monomials in η_i's, as z_1, z_2, ... and we let A_λ be the matrix of $n+1$ columns and infinitely many rows whose (i,j)-th entry is $\langle \psi_{c^\lambda}(w_j),\ z_i\rangle$. Then A_0 is full column rank in the sense that there is no nonzero $(n+1)$-vector v such that $A_0\, v = 0$. For any integer i, let $\mathcal{B}_i$ be the subspace of $\mathbb{R}^{n+1}$ spanned by the first i row vectors of A. Then we have

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}_i \subseteq \cdots \ .$$

Since $\operatorname{rank} B_i$ is bounded by $n+1$ for any i, it follows that there exists some k such that $\mathcal{B}_i = \mathcal{B}_k$ for any $i \geq k$, which implies that $A = TB_k$ for some matrix T of infinite rows and $n+1$ columns, where B_k is the $k \times (n+1)$ submatrix of A consisting the first k rows of A. Hence there is no nonzero $(n+1)$-vector v such that $B_k v = 0$, that is, $\operatorname{rank} B_k = n+1$.

Now let the B_λ be the submatrix of A_λ consisting of the first k rows of A_λ. Note that for the (ij)-th entry a_{ij}^λ of B_λ, we have

$$a_{ij}^\lambda \ = \ \langle \psi_{c^\lambda}(w_j),\ z_i\rangle \ = \ \langle c^\lambda,\ w_j z_i\rangle \ = \ F_{(w_j z_i)^{-1}c}[u_0](\lambda).$$

Thus the entries of matrix B_λ are continuous functions of λ, from which it follows that the rank of B_λ is a semicontinuous function of λ. Therefore,

$$\operatorname{rank}\ B_\lambda \ = \ n+1 \tag{29}$$

for $\lambda \in (0,\, \varepsilon)$ for some $\varepsilon > 0$. It follows immediately from (29) that the series

$$\psi_{c^\lambda}(w_1),\ \psi_{c^\lambda}(w_2)\ \ldots, \psi_{c^\lambda}(w_{n+1})$$

are linearly independent. This contradicts the fact that $\rho(c^\lambda)$ is bounded by n. Therefore, the Lie rank $\rho(c)$ is bounded by n. Applying the results in [3], [6], [12] cited above, we conclude that F_c is locally realizable by an analytic system. Furthermore, since $\rho(c) \leq n$, we know that also F_c is locally realizable by a system of dimension less than or equal to n. ∎

4 Analytic I/O Equations

In complete analogy with the algebraic case ([15]), an *analytic input/output equation of order* k is an equation of the type

$$A\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0, \tag{30}$$

where A is an analytic function defined on $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{(k+1)}$ and nontrivial in the last variable. The latter means that there exists some point

$$\bar{\mu}_0,\, \bar{\mu}_1,\, \ldots, \bar{\mu}_k,\, \bar{\nu}_0,\, \ldots,\, \bar{\nu}_{k-1}$$

in $\mathbb{R}^{m(k+1)} \times \mathbb{R}^k$ such that

$$A\left(\bar{\mu}_0,\, \bar{\mu}_1,\, \ldots, \bar{\mu}_k,\, \bar{\nu}_0,\, \ldots,\, \bar{\nu}_{k-1},\, \nu_k\right)$$

is not a constant function. Generalizing the notions in the algebraic case, we also consider two special classes of i/o equations, as follows. First, *meromorphic* equations are those for which $A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k)$ takes the form:

$$A_0(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1})\nu_k + A_1(\mu_0, \ldots, \mu_k, \nu, \ldots, \nu_{k-1}), \tag{31}$$

In other words,

$$\frac{\partial^2 A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k)}{\partial \nu_k^2} = 0$$

identically, the term "meromorphic" referring to the fact that ν_k is meromorphic in the remaining variables. Second, we consider *analytically recursive* equations, those for which $A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k)$ takes the form

$$A_0(\mu_0, \ldots, \mu_k)\nu_k + A_1(\mu_0, \ldots, \mu_k, \nu, \ldots, \nu_{k-1}); \tag{32}$$

the term is used because ν_k can be expressed analytically in terms of the other ν_i's. Given an i/o operator F_c, where c is a convergent series, F_c will be said to *satisfy* the analytic i/o equation (30) if (30) holds for each $\mathcal{C}^k$ i/o pair (u, y) of F_c (and then we call (30) an *analytic i/o equation for* F_c, and in particular we say that the operator F_c satisfies an *analytically recursive i/o equation* if there is some such equation for which A is analytically recursive, and that F_c satisfies a *meromorphic i/o equation* if A can be chosen meromorphic and minimal in the sense that $A_0 = 0$ is not an i/o equation of F_c.)

Lemma 4.1 Let $A : \mathbb{R}^{m(k+1)} \times \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ be real-analytic, and assume that F_c is an i/o operator. Then, the property

$$A\left(\mu_0, \ldots, \mu_k, F_c, F_{c_1(\mu_0)}, \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) = 0, \tag{33}$$

holds for each $\mu_0, \mu_1, \ldots, \mu_k \in \mathbb{R}^m$ if and only if F_c satisfies the i/o equation (30).

Remark 4.2 Equation (33) means that

$$A\left(\mu_0, \ldots, \mu_k, F_c[u](t), F_{c_1(\mu_0)}[u](t), \ldots, F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](t)\right) = 0, \tag{34}$$

for any $u \in \mathcal{V}_T$ and any $t \in [0, T]$. By definition, (34) holds for those u such that $u\#_t\omega_\mu \in \mathcal{C}^k$. But in general, $u\#_t\omega_\mu$ is not of class $\mathcal{C}^k$. The following proof of Lemma 4.1 in fact shows that equation (30) holds for all $\mathcal{C}^k$ i/o pairs of F_c if and only if it holds at any point at which $u(t)$ is $\mathcal{C}^k$, for every $u \in \mathcal{V}_T$. □

To prove Lemma 4.1, we need the following Lemma:

Lemma 4.3 Assume $f \in \mathcal{C}[0, 1]$ and $f(0) = 0$. Then for each given integer $n \geq 0$, there exists a sequence of polynomial functions $\{f_k\}$ such that

$$f_k \longrightarrow f, \quad \text{as } k \to \infty$$

uniformly and $f_k^{(i)}(0) = 0$ for all k and $0 \leq i \leq n-1$.

Proof. Suppose $f \in \mathcal{C}[0, 1]$ and $f(0) = 0$. Let $\hat{f}(x) = f(x^{1/n})$. Then, by Weierstrass' Theorem, there exists a sequence of polynomials approaching to $\hat{f}$; in particular, the sequence of polynomials can be chosen as the Bernstein polynomials

$$\hat{f}_k(x) = \sum_{j=0}^{k} \hat{f}\left(\frac{j}{k}\right)\binom{k}{j} x^j (1-x)^{k-j}.$$

Notice that

$$\hat{f}_k(0) = 0$$

for all k. Now let $f_k(x) = \hat{f}_k(x^n)$. Then

$$f_k(x) \longrightarrow f(x) \quad \text{as } k \to \infty$$

uniformly and f_k's are polynomials of x^n. Since $f_k(0) = 0$, it follows that $f_k^{(i)}(0) = 0$ for $0 \le i \le n-1$. ∎

We are now ready to prove Lemma 4.1.

Proof. Assume equation (34) holds for any $u \in \mathcal{V}_T$ and $t \in [0, T]$. Take an input

$$u \in \mathcal{V}_T \bigcap \mathcal{C}^{k-1}$$

and pick any $t \in [0, T]$. Assume

$$u^{(s)}(t) = \mu_s$$

for $0 \le s \le k$. Then

$$y^{(s)}(t) = \frac{\partial^s}{\partial t^s} F_c[u](t) = F_{c_s(\mu_0, \ldots, \mu_{s-1})}[u](t)$$

for $0 \le s \le k$. Therefore,

$$P\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0.$$

Since u and t can be picked arbitrarily, it follows that $A = 0$ is an i/o equation for F_c.

Now assume $A = 0$ is an i/o equation for F_c. Take any fixed $u \in \mathcal{V}_T$ and consider

$$\hat{u} := u \#_t \omega_\mu,$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_k)$. We shall prove the Lemma by first showing that (34) holds for all $u \in \mathcal{V}_T$ and t and all those μ's such that $|\mu_0| < 1$. For this purpose, we will first find a sequence $\{v_j\} \in \mathcal{C}^k$ such that

$$\|v_j - \hat{u}\|_1 \to 0, \text{ as } j \to \infty, \tag{35}$$

$$v_j(s) = \hat{u}(s) = \omega(s-t) \quad \text{for } s \ge t,$$

and there exists some fixed $\delta > 0$ such that

$$|v_j(s)| < 1 \quad \text{for } s \in [0, t+\delta]$$

for all large enough j.

Assuming for now that there exists such a sequence $\{v_j\}$, we show how to complete the proof. The output $F_c[v_j]$ is defined and differentiable in $[0, t+\delta]$ for large enough j. Applying

(30) to the $\mathcal{C}^k$ pair (v_j, y_j) at time t, we get

$$A\left(v_j(t), \ldots, v_j^{(k)}(t), y_j(t), \ldots, y_j^{(k)}(t)\right) = 0, \tag{36}$$

where $y_j = F_c[v_j]$.

By Lemma 2.2, we know that for $t \le \tau \le t + \delta$ for some $\delta > 0$

$$y_j^{(s)}(\tau) \;\to\; \frac{d^s}{dt^s} F_c[\hat{u}](\tau) \quad \text{as} \quad j \to \infty.$$

Letting

$$j \;\to\; \infty \quad \text{and} \quad \tau \;\to\; t^+$$

and taking the limits on both sides of (36), we get

$$A\left(\mu_0, \ldots, \mu_{k-1}, F_c[\hat{u}](t),\; \frac{d}{d\tau}\bigg|_{\tau=0^+} F_c[u](t+\tau),\; \ldots,\; \frac{d^k}{d\tau^k}\bigg|_{\tau=0^+} F_c[\hat{u}](t+\tau)\right) = 0.$$

Since u and t can be chosen arbitrarily, we get (34) under the assumption that $|\mu_0| < 1$. To remove this assumption, notice that for any fixed $u \in \mathcal{V}_T$ and t, the function

$$A\left(\mu_0, \ldots, \mu_{k-1}, F_c[u](t), F_{c_1(\mu_0)}[u](t), \ldots, F_{c_k(\mu_0, \ldots, \mu_{k-1})}[u](t)\right) \tag{37}$$

is analytic in μ. Hence (34) holds for *all* μ, not only for those with $|\mu_0| < 1$. Again, as u and t can be chosen arbitrarily, we get (33).

Now we return to prove the existence of $\{v_j\}$. Take $u \in \mathcal{V}_T$ and $\mu_0, \ldots, \mu_k \in \mathbb{R}^m$, where $|\mu_0| < 1$. Notice that the set

$$\hat{\mathcal{V}}_T := \left\{u \in \mathcal{V}_T \cap \mathcal{C}[0,\, T] : u(t) = \mu_0,\; \|u\|_\infty < 1\right\}$$

is dense in $\mathcal{V}_T$ (still using the L^1 norm), so we may assume that $u \in \hat{\mathcal{V}}_T$. Now let

$$\tilde{u}(s) = \begin{cases} u(s) - \omega_\mu(s-t) & \text{if } s \le t, \\ 0 & \text{if } t < s \le T. \end{cases}$$

By Lemma 4.3, there exists a sequence $\tilde{v}_j$ in $\mathcal{C}^k[0,\, T]$ such that

$$\tilde{v}_j(s) \;\to\; \tilde{u}(s)$$

uniformly and $\tilde{v}_j^{(i)}(t) = 0$ for $0 \le i \le k$. Let

$$v_j(s) = \tilde{v}_j(s) + \omega_\mu(s-t).$$

Then $\{v_j\} \to u\#_t\omega_\mu$ uniformly. Since

$$|u\#_t\omega_\mu(s)| < 1 \quad \text{for} \quad s \in [0,\, t+\delta]$$

for some small δ, it follows that $|v_j(s)| < 1$ for $s \in [0,\, t+\delta]$ if j is large enough. ∎

Similar to the algebraic case discussed in [15], we have the following result:

Lemma 4.4 If F_c satisfies an analytic i/o equation, then it satisfies a meromorphic i/o equation.

Proof. Assume that F_c satisfies an analytic i/o equation of order k;

$$A\left(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)\right) = 0.$$

Without loss of generality, we may assume that k is smallest possible among all analytic i/o equations for F_c. Now let, for each $i \geq 0$,

$$Q_i(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k) = \frac{\partial^i}{\partial \nu_k^i} A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_k).$$

Claim: There exists some i such that $Q_i = 0$ is not an i/o equation for F_c.

Suppose by way of contradiction that $Q_i = 0$ is an i/o equation of F_c for all i, i.e.,

$$Q_i(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}(t)) = 0 \tag{38}$$

for all $\mathcal{C}^k$ i/o pairs (u, y) of F_c and for all $i \geq 0$. For each fixed u and each fixed t, we let

$$(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_k) = (u(t), \ldots, u^{(k)}(t), y(t), \ldots, \ldots, y^{(k)}(t)).$$

Then (38) means that

$$\frac{\partial^i}{\partial \nu_k} A(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1}, \bar{\nu}_k) = 0$$

for all i. It then follows from the analyticity of A that

$$A(\bar{\mu}_0, \ldots, \bar{\mu}_k, \bar{\nu}_0, \ldots, \bar{\nu}_{k-1}, \alpha) = 0$$

for all $\alpha \in \mathbb{R}$. Since u and t can be chosen arbitrarily, one concludes that

$$A(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t), \alpha) = 0$$

for any $\mathcal{C}^k$ i/o pair (u, y) of F_c and any constant α. Choose an α such that the function

$$A_1(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}) := A(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}, \alpha)$$

is not identically zero. (Such an α exists because A is not identically zero, by definition of i/o equations.) It follows immediately that

$$A_1(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k-1)}(t)) = 0 \tag{39}$$

for all i/o pairs of F_c, and this is a nontrivial equation, by the choice of α. We may assume that

$$\frac{\partial}{\partial \nu_j} A_1(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}) \neq 0$$

for some $j = 0, 1, \ldots, k-1$. Otherwise, if

$$\frac{\partial}{\partial \nu_i} A_1(\mu_0, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1}) = 0$$

for any $i = 0, 1, \ldots, k-1$, then there exist some $\nu_0, \ldots, \rho_{k-1}$ so that

$$A_2(\mu_0, \mu_1, \ldots, \mu_k) = A_1(\mu_0, \mu_1, \ldots, \mu_k, \nu_0, \ldots, \nu_{k-1})$$

is not identically zero and

$$A_2(u(t), u'(t), \ldots, u^{(k)}(t)) = 0$$

holds for all input functions and all t, which is impossible. Let j be as large as possible. Noting the fact that $y^{(j)}$ does not depend on $u^{(i)}$ for any $i > j$, one knows that there exists some analytic function

$$A_3(\mu_0, \ldots, \mu_j, \nu_0, \ldots, \nu_j)$$

such that $A_3 = 0$ is an i/o equation for F_c of order $j < k$, which contradicts the assumed minimality of k.

Thus we proved by induction that there exists some i such that $Q_i = 0$ is not an i/o equation of F_c. Now let $r \geq 1$ be the smallest number for which $Q_r = 0$ is not an i/o equation of F_c. Then

$$Q_{r-1}(u(t), \ldots, u^{(k)}(t), y(t), \ldots, y^{(k)}) = 0 \tag{40}$$

is an i/o equation of F_c. Taking derivative with respect to time t on both sides of (40), one sees that F_c satisfies the following meromorphic i/o equation:

$$\begin{aligned} Q_r \left(u(t), \ldots, u_k(t), y(t), \ldots, y^{(k)}\right) y^{(k+1)} \\ = P_3 \left(u(t), \ldots, u_k(t), y(t), \ldots, y^{(k)}\right) \,, \end{aligned}$$

where P_3 is some analytic function defined in $\mathbb{R}^{m(k+1)} \times \mathbb{R}^{(k+1)}$. ∎

For any given series c, the observation space $\mathcal{F}_2(c)$ is defined to be the $\mathbb{R}$-space spanned by $c_n(\mu_0, \ldots, \mu_{n-1})$ for all n and all values of $\mu_0, \ldots, \mu_{n-1}$. For the definition of $c_n(\mu_0, \ldots, \mu_{n-1})$ we refer the readers to [16]. Roughly speaking, $c_n(\mu_0, \ldots, \mu_{n-1})$ was defined in such a way that for any convergent power series c,

$$\frac{d^n}{dt^n} F_c[u](t) = F_{c_n(u(t), \ldots, u^{n-1}(t))}[u](t), \tag{41}$$

for any continuous $u \in \mathcal{V}_T$ such that T is admissible for c. The observation algebra $\mathcal{A}_2(c)$ is defined to be the $\mathbb{R}$-algebra generated by the elements of $\mathcal{F}_2(c)$, and the observation field $\mathcal{Q}_2(c)$ is defined to be the quotient field of $\mathcal{A}_2(c)$. Again, $\mathcal{Q}_2(c)$ is always well defined since $\mathcal{A}_2(c)$ is an integral domain (cf [15]).

The following Theorem relates the existence of analytic i/o equations to analytic finiteness properties of the observation algebra and field. An analogous result holds in the algebraic case (see [15]).

Theorem 2 *Assume c is a convergent power series. Then*

(a) $\mathcal{Q}_2(c)$ is meromorphically finitely generated if F_c satisfies an analytic i/o equation;

(b) $\mathcal{A}_2(c)$ is analytically finitely generated if F_c satisfies an analytically recursive i/o equation.

Proof. We shall only provide the proof of part (a). Part (b) can be proved by the same approach.

Assume F_c satisfies an analytic i/o equation. By Lemma 4.4, F_c satisfies a meromorphic i/o equation

$$\begin{aligned} A_0\left(u(t),\,\ldots,\,u^{(k+1)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right)\,y^{(k+1)}(t) \\ = A_1\left(u(t),\,\ldots,\,u^{(k+1)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right). \end{aligned} \tag{42}$$

Taking derivative with respect to time t on both sides of the equation, one gets

$$\begin{aligned} & A_0\left(u(t),\,\ldots,\,u^{(k+1)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right)\,y^{(k+2)}(t) \\ &= \tilde{A}_2\left(u(t),\,\ldots,\,u^{(k+2)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right) \\ &+ \hat{A}_2\left(u(t),\,\ldots,\,u^{(k+2)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right)\,y^{(k+1)}(t), \end{aligned} \tag{43}$$

where

$$\begin{aligned} \tilde{A}_2 &= \sum_{i=0}^{k+1} \frac{\partial}{\partial \mu_i}(A_1 - A_0)\,\mu_{i+1} + \sum_{i=0}^{k-1} \frac{\partial}{\partial \nu_i}(A_1 - A_0)\,\nu_{i+1}\,, \\ \hat{A}_2 &= \frac{\partial}{\partial \nu_k}\,(A_1 - A_0)\,. \end{aligned}$$

Multiplying by A_0 on both sides of (43) and replacing $y^{(k+1)}$ by (42), one knows that there exists some analytic function A_2 such that

$$\begin{aligned} A_0^2\left(u(t),\,\ldots,\,u^{(k+1)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right)\,y^{(k+2)}(t) \\ = A_2\left(u(t),\,\ldots,\,u^{(k+2)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right). \end{aligned}$$

Using the above arguments repeatedly, one proves that for each $r > 0$ there exists some analytic function A_r so that

$$\begin{aligned} A_0^r\left(u(t),\,\ldots,\,u^{(k+1)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right)\,y^{(k+r)}(t) \\ = A_r\left(u(t),\,\ldots,\,u^{(k+r)}(t),\,y(t),\,\ldots,\,y^{(k)}(t)\right). \end{aligned}$$

According to Lemma 4.1, we have

$$\begin{aligned} & A_0^r\left(\mu_0,\,\ldots,\,\mu_{k+1},\,F_c,\,\ldots,\,F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right)\,F_{c_{k+r}(\mu_0,\ldots,\mu_{k+r-1})} \\ &= A_r\left(\mu_0,\,\ldots,\,\mu_{k+r},\,F_c,\,\ldots,\,F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) \end{aligned} \tag{44}$$

for any $r > 0$ and any $\mu_0,\,\ldots,\,\mu_{k+r} \in \mathbb{R}^m$. Let

$$\begin{aligned} \Omega = \Big\{(\mu_0,\,\ldots,\,\mu_{k+1}) : \\ A_0\left(\mu_0,\,\ldots,\,\mu_{k+1},\,F_c,\,\ldots,\,F_{c_k(\mu_0,\ldots,\mu_{k-1})}\right) \neq 0\Big\}. \end{aligned}$$

It follows from the fact that $A_0 = 0$ is not an i/o equation of F_c that there exists some

$$(\mu_0,\, \ldots,\, \mu_{k+1}) \in \mathbb{R}^{m(k+2)},$$

some $u \in \mathcal{V}_T$, and some $\tau \in [0,\, T]$ so that

$$A_0\left(\mu_0,\, \ldots,\, \mu_{k+1},\, F_c[u](\tau),\, \ldots,\, F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](\tau)\right) \neq 0.$$

Since the function

$$\begin{aligned}&\psi(\mu_0,\, \mu_2\, \ldots,\, \mu_{k+1}) \\ &\qquad := A_0\left(\mu_0,\, \ldots,\, \mu_{k+1},\, F_c[u](\tau),\, \ldots,\, F_{c_k(\mu_0,\ldots,\mu_{k-1})}[u](\tau)\right)\end{aligned}$$

is an analytic function,

$$\Omega_1 := \left\{(\mu_0,\, \ldots,\, \mu_{k+1}) :\ \psi(\mu_0,\, \ldots,\, \mu_{k+1}) \neq 0\right\}$$

is an open dense subset of $\mathbb{R}^{m(k+2)}$. As $\Omega_1 \subseteq \Omega$, Ω is itself an open dense set of $\mathbb{R}^{m(k+2)}$.

Now we let Φ be the set of all the coefficients of μ_{ij} that appear in $c_n(\mu_0, \ldots, \mu_{n-1})$, seen as a polynomial in μ_{ij} over the ring of power series in variables $\eta_0, \ldots, \eta_m$ for all $n \leq k+1$. Note that Φ is a finite set of power series.

Pick up an arbitrary $r \geq 2$. Equation (44) implies that $c_{k+r}(\mu_0,\, \ldots,\, \mu_{k+r-1})$ is meromorphically generated by the elements of Φ if

$$(\mu_0,\, \mu_1,\, \ldots,\, \mu_{k+1}, \mu_{k+2},\, \ldots,\, \mu_{k+r-1}) \in \Omega \times \mathbb{R}^{r-2}\ .$$

Since Ω is dense in $\mathbb{R}^{k+2}$, it follows that $\Omega \times \mathbb{R}^{r-2}$ is dense in $\mathbb{R}^{k+r}$. By Lemma 12.11 in [10], we know that for any choice of $\mu_0, \ldots, \mu_{k+r-1}$,

$$c_{k+r}(\mu_0,\, \mu_1, \ldots, \mu_{k+r}) \tag{45}$$

is a linear combination of the elements in the set

$$\mathcal{B} := \left\{c_{k+r}(\mu_0,\, \ldots,\, \mu_{k+r-1}) :\ (\mu_0,\, \ldots, \mu_{k+r-1}) \in \Omega \times \mathbb{R}^{r-2}\right\}\ .$$

It follows immediately that (45) is meromorphically generated by the elements of Φ for any $\mu_0, \ldots, \mu_{k+r-1}$. Since r can be chosen arbitrarily, we get our conclusion that all of $\mathcal{Q}_2(c)$ is meromorphically generated by the finite set Φ. ∎

5 Main Results

In the previous section we showed that the realizability of an operator is closely related to the finiteness properties of $\mathcal{A}_1(c)$ and $\mathcal{Q}_1(c)$, while the existence of i/o equations is closely related to the finiteness properties of $\mathcal{A}_2(c)$ and $\mathcal{Q}_2(c)$. One of the main results in [16] shows that $\mathcal{F}_1(c) = \mathcal{F}_2(c)$, which implies that $\mathcal{A}_1(c) = \mathcal{A}_2(c)$ and consequently, $\mathcal{Q}_1(c) = \mathcal{Q}_2(c)$. Combining Theorem 2 and Theorem 1 we get our main results in this work:

Theorem 3 *Assume that c is a convergent series. Then*

(a) *F_c is realizable by an analytic system if F_c satisfies an analytically recursive i/o equation.*

(b) *F_c is locally realizable by an analytic system if F_c satisfies an analytic i/o equation.* ∎

6 Remarks

We have shown in [15] that the existence of an algebraic i/o equation is *equivalent* to realizability by a singular polynomial system. However, in contrast to its algebraic analogue, the converse of Theorem 3 does not hold in the analytic case. By a small modification of a construction due to Respondek, it is shown in [14] that the following system defines an i/o operator F so that $F[u]$ is defined for $0 \le t \le 1$ and all u for which $\|u_2\|_\infty \le 1$, but F does not satisfy any analytic i/o equation. The example, with $\mathcal{M} = \mathbb{R}^3$, is as follows:

$$\begin{aligned} x_1' &= u_1 \\ x_2' &= u_2 \\ x_3' &= u_3 \end{aligned}$$

$$h(x) \;=\; e^{x_1} \sum_{k=0}^{\infty} a_k f_k(x_2) \, \frac{x_3^k}{k!} \;,$$

with initial state $x(0) = 0$. The functions f_k and coefficients a_k are defined via

$$f_k(x) \;=\; \underbrace{\exp(\exp(\cdots(\exp(x))\cdots))}_{k}$$

for $k \ge 1$, and $f_0(x) = 1$, and $a_k = (f_k(1))^{-1}$, $k = 0, 1, \ldots$.

It is possible to provide partial converses to Theorem 3, however. One possibility is to consider the mapping sending vectors

$$(x, u(0), u'(0), \ldots, u^{(n-1)}(0)) \;,$$

consisting of states and $(n-1)$-jets of inputs, into n-jets of outputs and $(n-1)$-jets of inputs:

$$(y(0), y'(0), \ldots, y^{(n)}(0), u(0), u'(0), \ldots, u^{(n-1)}(0)) \;.$$

This map is analytic, and hence the image of compact sets is a finite union of embedded submanifolds (from results in subanalytic set theory). This implies that, for bounded states and bounded controls (in an appropriate Whitney topology), a finite number of local analytic equations (as opposed to a single global equation) is satisfied by i/o pairs. Details will be given in a future paper. An alternative, but closely related, idea is being developed by W. Respondek (personal communication).

7 Comparison With Algebraic Results

Throughout the paper we have mentioned analogies with the algebraic case considered in [15]. In this section, we summarize these analogies and we point out the major differences with the analytic case.

The definition of input/output operator is the same in both papers, but while here we deal with analytic systems (17), in [15] the interest is in systems defined by rational equations. Essentially, the latter systems are defined by asking that the vector fields g_i, and the output function h, be expressible as rational functions of the state –technically one must add a non-degeneracy condition to deal with the possible poles of these rational functions, and hence the name "singular polynomial systems."

Theorem 1 has an analogue in the algebraic case: in part (a), realizability by an analytic system is replaced by realizability by a rational system, and "analytically finitely generated" is replaced by the standard finite generation notion for algebras; part (b) has a similar analogue, except that local realizability is replaced by *global* rational realizability when the field $\mathcal{Q}_1(c)$ is a finitely generated field extension of $\mathbb{R}$. Note that neither result contains the other, as global realization by rational systems does not imply local analytic realizability, due to the possibility of poles.

Analytic i/o equations (30) are replaced in [15] by polynomial equations, meromorphic equations are replaced by "rational equations" in which the highest derivative of the output is expressible as a rational function of lower order derivatives, and analytically recursive equations are replaced by "recursive" equations in which the latter expression contains no derivatives of outputs in the denominator. The analogue of Theorem 2 is somewhat easier to prove in the algebraic case: algebraic i/o equations give rise to finitely generated (rather than meromorphically finitely generated) field extensions, and recursive i/o equations imply finitely generated (rather than analytically finitely generated) algebras.

Finally, the form of the main Theorem 3 is similar to that in [15], except that the converse of part (b) holds for algebraic equations and rational systems. This was discussed in the previous section, and is due to the lack of a strong property of elimination in the analytic case.

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