

## SYLVESTER DOMAINS

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### 0. Outline

The *inner rank* of an  $m \times n$  matrix  $A$  over a ring is defined as the least integer  $r$  such that  $A$  can be expressed as the product of an  $m \times r$  and an  $r \times n$  matrix. For example, over a skew field this concept coincides with the usual notion of rank. A ring homomorphism  $R \rightarrow S$  is said to be *rank-preserving* if each matrix over  $R$  has the same inner rank over  $R$  as its image has over  $S$ . Such a homomorphism is necessarily injective, since a matrix has inner rank 0 if and only if all its entries are 0.

It is a theorem of P.M. Cohn [10] that every semifir has a rank-preserving homomorphism to a skew field, and in this paper we study the rings which share this property. In Section 1, Cohn's prime matrix ideal theorem [8] is used to show that the rings in question are precisely those for which Sylvester's law of nullity holds. For this reason we will call such rings Sylvester domains.

In [3], Bedoya and Lewin show that a two-sided Noetherian domain  $R$  is a Sylvester domain if and only if the global dimension of  $R$  is at most 2 and  $R$  is *projective-free* (that is, every finitely generated projective  $R$ -module is free of unique rank). For the "only if" part, essentially stronger statements can be made in general. In Section 2 it will be proved that every Sylvester domain has weak global dimension at most 2, and all its flat modules are directed unions of free submodules; further a Sylvester domain must have IBN (that is, invertible matrices are

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square, or equivalently, every free module has unique rank). Let us see how close these three necessary conditions come to being sufficient. Consider those rings with IBN over which every flat module is a directed union of free submodules: where the global dimension is 0 these are precisely the skew fields; where the global dimension is 1 these are precisely the semifirs that are not skew fields, and by Cohn's Theorem [10] these are all Sylvester domains; where the weak global dimension is 2 the rings can fail to be Sylvester domains, as will be shown in Section 6 by examples provided by G.M. Bergman. Thus the problem is to determine which rings of weak global dimension 2 are Sylvester domains.

To decide whether a one-sided Ore domain has a rank-preserving homomorphism to a skew field it suffices to consider the usual skew field of (left or right) fractions. We show in Section 2 that a two-sided Ore domain is a Sylvester domain if (and only if) the weak global dimension is at most 2 and every flat module is a directed union of free submodules. This gives an answer to the problem raised by G.M. Bergman [4, p. 150], [8, Exercise 5.5.12°] of characterizing commutative Sylvester domains. In Section 3 we deduce that a right coherent two-sided Ore domain is a Sylvester domain if (and only if) it is projective-free and has weak global dimension at most 2. This generalizes the main result of [3]. In Section 4 we briefly discuss some commutative Sylvester domains.

The immediate example of a ring satisfying the conditions of the Bedoya-Lewin result is the polynomial ring in one indeterminate over a principal ideal domain, since this is Noetherian of global dimension at most 2 and projective-free. The latter fact is Seshadri's Theorem [18] and the fascinating aspect here is that an analysis of Seshadri's proof uncovers a rank-preserving homomorphism. Although a rank-preserving homomorphism was to be expected, and gives no information beyond the fact that projectives are free, the extension of Seshadri's argument to free algebras over a principal ideal domain, cf. [2, p. 212], does give new information and enables us to show, in Section 5, that such free algebras are Sylvester domains. Besides providing further interesting examples of Sylvester domains, such as all free rings, this has applications to free radical rings in Appendix II.

In Section 6 it is proved that the coproduct of Sylvester domains amalgamating a skew field is again a Sylvester domain. This is quite a natural result in light of the fact that interest in rank-preserving homomorphisms to skew fields developed (via semifirs) from Cohn's investigation of the coproduct of skew fields amalgamating a skew field.

Appendix I consists of some work of G.M. Bergman related to Sylvester's law of nullity. His results for example give a surprising connection between finitely generated projectives and mappings to skew fields, for (semi) hereditary rings.

This article evolved from one of the authors (EDS) obtaining, independently of [3], the main result of [3]. His argument suggested the statements and proofs in Sections 2 and 3 to the other author (WD) who then carried on to obtain the other results.

We are grateful to G.M. Bergman for his permission to include Appendix I, and we thank him and P.M. Cohn for their comments, from which this work has greatly benefitted.

## 1. A characterization of Sylvester domains

We begin by recalling the essential result of Cohn [8, Chapter 7], referring the reader to [8] or [16] for a proof.

Given a ring  $R$  and a homomorphism from  $R$  to a skew field, we can form the set  $P$  of all square matrices over  $R$  that are mapped to singular matrices over the skew field. This set  $P$  is easily seen to have the following properties:

- (1) The  $1 \times 1$  matrix  $(1)$  is not in  $P$ .
- (2) For any square matrices  $A, B$  their *diagonal sum*  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  is in  $P$  if and only if  $A$  or  $B$  is in  $P$ .
- (3) For any matrix  $A$  and columns  $b, c$  if  $(A \ b), (A \ c)$  are square matrices that belong to  $P$ , then  $(A \ b+c)$  is in  $P$ . Similarly for columns in positions other than the last. Similarly for rows.
- (4) Every non-full square matrix is in  $P$ ,

where a square matrix is called *full* if its inner rank equals its order, and *non-full* otherwise.

Over a ring  $R$ , a set of square matrices satisfying (1)–(4) is called a *prime matrix ideal*. Given a prime matrix ideal  $P$  over  $R$  we can consider the homomorphism  $R \rightarrow R_P$  which is universal with the property that each square matrix over  $R$  that is not in  $P$  becomes invertible over  $R_P$ . What Cohn has proved [8, Theorems 7.5.3, 7.2.2] is that  $R_P$  is a local ring, and that each element of  $P$  remains noninvertible over  $R_P$ . Thus if we write  $R_P/P$  to denote the residue skew field of  $R_P$ , then  $P$  is precisely the set of those square matrices that are mapped to singular matrices over  $R_P/P$ . It follows that there is a bijective correspondence between the set of prime matrix ideals,  $P$ , and the set of (the usual equivalence classes of) ring epimorphisms,  $R \rightarrow R_P/P$ , from  $R$  to skew fields.

A homomorphism from  $R$  to a skew field is called *fully-inverting* if the image of each full matrix is invertible. By Cohn's Theorem then,  $R$  has a fully-inverting homomorphism to a skew field if and only if the set of non-full square matrices is a prime matrix ideal over  $R$ . This is clearly the smallest set for which (4) is satisfied, and the conditions under which (1)–(3) are also satisfied by the set of non-full square matrices are as follows:

- (5)  $R$  is nonzero.

- (6) The diagonal sum of square full matrices is full.
- (7) If  $(A \ b)$ ,  $(A \ c)$  are square non-full matrices, then  $(A \ b + c)$  is non-full, and similarly for rows.

These rather technical conditions will be used in the proof of the main result of this section. We will also need the following result that has proved extremely useful on many occasions, cf. [8, 10]. An  $m \times n$  matrix is *left full* if its inner rank is  $m$ .

**Lemma 1** (Cohn). *If  $A$  is a left full matrix that does not remain left full when the first column is deleted then there is a factorization*

$$(8) \quad A = B \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$

where  $B$  is square and  $C$  has one less row and column than  $A$ . When this happens  $B$  is full and  $C$  is left full.

**Proof.** Say  $A$  is  $m \times n$ , and write  $A = (d \ D)$  where  $d$  is  $m \times 1$ ,  $D$  is  $m \times n - 1$ , not left full. Then  $D = EC$  where  $E$  is  $m \times m - 1$ ,  $C$  is  $m - 1 \times n - 1$ . Now

$$A = (d \ D) = (d \ E) \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$

is the desired factorization. It is clear from (8) that if  $B$  is not full or  $C$  is not left full then  $A$  is not left full.

The following result, abstracted from [10, Theorem 2], will enable us to show that every fully-inverting homomorphism to a skew field is rank-preserving.

**Proposition 2.** *If  $R$  is a ring such that the set of full matrices over  $R$  is closed under products (where defined) and diagonal sums, then the inner rank of a matrix over  $R$  is the maximum of the orders of its full submatrices.*

**Proof.** Without loss of generality  $R$  is nonzero. Let  $A$  be an  $m \times n$  matrix of inner rank  $r$ , say  $A = BC$  is a minimal factorization (that is,  $B$  is  $m \times r$ ,  $C$  is  $r \times n$ ). The proof that  $A$  has a full  $r \times r$  submatrix (clearly the largest possible) will be by induction on  $n + m$ . Since the product of two full matrices is again full by assumption, it suffices to show that  $B$ ,  $C$  have full  $r \times r$  submatrices, and by symmetry it suffices to consider  $C$ . Now  $C$  is left full, and if it remains left full when the first column is deleted then the induction hypothesis guarantees that the truncated matrix, and hence  $C$ , has a full  $r \times r$  submatrix. Thus we may assume that  $C$  does not remain left full when the first column is deleted, so by Lemma 1 there is a factorization  $C = D \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$  where  $D$  is full and  $E$  is left full. By the induction hypothesis  $E$  has a full  $r - 1 \times r - 1$  submatrix, so by the assumption that the

diagonal sum of full matrices is full,  $\begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$  has a full  $r \times r$  submatrix. Now the product with  $D$  is the desired  $r \times r$  full submatrix of  $C$ .

Let us now introduce the rings that will turn out to have a rank-preserving homomorphism to a skew field. A nonzero ring  $R$  is called a *Sylvester domain* if it satisfies *Sylvester's law of nullity*:

$$(9) \quad \text{If } A \text{ is an } m \times n, \text{ and } B \text{ an } n \times s \text{ matrix over } R, \text{ then} \\ \rho(AB) \geq \rho(A) + \rho(B) - n,$$

where  $\rho$  denotes the inner rank. For  $n = 1$  this says that  $R$  has no proper zerodivisors.

Examples (i) (Sylvester [19]) A field is a Sylvester domain.

(ii) (Cohn [8, Proposition 5.5.5]) A semifir is a Sylvester domain. (Bergman remarks that "sylvester" meaning "of the forest" is a very appropriate name for a generalization of "fir".)

It is more practical to have Sylvester's law of nullity expressed in the following apparently weaker form:

$$(10) \quad \text{If } A \text{ is an } m \times n, \text{ and } B \text{ an } n \times s \text{ matrix over } R \text{ such that } AB = 0 \text{ then} \\ n \geq \rho(A) + \rho(B).$$

Clearly (9) implies (10), and to see the converse suppose that  $C$  is an  $m \times n$  and  $D$  an  $n \times s$  matrix over  $R$  and let  $EF$  be a minimal factorization of  $CD$ . Taking

$$A = (C \quad E), B = \begin{pmatrix} D \\ -F \end{pmatrix}$$

we can apply (10) and the conclusion is  $n + \rho(CD) \geq \rho(A) + \rho(B)$  since the number of columns of  $A$  is  $n + \rho(CD)$  by choice of  $E, F$ . Now  $\rho(A) + \rho(B) \geq \rho(C) + \rho(D)$  is clear, so (9) is verified.

Let us now prove the following consequences of Sylvester's law of nullity:

$$(11) \quad \text{For any matrices } A, B \text{ over a Sylvester domain}$$

$$\rho \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \rho(A) + \rho(B).$$

$$(12) \quad \text{Over a Sylvester domain, if three matrices } A, B, C \text{ have the same number} \\ \text{of rows, and if } \rho(A \ B) = \rho(A \ C) = \rho(A), \text{ then } \rho(A \ B \ C) = \rho(A).$$

By partitioning a minimal factorization of  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  we may write

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} X \\ X' \end{pmatrix} (Y \quad Y')$$

where the number of columns of  $X$  is  $\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ . Now  $XY' = 0$  so by (10) we have

$$\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \geq \rho(X) + \rho(Y') \geq \rho(XY) + \rho(X'Y') = \rho(A) + \rho(B) \geq \rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

which proves (11). To see (12), partition minimal factorizations of  $(A \ B)$ ,  $(A \ C)$ , say  $(A \ B) = D(E \ E')$ ,  $(A \ C) = F(G \ G')$ , and  $A = DE = FG$  are minimal factorizations of  $A$ . Thus the number of columns of  $(D \ F)$  is  $2\rho(A)$  and

$$(D \ F)\begin{pmatrix} E \\ -G \end{pmatrix} = 0$$

so by (10) we have

$$\begin{aligned} 2\rho(A) &\geq \rho(D \ F) + \rho\begin{pmatrix} E \\ -G \end{pmatrix} \geq \rho(D) + \rho(G) \geq \rho(DE) + \rho(FG) \\ &= \rho(A) + \rho(A). \end{aligned}$$

It follows that  $\rho(D \ F) = \rho(A)$ , say  $(D \ F) = H(J \ K)$  where the number of columns of  $H$  is  $\rho(D \ F) = \rho(A)$ . Then

$$\begin{aligned} (A \ B \ C) &= (DE \ DE' \ FG') = (HJE \ HJE' \ HKG') \\ &= H(JE \ JE' \ KG') \end{aligned}$$

which shows  $\rho(A \ B \ C)$  is at most the number,  $\rho(A)$ , of columns of  $H$ . This proves (12).

Let us digress briefly to recall why semifirs are Sylvester domains. One of the characterizations of semifirs is that they are the nonzero rings which have the following property:

- (13) If  $AB = 0$  then there exists an invertible square matrix  $U$  such that  $AU = (C \ 0)$ ,  $U^{-1}B = \begin{pmatrix} 0 \\ D \end{pmatrix}$ , and the product of these is “trivially” zero,

and this condition clearly implies (10).

Returning now to arbitrary rings, we can state and prove the main result of this section.

**Theorem 3.** *For any ring  $R$  the following are equivalent:*

- (i) *The set of non-full matrices over  $R$  is a prime matrix ideal.*
- (ii)  *$R$  has a fully-inverting homomorphism to a skew field.*
- (iii)  *$R$  has a rank-preserving homomorphism to a skew field.*
- (iv)  *$R$  is a Sylvester domain.*

**Proof.** (i)  $\Rightarrow$  (ii) is Cohn’s Theorem [8] mentioned at the beginning of this section.

(ii)  $\Rightarrow$  (iii). Let  $R \rightarrow K$  be a fully-inverting homomorphism to a skew field. Then the set of full matrices over  $R$  is closed under products (where defined) and

diagonal sums, since the same is true for the set of invertible matrices over  $K$ . By Proposition 2 the inner rank over  $R$  is then determined by the full submatrices, and as these are preserved by  $R \rightarrow K$ , this homomorphism is rank-preserving.

(iii)  $\Rightarrow$  (iv) follows from the fact that a skew field is a Sylvester domain.

(iv)  $\Rightarrow$  (i). Since Sylvester domains are nonzero by definition, it remains to verify (6), (7). From (11), (6) is immediate. To see (7), let  $A$  be an  $n \times n - 1$  matrix,  $b, c$  be  $n \times 1$  matrices and suppose  $(A \ b), (A \ c)$  are non-full. We wish to show that  $(A \ b + c)$  is non-full and this is immediate if  $\rho(A) \leq n - 2$ , so we may assume  $\rho(A) = n - 1$ . But then  $\rho(A \ b) = \rho(A \ c) = \rho(A) = n - 1$ , so by (12),  $\rho(A \ b \ c) = n - 1$ . Now

$$(A \ b + c) = (A \ b \ c) \begin{pmatrix} I & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

has inner rank at most  $n - 1$  so is non-full. By symmetry, the analogous result holds for rows.

The proof of (ii)  $\Rightarrow$  (iii) above is similar to Cohn's proof [10] that a semifir has a rank-preserving homomorphism to a skew field; the proof of (iv)  $\Rightarrow$  (i) is an extension of Cohn's proof [8] that a semifir has a fully-inverting homomorphism to a skew field.

In [8, Theorem 7.6.5] Cohn shows that if  $R$  is a Sylvester domain and  $P$  is the set of non-full square matrices over  $R$  then the local ring  $R_P$  is already a skew field. Let us now use a proof similar to Cohn's to get a generalization that will be used in both Appendix I and Appendix II. The statement is rather complicated since it has been set up to apply to two quite different situations.

We recall the following from [8, Section 7.1]: For any set  $\Sigma$  of matrices over a ring  $R$  there is a *universal  $\Sigma$ -inverting* homomorphism  $R \rightarrow R_\Sigma$  which is universal with the property that the image of each element of  $\Sigma$  is invertible. If  $1 \in \Sigma$  and whenever  $A, B \in \Sigma$  then

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma$$

for any matrix  $C$  of the appropriate size, then we say  $\Sigma$  is *multiplicative*; and when this holds, every element of  $R_\Sigma$  is an entry in the inverse of some element of  $\Sigma$ , by the proof of [8, Theorem 7.1.2].

**Proposition 4.** *Let  $\Sigma$  be a multiplicative set of square matrices over a ring  $R$ , and let  $R \rightarrow S \rightarrow R_\Sigma \rightarrow T$  be ring homomorphisms where  $R \rightarrow R_\Sigma$  is the universal  $\Sigma$ -inverting map. Suppose the following conditions hold:*

(14) *Over  $S$  the image of each element of  $\Sigma$  is full.*

(15) Each square matrix over  $S$  that is a left factor of the image of some element of  $\Sigma$  becomes invertible over  $R_\Sigma$ .

(16) Each square matrix over  $R$  that is full over  $S$  remains full over  $T$ .

Then  $R_\Sigma \rightarrow T$  is an embedding.

In applications  $S$  will be either  $R$  or  $R_\Sigma$ , and (14), (15) will be readily verifiable. For example, suppose  $P$  is a prime matrix ideal and  $\Sigma$  is the set of square matrices not in  $P$ , so  $R_\Sigma = R_P$ . If we take  $S = R_\Sigma$  then, as this is a local ring, (14), (15) are clearly satisfied. If we take now  $T = R_P/P$  the proposition says that if each square matrix over  $R$  that does not become invertible over  $R_P$  becomes non-full, then  $R_P$  is already a skew field (and conversely). This obviously applies to the above mentioned case dealt with by Cohn where every element of  $P$  is already non-full over  $R$ .

**Proof of Proposition 4.** Let  $r$  be an element of  $R_\Sigma$  that is mapped to zero in  $T$ . By the remarks preceding the statement of the proposition, there is, for some integer  $n$ , an  $n \times n$  matrix  $A \in \Sigma$  and an equation  $Ax + a = 0$  over  $R_\Sigma$  with  $a \in {}^nR$ ,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in {}^nR_\Sigma,$$

$x_i = r$  for some  $i$ , say  $i = 1$ . Here, and throughout the proof, we use the same symbol to denote an element and its image; for this to be meaningful it is necessary to specify the ring where the image is being considered. In  $R$ , partition  $A = (a_1 \ A_1)$  where  $a_1$  is the first column of  $A$ , and write  $A^* = (a_1 \ a \ A_1)$ . Then in  $R_\Sigma$ ,  $A^*$  left annihilates the transpose of  $(x_1 \ 1 \ x_2 \ \dots \ x_n)$ . Since  $x_1 (=r)$  is assumed to map to 0 in  $T$ ,  $(a \ A_1)$  becomes non-full over  $T$ , and hence is already non-full over  $S$  by (16). Since  $A \in \Sigma$ ,  $A$  and hence  $A^*$  is left full over  $S$  by (14); but we have seen that  $A^*$  does not remain left full over  $S$  when the first column is deleted. Hence by Lemma 1 there is a factorization

$$A^* = B \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$$

over  $S$ , where  $B$  is square and, as a square factor of  $A$  over  $S$ , becomes invertible over  $R_\Sigma$  by (15). Hence over  $R_\Sigma$ ,

$$\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} = B^{-1}A^*$$

left annihilates the transpose of  $(x_1 \ 1 \ x_2 \ \dots \ x_n)$  and so  $x_1 = 0$  in  $R_\Sigma$ . This shows that  $R_\Sigma \rightarrow T$  is injective.



## 2. Rings over which all full matrices are non-zero-divisors

The problem remains that Theorem 3 is not readily applicable even in the commutative case, and it would be useful to have a characterization as convenient as the Bedoya–Lewin result. An approach that turns out to be illuminating is to consider the class of rings whose full matrices are non-zero-divisors. Clearly this class contains the class of all Sylvester domains, and for example these two classes have the same intersection with the class of all domains with a one-sided Ore condition, since over such domains the matrices that are non-zero-divisors are the ones that become invertible in the usual skew field of fractions.

To make the statement of the next result more manageable we introduce the following terminology. A module will be called *spacial* if every finite dependent subset lies in a submodule generated by fewer elements; that is, if a dependent subset has  $n$  elements then it lies in an  $n - 1$  generator submodule. A module with the weaker property of being a directed union of free submodules will be called *locally-free*. A module is said to have *local-rank*  $n$  if  $n$  is the least integer such that any finite subset lies in an  $n$ -generator submodule. For example, the local-rank of a finitely generated module is the minimum number of generators. Another example is that of a locally-cyclic module, such as the field of fractions of an integral domain, which has local-rank 1.

Recall that for a ring  $R$  the *right annihilator* of a subset  $X$  of  $R^n$  is the set of all  $y \in {}^nR$  such that  $xy = 0$  for all  $x \in X$ . The *left annihilator* of a subset  $Y$  of  ${}^mR$  is defined dually. By the *right annihilator* of a matrix  $A$ , we mean the right annihilator of the set of rows of  $A$ , and dually for the *left annihilator* of  $A$ . If the right annihilator of  $A$  is 0 then  $A$  is said to be *right regular*, and dually for *left regular*. If  $A$  is left and right regular it is said to be *regular* or a *non-zero-divisor*.

**Theorem 5.** *For any ring  $R$  the following are equivalent:*

- (i) *Every full matrix is left regular.*
- (ii) *Every left full matrix is left regular.*
- (iii) *Every free left module is spacial.*
- (iv) *Every flat left module is spacial.*
- (v) *The right annihilator of every nonzero matrix  $A$  has local-rank less than the number of columns of  $A$ .*
- (vi) *The right annihilator of every nonzero row vector  $\alpha$  has local-rank less than the length of  $\alpha$ .*

*Further, when these hold, the kernel of any homomorphism between spacial right modules is again spacial.*

**Proof.** (i)  $\Rightarrow$  (ii). Assume (i) and let  $A$  be an  $m \times n$  matrix of inner rank  $m$ . We will prove by induction on  $n$  that  $A$  is left regular. If the matrix obtained by deleting the first column of  $A$  is left full then  $A$  is left regular by the induction hypothesis. In the contrary case, Lemma 1 implies that there is a factorization  $A = B \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$  where  $B$  is

full, so left regular by (i), and  $C$  is left full so left regular by the induction hypothesis. As a product of left regular matrices,  $A$  is then left regular.

(ii)  $\Rightarrow$  (iv). Let  $M$  be a flat left  $R$ -module. Any finite dependent subset of  $M$  can be arranged to form a column, say  $X \in {}^n M$ , and by dependence there is a nonzero  $A \in R^n$  such that  $AX = 0$ . By flatness this comes from an  $R$ -relation, say  $AB = 0$ ,  $X = BY$  where  $B$  is an  $n \times m$  matrix over  $R$  and  $Y \in {}^m M$ , for some  $m$ . Since  $B$  is not left regular it is not left full, by (ii). Say  $B = CD$  where  $C$  is  $n \times n-1$  and  $D$  is  $n-1 \times m$ . Thus  $X = BY = C(DY)$  and the elements of  $X$  lie in the  $n-1$  generator submodule of  $M$  generated by the entries of  $DY$ . This proves that  $M$  is spacial.

(iv)  $\Rightarrow$  (iii) is obvious.

(iii)  $\Rightarrow$  (ii), since each of the following statements implies the next. Every free left  $R$ -module is spacial. For every  $m, n$  any set of  $m$  dependent elements of  $R^n$  lies in an  $m-1$  generator submodule. For every  $m, n$  any  $m \times n$  matrix that is not left regular is the product of an  $m \times m-1$  and an  $m-1 \times n$  matrix. Every matrix that is not left regular is not left full. Every left full matrix is left regular.

(Alternatively we could have proved (ii)  $\Leftrightarrow$  (iii) by this argument, and proved (iii)  $\Rightarrow$  (iv) (hence (iii)  $\Leftrightarrow$  (iv)) by using the fact that a flat module is a direct limit of free modules [15], and a direct limit of spacial modules is spacial. The approach used actually proves the small part of Lazard's argument that we really need.)

(ii)  $\Leftrightarrow$  (v). Let  $A$  be a nonzero  $m \times n$  matrix. Any finite set of columns in the right annihilator of  $A$  can be arranged to form a matrix  $B$  such that  $AB = 0$ . Now for any minimal factorization  $B = CD$ ,  $D$  is left full so left regular by (ii) so  $AC = 0$ . Thus  $C$  is not left regular so by (ii) is not left full. It follows that  $C$  has at most  $n-1$  columns. Now the columns of  $B = CD$  lie in the submodule (of the right annihilator of  $A$ ) generated by the columns of  $C$ , which shows that the right annihilator of  $A$  has local-rank at most  $n-1$ .

(v)  $\Rightarrow$  (vi) is clear.

(vi)  $\Rightarrow$  (i). Let  $A$  be an  $n \times n$  matrix and suppose that  $\alpha A = 0$  for some nonzero  $\alpha \in R^n$ . Then the columns of  $A$  lie in the right annihilator of  $\alpha$ , which has local-rank less than  $n$ . Then the columns of  $A$  lie in an  $n-1$  generator submodule of  ${}^n R$ , so  $A$  is not full.

Assume now that (i)-(vi) hold and that  $\alpha: M \rightarrow N$  is a homomorphism between spacial right  $R$ -modules. Any finite dependent subset of  $\text{Ker } \alpha$  can be arranged to form a row, say  $X \in (\text{Ker } \alpha)^m$ . Factor  $X = YA$  with  $Y \in M^r$ ,  $A \in {}^r R^m$  and  $r$  minimal. Then  $A$  has inner rank  $r$  (so is left regular), and since  $M$  is spacial  $r < m$  and the elements of  $Y$  are independent. Now  $\alpha(Y) \in N^r$  and again there is a factorization  $\alpha(Y) = ZB$  where the elements of  $Z$  are independent. Then

$$0 = \alpha(X) = \alpha(YA) = \alpha(Y)A = ZBA.$$

Since  $Z$  is independent and  $A$  is left regular it follows that  $B = 0$  so  $\alpha(Y) = 0$ . Thus  $Y \in (\text{Ker } \alpha)^r$  which proves that  $\text{Ker } \alpha$  is spacial.  $\square$

Thus we have a substantial number of conditions equivalent to every full matrix being regular; we leave the formulation of these to the interested reader, and for now only record the following corollary.

**Theorem 6.** *If  $R$  is a Sylvester domain then  $R$  has weak global dimension at most 2, and every flat  $R$ -module is locally-free.*

**Proof.** Over a Sylvester domain every full matrix is regular, so the conditions of Theorem 5, and their duals, hold, so every flat module is spacial, hence locally free, and the kernel of any homomorphism between spacial modules is spacial, which says that the kernel of any homomorphism between flat modules is flat. Thus the weak global dimension is at most 2.

Together with the obvious fact that Sylvester domains have IBN, this shows that Sylvester domains are projective-free.

In Section 6 we will see examples that illustrate the distinctions between being a Sylvester domain, having every full matrix regular, and having every full matrix left regular. These examples will show in particular that the converse of Theorem 6 can fail in an interesting way.

Now let us observe that the converse of Theorem 6 holds for two-sided Ore domains.

**Theorem 7.** *For any two-sided Ore domain  $R$  the following are equivalent:*

- (i)  *$R$  is a Sylvester domain.*
- (ii) *The weak global dimension of  $R$  is at most 2 and every flat right  $R$ -module is locally-free.*
- (iii) *The right annihilator of every matrix is locally-free.*
- (iv) *The right annihilator of every row vector is locally-free.*
- (v) *Every full matrix is left regular.*

*These statements are further equivalent to their left-right duals.*

**Proof.** (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) for any ring  $R$ .

(iv)  $\Rightarrow$  (v) for any right Ore domain  $R$ . Let  $K$  denote the skew field of right fractions, and let  $\alpha$  be a nonzero row vector of length  $n$  over  $R$ . If the right annihilator of  $\alpha$  contains  $n$  right  $R$ -independent elements, then these are right  $K$ -independent and this is impossible since  $\alpha$  is nonzero. So the right annihilator of  $\alpha$  does not contain  $n$  right  $R$ -independent elements, and as it is locally-free by (iv), it has local-rank at most  $n - 1$ . Now by Theorem 5, (v) holds.

(v)  $\Rightarrow$  (i) for any left Ore domain  $R$ . Any full matrix over  $R$  is left regular over  $R$  so remains left regular over the skew field of left fractions of  $R$ , so becomes invertible. Thus  $R$  has a fully-inverting homomorphism to a skew field, which implies that  $R$  is a Sylvester domain, by Theorem 3.

It seems worthwhile at this point to record the following consequence of the above arguments (and also implicit in [3]) which provides a source of examples of matrices that cannot have the same inner rank over a ring  $R$  and over a skew field containing  $R$ . Suppose that some left  $R$ -module  $M$  has a free resolution

$$\dots \rightarrow R^{n_2} \xrightarrow{A_1} R^{n_1} \xrightarrow{A_0} R^{n_0} \rightarrow M \rightarrow 0.$$

If  $M$  has projective dimension at least 3, or more generally, if the kernel of  $A_0$  is not free, then  $A_1$  cannot be written  $A_1 = BC$  where  $B$  is right regular and  $C$  is left regular. For if it can, then as  $B$  is right regular the rows of  $C$  lie in, and hence generate, the left annihilator of  $A_0$ ; and as  $C$  is left regular the rows of  $C$  are left  $R$ -independent. Thus the kernel of  $A_0$  is free, a contradiction. For example, if  $k$  is a field, the Koszul resolution for the ring  $R = k[x, y, z]$  is a minimal resolution

$$0 \rightarrow R \rightarrow R^3 \rightarrow R^3 \rightarrow R \rightarrow k \rightarrow 0$$

and if bases are chosen one obtains matrices, for example

$$(y \quad z \quad x), \quad \begin{pmatrix} -z & 0 & x \\ y & -x & 0 \\ 0 & z & -y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and the middle matrix is full, cf [3].

### 3. Coherence and duality

Recall that for any ring  $R$  the *dual* of a left  $R$ -module  $M$  is the right  $R$ -module  $M^* = \text{Hom}_R(M, R)$ , and the dual of a right  $R$ -module is defined similarly.

If  $M$  is a finitely presented left  $R$ -module, say  $A$  is an  $m \times n$  matrix and

$$R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0$$

is exact, then

$$0 \rightarrow M^* \rightarrow {}^n R \xrightarrow{A} {}^m R$$

is again exact. This permits us to use the concepts “dual of a finitely presented left module” and “right annihilator of a matrix” interchangeably.

It is somewhat surprising that the Noetherian case of the following lemma can be viewed as Bass’s Global Dimension Two Theorem [1], [13, p. 379].

**Lemma 8.** *A ring has weak global dimension at most 2 if and only if the dual of every finitely presented left module is flat.*

Since having weak global dimension at most 2 is a left-right symmetric condition, this is also equivalent to the dual of every finitely presented right module being flat.

**Proof.** As remarked above, the dual of a finitely presented left module is the kernel of a homomorphism between flat (indeed, finitely generated free) right modules, so is flat if the weak global dimension is at most 2. Conversely, suppose that  $R$  is a ring such that the right annihilator of every matrix is flat. To prove the lemma it suffices to show that any homomorphism  $F_1 \rightarrow F_0$  of free right  $R$ -modules has flat kernel. And since flatness is a “local” condition, it suffices to verify that any finite subset  $X$  of the kernel  $K$  lies in a flat submodule of  $K$ . Now  $X$  lies in a finitely generated free submodule  $F'_1$  of  $F_1$ , and the image of  $F'_1$  lies in a finitely generated free submodule  $F'_0$  of  $F_0$ . Thus the kernel of  $F'_1 \rightarrow F'_0$  is the right annihilator of a matrix, so flat, and as a submodule of  $K$  containing  $X$  it has the requisite properties.

The above lemma combines well with the concept of “coherence”. Recall that a ring  $R$  is said to be *right coherent* if the following equivalent conditions hold:

- (17) Every finitely generated right ideal of  $R$  is finitely related (i.e. the right annihilator of any row vector is finitely generated).
- (18) The direct product of any family of copies of  $R$  is flat as left  $R$ -module.
- (19) Every finitely generated submodule of a free right  $R$ -module is finitely related.

For proofs that these are equivalent the reader is referred to [7] or [13, p. 439].

Since  $\text{Hom}_R(-, R)$  converts direct sums into direct products, it is clear that (18) is equivalent to

- (20) The dual of every free right  $R$ -module is flat.

Further, it is not difficult to show that (19) is equivalent to

- (21) The dual of every finitely presented left  $R$ -module is finitely presented.

**Proposition 9.** *For any ring  $R$  the following are equivalent:*

- (i)  $R$  is right coherent and has weak global dimension at most 2.
- (ii) The dual of every finitely presented left  $R$ -module is finitely generated projective.
- (iii) The dual of every right  $R$ -module is flat.

C.U. Jensen [14, Theorem 5A] had previously observed the equivalence of (i) and (iii).

**Proof.** (i)  $\Leftrightarrow$  (ii). Using right coherence in the form (21) we see, by Lemma 8, that (i) is equivalent to the dual of every finitely presented left  $R$ -module being finitely presented and flat, which is equivalent to (ii).

(i)  $\Rightarrow$  (iii). Observe that for any right  $R$ -module  $M$  there is a presentation  $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  where  $F_1, F_0$  are free right  $R$ -modules. Dualizing gives  $M^*$  as the kernel of a homomorphism  $F_0^* \rightarrow F_1^*$  between modules that are flat by (20), and hence  $M^*$  is flat, since the weak global dimension is at most 2.

(iii)  $\Rightarrow$  (i) since (iii) implies right coherence in the form (20), and implies weak global dimension at most 2 by the left-right dual of lemma 8.  $\square$

Our purpose in obtaining this proposition is to apply it to the Ore case although there are some interesting statements that can be made by combining Proposition 9, Theorem 5, and their duals. For example the following are equivalent for any ring  $R$ :

- (22)  $R$  is left and right coherent and every full matrix is regular.
- (23) The right annihilator of every nonzero row vector  $\alpha$  is free of rank less than the length of  $\alpha$ , and dually for column vectors.
- (24) The right annihilator of every nonzero matrix  $A$  is free of rank less than the number of columns of  $A$ , and dually for the left annihilator.

Let us note one close connection between right coherence and full matrices that is not yet immediate from our results: It is apparent from Chase's arguments [7] that a ring  $R$  is right coherent if and only if for any matrix  $A$ , say  $m \times n$  and right infinite matrix  $B$ , say  $n \times \gamma$ , if  $AB = 0$  then there is a matrix  $C$  and a right infinite matrix  $D$  such that  $AC = 0$ ,  $B = CD$ . Extending the notion of left full to right infinite matrices we see that  $R$  is right coherent and every full matrix is left regular if and only if every left full (possibly right infinite) matrix is left regular. To some extent this explains why coherence is a natural property to impose on Sylvester domains.

By putting in right coherence we get the following natural consequence of Theorem 7.

**Theorem 10.** *For any two-sided Ore domain  $R$  the following are equivalent:*

- (i)  $R$  is a right coherent Sylvester domain.
- (ii) The dual of every right  $R$ -module is locally-free.
- (iii)  $R$  is right coherent, projective-free and has weak global dimension at most 2.
- (iv) The dual of every finitely presented left  $R$ -module is free.
- (v) The right annihilator of every matrix is free.
- (vi) The right annihilator of every row vector is free.

**Proof.** (i)  $\Rightarrow$  (ii). If  $R$  is a right coherent Sylvester domain then it has weak global dimension at most 2 so by Proposition 9 the dual of every right  $R$ -module is flat, and thus locally-free.

(ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) follow easily from Proposition 9.

(iv)  $\Rightarrow$  (v)  $\Rightarrow$  (vi) are clear.

(vi)  $\Rightarrow$  (i). If the right annihilator of every row vector is free then  $R$  is a Sylvester domain by Theorem 7, and is right coherent in the form (17), since over a right Ore domain a free submodule of a free right module of finite rank again has finite rank.

#### 4. The literature on commutative Sylvester domains

For rings with weak global dimension at most 1 the situation is quite straightforward, even in the non-commutative case.

**Proposition 11.** *For any ring  $R$  of weak global dimension at most 1 the following are equivalent:*

(i)  $R$  is a semifir.

(ii)  $R$  is a Sylvester domain.

(iii)  $R$  is right coherent projective-free.

*If further  $R$  is commutative, these are equivalent to*

(iv)  $R$  is a projective-free integral domain.

**Proof.** (i)  $\Rightarrow$  (ii) is Cohn's Theorem, cf. (13), and (ii)  $\Rightarrow$  (i) by Theorem 6. Also (i)  $\Rightarrow$  (iii) is clear, and (iii)  $\Rightarrow$  (i) holds for if every finitely generated right ideal is finitely related flat then it is projective and hence free of unique rank. Finally, if  $R$  is a projective-free integral domain then from Endo's Theorem [13, Exercise 11.11] that over an integral domain every finitely generated flat module is projective, it follows that  $R$  is a semifir.

Thus among commutative rings of weak global dimension at most 1, Sylvester domains are characterized by being projective-free integral domains. The usual term for a commutative semifir is *Bezout domain*, and the local Bezout domains are precisely the valuation rings.

It seems unlikely that the two-dimensional case has much in common with the one-dimensional case; but let us see what can be said about projective-free integral domains  $R$  of weak global dimension 2:

If  $R$  is coherent then  $R$  is a Sylvester domain by Theorem 10. This subsumes the case where  $R$  also has global dimension 2, since such rings are coherent, cf. [20, 6.1].

If  $R$  is not coherent then  $R$  is either a Sylvester domain or a non Sylvester domain, and it would be interesting to have one example of each. However we have only succeeded in finding examples of the former, as in the following curious case.

**Theorem 12** (after Dobbs [12]). *Let  $R$  be a local integral domain with maximal*

ideal  $M$  such that the ideals  $mM, m \in M$ , form a chain under inclusion. Then the following are equivalent:

- (i)  $R$  is a Sylvester domain.
- (ii)  $R$  has weak global dimension at most 2.
- (iii)  $M$  is flat as  $R$ -module.
- (iv)  $M$  is idempotent or principal.
- (v)  $M$  is locally-cyclic as  $R$ -module.

Examples of rings  $R$  with the given property can be constructed as follows: Let  $V$  be a valuation ring,  $M$  an arbitrary ideal of  $V$ , and  $R$  a subring of  $V$  that is the preimage of a subfield  $R/M$  of  $V/M$ . Then  $R$  is local, and the ideals  $mM$  form a chain under inclusion. (Notice that if  $M$  is nonzero and  $R/M$  is a proper subfield of  $V/M$  then  $R$  is not a valuation ring, so has weak global dimension at least 2.) Conversely, every such  $R$  arises this way, since the conductor of  $M$  in the field of fractions of  $R$  is a valuation ring, as can be easily verified.

**Proof of Theorem 12.** (i)  $\Rightarrow$  (ii) is clear. Observe that (i)–(v) are all true if  $R$  is a valuation ring, so we may assume that  $R$  is not a valuation ring, so there exist incomparable principal ideals  $aR, bR$ . By symmetry we may assume  $bM \supseteq aM$  so  $aR \supset aR \cap bR \supseteq aM$ , and as  $aM$  is a maximal submodule of  $aR, aR \cap bR = aM$ . Since the kernel of  $aR \oplus bR \rightarrow R, (ax, by) \mapsto ax - by$  may be identified with  $aR \cap bR$ , we have an exact sequence  $0 \rightarrow aM \rightarrow aR \oplus bR \rightarrow R$ . With this available we can proceed with the next two steps of the proof.

(ii)  $\Rightarrow$  (iii). If  $R$  has weak global dimension at most 2 then  $aM$ , and hence  $M$ , is flat.

(iii)  $\Rightarrow$  (iv). If  $M$  is flat as  $R$ -module then

$$0 \rightarrow aM \otimes_R M \rightarrow aM \oplus bM \rightarrow M$$

is exact, which means that  $aM^2 = aM \cap bM$ , and as this equals  $aM$ , so  $M^2 = M$ .

(iv)  $\Rightarrow$  (v). If  $M = M^2$  then from the fact that  $M^2 = \bigcup mM$  we see that any finite subset of  $M$  lies in some  $mM \subseteq mR$ , so  $M$  is locally-cyclic.

(v)  $\Rightarrow$  (i). Suppose that  $M$  is locally-cyclic. We will show that (iv) of Theorem 7 holds, by proving that the right annihilator  $A$  of an arbitrary nonzero row vector  $\alpha \in R^n$  is isomorphic to  ${}^{n-i}R \oplus {}^{i-1}M$  for some  $i \leq n$ , and hence is locally-free. Consider first the case where  $A$  does not lie in  ${}^nM$ . Here some  $x \in A$  has a unit as one of its entries, so there is a decomposition  ${}^nR = xR \oplus B$  where  $B$  is free of rank  $n - 1$ . Then  $A = xR \oplus A'$  where  $A'$  is the kernel of  $B \rightarrow {}^nR \xrightarrow{\alpha} R$ . By induction on  $n$  we may assume that  $A'$ , and hence  $A$ , is of the desired form. This leaves the case where  $A$  lies in  ${}^nM$ , and for this to happen  $\alpha$  must lie in  $M^n$ , say  $\alpha = (m_1, \dots, m_n)$ . One of the ideals  $m_1M, \dots, m_nM$  contains all the others, say  $m_1M$ , and  $m_1$  is not a zerodivisor so the projection  $A \rightarrow {}^{n-1}M$ , onto the last  $n - 1$  coordinates, is an isomorphism and  $A$  has the desired form.  $\square$



In the above theorem, the given ring is coherent if and only if it is a valuation ring or  $M$  is finitely generated. For if  $R$  is coherent and is not a valuation ring then

$$M \cong aM = aR \cap bR$$

is finitely generated; and conversely, if  $M$  is finitely generated then from the description of right annihilators of row vectors given in the above proof, (17) is satisfied and so  $R$  is coherent. But it is clear that if  $M$  is finitely generated then  $M^2 = mM$  for some  $m \in M$ , so if  $M$  is idempotent as well then it must be 0. It follows that the coherent Sylvester domains occurring in Theorem 12 are the valuation rings. Thus if  $V$  is a valuation ring,  $M$  a nonzero idempotent ideal of  $V$ , and  $R/M$  a proper subfield of  $V/M$  then  $R$  is a noncoherent Sylvester domain.

Another source of noncoherent Sylvester domains is the following.

**Example.** Let  $B$  be a Bezout domain and  $S$  a multiplicative subset of  $B$  closed under taking factors. Write  $B_S$  for the ring obtained by inverting the elements of  $S$ , and  $R = B + xB_S[x]$  for the subring of  $B_S[x]$  consisting of all polynomials whose constant term lies in  $B$ . Such a ring need not be coherent since for any  $b \in B$ ,  $bR \cap xR$  is the directed union of the ideals  $(bx/s)R$ , where  $s$  ranges over the factors of  $b$  in  $S$ , and this need not be finitely generated. For example, if  $b$  is nonzero and  $S$  is the complement of a nonmaximal prime ideal properly containing  $b$ .

Now  $R$  is the direct limit of the directed system of subrings  $B[x/s]$ ,  $s \in S$ , each of which is isomorphic to  $B[x]$ . So if  $B[x]$  is a Sylvester domain then so is  $R$ ; unfortunately we do not know when  $B[x]$  is a Sylvester domain (although some partial results will be obtained in the next section). According to [20, 8.2(b)],  $B[x]$  is always coherent, so it remains to decide when  $B[x]$  is projective-free.

To obtain a related projective-free ring we can localize  $B[x]$  at some prime ideal of the form  $P[x]$ . This then gives a Sylvester domain, and the above mentioned direct limit is again a Sylvester domain, and it need not be coherent.

## 5. Free algebras over Bezout domains

The following result arises from the proof of Seshadri's Theorem given in [2]. We are indebted to G.M. Bergman for pointing out the present strong form of the statement, and some subsequent applications.

**Theorem 13.** *Let  $R$  be an integral domain, and let  $S$  be a set consisting of nonzero elements  $p$  of  $R$  such that  $R/pR$  is a field. Let  $K$  denote the ring obtained from  $R$  by inverting the elements of  $S$ . Then for any set  $X$ , the homomorphism  $R\langle X \rangle \rightarrow K\langle X \rangle$ , of free algebras on  $X$ , is rank-preserving.*

**Proof.** Let  $A$  be a matrix over  $R\langle X \rangle$  and suppose that we are given a factorization  $A = BC$  over  $K\langle X \rangle$ . We will prove that there exists an invertible matrix  $U$  over

$K\langle X \rangle$  such that  $BU^{-1}$ ,  $UC$  have entries in  $R\langle X \rangle$ ; it is then immediate that  $R\langle X \rangle \rightarrow K\langle X \rangle$  is rank-preserving.

By multiplying  $C$  and dividing  $B$  by elements of  $S$ , we may transform the factorization to the form  $A = s^{-1}BC$  where  $B, C$  have entries in  $R\langle X \rangle$  and  $s$  is a product of elements of  $S$ . If  $s$  is the empty product we are finished, so we may assume that  $s = pt$  where  $p$  is an element of  $S$  and  $t$  is a shorter product of elements of  $S$ . Denote the field  $R/pR$  by  $k$ , and the homomorphism  $R\langle X \rangle \rightarrow k\langle X \rangle$  by  $f \mapsto \bar{f}$ . Then  $\bar{B}\bar{C} = 0$ . By [8, Theorem 1.3.1] and the proof of [8, Theorem 2.2.4] there is a matrix  $\bar{U}$  over  $k\langle X \rangle$  that is a product of matrices that differ from the identity matrix by one off-diagonal entry, such that

$$\bar{B}\bar{U}^{-1} = (\bar{B}_1 \quad 0), \quad \bar{U}\bar{C} = \begin{pmatrix} 0 \\ \bar{C}_2 \end{pmatrix}.$$

Lifting  $\bar{U}$  back to a product  $U$  of elementary matrices over  $R\langle X \rangle$  we can express this as

$$BU^{-1} = (B_1 \quad B_2p), \quad UC = \begin{pmatrix} pC_1 \\ C_2 \end{pmatrix}.$$

Thus we have transformed  $s^{-1}BC$  to

$$s^{-1}(B_1 \quad B_2p) \begin{pmatrix} pC_1 \\ C_2 \end{pmatrix},$$

which can be further transformed to

$$s^{-1}(B_1p \quad B_2p) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

which is of the form  $t^{-1}B'C'$  where  $B', C'$  have entries in  $R\langle X \rangle$  and  $t$  is shorter than  $s$ . Continuing in this way eventually transforms the factorization to  $R\langle X \rangle$ .

To see why this argument gives Seshadri's Theorem the reader can consult Appendix I, which discusses inner rank and finitely generated projectives, cf. Proposition 19 and Corollary 20.

**Corollary 14.** *A free algebra over a principal ideal domain is a Sylvester domain.*

**Proof.** Let  $R$  be a principal ideal domain and  $S$  be the set of atoms of  $R$ . Then inverting the elements of  $S$  gives the field of fractions,  $K$ , of  $R$ . By Theorem 13,  $R\langle X \rangle \rightarrow K\langle X \rangle$  is rank-preserving, but  $K\langle X \rangle$  is a semifir so has a rank-preserving homomorphism to a skew field, and hence so does  $R\langle X \rangle$ .

Combined with Theorem 6 this gives a strengthening of Seshadri's Theorem, namely, for a free algebra over a principal ideal domain every flat module is a directed union of free submodules.

It would be interesting to have a characterization of the commutative rings  $R$  for which all free  $R$ -algebras are Sylvester domains. It can be seen that the class of all such rings is contained in the class of all Bezout domains, contains the class of all principal ideal domains, and is closed under taking direct limits. Bergman observes that Theorem 13 applies to inverting the set  $S$  of atoms in any Bezout domain  $R$ ; thus a given Bezout domain  $R$  lies in our class of rings if and only if the same is true for the ring  $K$  obtained from  $R$  by inverting the atoms. (To see the “only if” part of this statement, observe that we are inverting central elements in the free algebras, and these can be cleared from denominators in matrices without affecting the inner rank.) For example, for any principal ideal domain  $Z$  with field of fractions  $Q$ , the set  $S$  of atoms of  $Z$  in the ring  $R = Z + xQ[x]$  satisfies the conditions of Theorem 13, and inverting them gives the principal ideal domain  $K = Q[x]$ , so  $R$  is in our class of rings. More generally, since these constructions behave well with respect to direct limits, the atom eliminating procedure can be repeated indefinitely and we may thus associate with any Bezout domain  $R$  an atomless Bezout domain  $K$ , possibly a field, such that free  $R$ -algebras are Sylvester domains if and only if free  $K$ -algebras are Sylvester domains. (Notice that  $K$  is obtained by inverting certain elements of  $R$ .)

(For the analogous problem of determining the commutative rings  $R$  for which the free power series  $R$ -algebras are Sylvester domains we know only the following: If the free power series  $R$ -algebras are Sylvester domains then  $R$  is a Bezout domain; and in the other direction, if  $R$  is a field then free power series  $R$ -algebras are semifirs so Sylvester domains; and if  $R$  is a principal ideal domain then the free power series  $R$ -algebra in one indeterminate is a Sylvester domain by the Bedoya-Lewin Theorem.)

A far more difficult task is to characterize the semifirs  $R$  which have the property that every ring  $R\langle X \rangle$  freely generated over  $R$  by a set  $X$  of  $R$ -centralizing indeterminates is a Sylvester domain. It is not even known if this is true for  $R$  a free algebra (in more than one indeterminate) over a field. A major difficulty here is the appearance of non-free projectives, observed by Ojanguren and Sridharan [17]. Let us examine their argument in greater detail. Suppose that in some given ring  $S$ ,  $a, b, c, d$  are non-zero-divisors such that  $ab - cd = 1$ . Then the exact sequence

$$0 \rightarrow Sb \cap Sd \rightarrow Sb \oplus Sd \rightarrow S \rightarrow 0$$

splits and the ideal

$$Sb \cap Sd = Sdab + Sbcd$$

is a 2-generator projective module. If  $S$  is projective-free then this ideal would have to be free of rank 1, and this amounts to  $S$  having elements  $p, q, r, s$  such that  $rb = sd$ ,  $da = qr$ ,  $bc = ps$ . Now if a ring  $R$  with no proper zero-divisors contains elements  $a, b, c$  such that  $ab - ca = 1$  then the ring  $R[x]$  has a relation  $a(x+b) - (x+c)a = 1$ , so if  $R[x]$  is to be projective-free then it must contain elements

$p, q, r, s$  such that

$$r(x + b) = sa, \quad a^2 = qr, \quad (x + b)(x + c) = ps.$$

From these equations,  $s, p$  are linear in  $x$  and their leading coefficients are units in  $R$ . Adjusting by a unit of  $R$  we may write  $p = (x + b')$ ,  $s = (x + c')$  and hence  $r = a$ ,  $rb = c'a$  so  $a$  has (left) inverse  $c' - c$ . Thus one condition that  $R$  must satisfy in order for  $R[x]$  to be projective-free is that 1, or any unit, can be expressed in the form  $ab - ca$  only if  $a$  is a unit. A semifir  $R$ , indeed a noncommutative principal ideal domain, where this fails is  $K[t]$ ,  $K$  a noncommutative skew field, say  $\alpha\beta - \beta\alpha = \gamma \neq 0$ , so  $(t + \alpha)\beta - \beta(t + \alpha) = \gamma$ ,  $t + \alpha$  a nonunit. This is the example given in [17].

### 6. Universal constructions: examples and counterexamples

In this section we examine how certain ring constructions relate to Sylvester domains. The first construction is that of a coproduct with amalgamation, and it is proved that the coproduct of Sylvester domains amalgamating a skew field is again a Sylvester domain. The other construction is that of adjoining to a skew field two matrices whose product is specified to be zero. This is found to produce a Sylvester domain whenever the two adjoined matrices cannot violate Sylvester's law of nullity.

Our arguments are built on the following result which arises from G.M. Bergman's work on coproducts [5] and would have fitted in naturally in [5, Section 2]. By a *completely reducible* ring  $K$  we mean a finite direct product of full matrix rings over skew fields. By a *faithful  $K$ -ring* we mean a ring  $R$  given with an injective ring homomorphism from  $K$  to  $R$ .

**Lemma 16.** *Let  $K$  be a completely reducible ring,  $\{R_\lambda\}_{\lambda \in \Lambda}$  be a set of faithful  $K$ -rings and  $R$  be their coproduct amalgamating  $K$ . Then any homomorphisms  $P' \xrightarrow{\alpha} P \xrightarrow{\beta} P''$  of finitely generated projective left  $R$ -modules such that  $\alpha\beta = 0$  can be extended to a commutative diagram*

$$(25) \quad \begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ \downarrow \alpha' & & \downarrow u & & \uparrow \beta' \\ \bigoplus_{\lambda} R \otimes_{R_\lambda} P'_\lambda & \xrightarrow{\bigoplus R \otimes \alpha_\lambda} & \bigoplus_{\lambda} R \otimes_{R_\lambda} P_\lambda & \xrightarrow{\bigoplus R \otimes \beta_\lambda} & \bigoplus_{\lambda} R \otimes_{R_\lambda} P''_\lambda \end{array}$$

where  $P'_\lambda \xrightarrow{\alpha_\lambda} P_\lambda \xrightarrow{\beta_\lambda} P''_\lambda$  are homomorphisms of finitely generated projective left  $R_\lambda$ -modules such that  $\alpha_\lambda \beta_\lambda = 0$ .

**Proof.** Assume, without loss of generality, that  $\Lambda$  is a finite set. By [5, Theorems 2.2, 2.3], there exists a commutative diagram

$$(26) \quad \begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ & & \downarrow u & & \uparrow \beta' \\ & & \bigoplus_{\lambda} R \otimes_{R_{\lambda}} P_{\lambda} & \longrightarrow & \bigoplus_{\lambda} R \otimes_{R_{\lambda}} \text{Im } \beta_{\lambda} \end{array}$$

where  $\beta_{\lambda}$  is the composite  $P_{\lambda} \xrightarrow{\sim} (P_{\lambda})u^{-1} \rightarrow P \xrightarrow{\beta} P''$ , and each  $P_{\lambda}$  is a finitely generated projective left  $R_{\lambda}$ -module. Since  $R$  is flat as right  $R_{\lambda}$ -module this extends to a commutative diagram

$$(27) \quad \begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ \downarrow \alpha' & & \downarrow u & & \downarrow \beta' \\ \bigoplus_{\lambda} R \otimes_{R_{\lambda}} \text{Ker } \beta_{\lambda} & \longrightarrow & \bigoplus_{\lambda} R \otimes_{R_{\lambda}} P_{\lambda} & \longrightarrow & \bigoplus_{\lambda} R \otimes_{R_{\lambda}} \text{Im } \beta_{\lambda} \end{array}$$

cf. [5, Corollary 2.17]. Now  $\text{Im } \alpha'$  is finitely generated as left  $R$ -module so lies in an  $R$ -submodule of  $\bigoplus_{\lambda} R \otimes_{R_{\lambda}} \text{Ker } \beta_{\lambda}$  generated by finitely many elements chosen from the  $\text{Ker } \beta_{\lambda}$ . Thus there is an  $R$ -linear map

$$\bigoplus_{\lambda} R \otimes_{R_{\lambda}} P'_{\lambda} \rightarrow \bigoplus_{\lambda} R \otimes_{R_{\lambda}} \text{Ker } \beta_{\lambda}$$

whose image contains  $\text{Im } \alpha'$  and the  $P'_{\lambda}$  are free  $R_{\lambda}$ -modules of finite rank. Since  $P'$  is projective,  $\alpha'$  factors through this map and we have a commutative diagram

$$(28) \quad \begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ \downarrow \alpha' & & \downarrow u & & \\ \bigoplus_{\lambda} R \otimes_{R_{\lambda}} P'_{\lambda} & \xrightarrow{\bigoplus R \otimes \alpha_{\lambda}} & \bigoplus_{\lambda} R \otimes_{R_{\lambda}} P_{\lambda} & & \end{array}$$

where each  $\alpha_{\lambda}u^{-1}\beta$  is zero. Dualizing gives a diagram

$$(29) \quad \begin{array}{ccccc} P''^* & \xrightarrow{\beta^*} & P^* & \xrightarrow{\alpha^*} & P'^* \\ & & \uparrow u^* & & \uparrow \alpha^* \\ & & \bigoplus_{\lambda} P_{\lambda}^* \otimes_{R_{\lambda}} R & \xrightarrow{\bigoplus \alpha_{\lambda}^* \otimes R} & \bigoplus_{\lambda} P'_{\lambda}^* \otimes_{R_{\lambda}} R \end{array}$$

where  $\alpha_\lambda^* u^{*-1} \beta^* = 0$  for each  $\lambda$ . Now the procedure that led from (26) to (28) leads from (29) to the dual of a diagram of the form (25). Dualizing again will give the desired result.

For our first application we have the following result.

**Theorem 17.** *Let  $K$  be a skew field,  $\{R_\lambda\}_{\lambda \in \Lambda}$  be a set of  $K$ -rings, and  $R$  be their coproduct amalgamating  $K$ . If each  $R_\lambda$  is a Sylvester domain then  $R$  is a Sylvester domain.*

**Proof.** It suffices to consider the case where  $\Lambda$  is finite, say  $\Lambda = \{1, \dots, p\}$ . Let  $A, B$  be matrices over  $R$  such that  $AB = 0$ . We may apply Lemma 16, and here all finitely generated projective  $R_\lambda$ -modules are free, so there exist factorizations

$$A = A' \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix} U^{-1}, \quad B = U \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_p \end{pmatrix} B'$$

where each  $A_\lambda B_\lambda = 0$  over  $R_\lambda$ , and  $U$  is a square matrix over  $R$ , by [5, Corollary 2.11]. Let  $n_\lambda$  denote the number of columns of  $A_\lambda$  (or rows of  $B_\lambda$ ). Since each  $R_\lambda$  satisfies (10), and ring homomorphisms cannot increase inner rank,  $\rho(A_\lambda) + \rho(B_\lambda) \leq n_\lambda$  over  $R$ . Hence

$$\rho(A) + \rho(B) \leq \sum_\lambda \rho(A_\lambda) + \sum_\lambda \rho(B_\lambda) \leq \sum_\lambda n_\lambda,$$

which proves that  $R$  satisfies (10).

Our next result illuminates some of the properties that were considered in Section 2.

Let us fix a field  $k$ . For any positive integers  $m, r, n$  let  $R(m, r, n)$  denote the  $k$ -algebra having an  $m \times r$  matrix  $X = (x_{ij})$ , and an  $r \times n$  matrix  $Y = (y_{jk})$  whose product is 0, such that the pair  $X, Y$  is universal with these properties. That is,  $R$  is the  $k$ -algebra presented on  $mr + rn$  generators  $x_{ij}, y_{jk}$  and  $mn$  relations  $\sum_{j=1}^r x_{ij} y_{jk} = 0$ .

**Theorem 18** (with G.M. Bergman). *Let  $m, r, n$  be positive integers.  $R(m, r, n)$  is a Sylvester domain if and only if  $r \geq m + n$ .  $R(m, r, n)$  has every full matrix regular if and only if  $r > \max(m, n)$ .  $R(m, r, n)$  has every full matrix left regular if and only if  $r > n$ .*

**Proof.** Write  $R$  for  $R(m, r, n)$ . Observe that  $\rho(X) = \min(m, r)$  since there are homomorphisms from  $R$  to  $k$  which send  $Y$  to 0 and  $X$  to a matrix of inner rank  $\min(m, r)$ , for example, let each  $x_{ij}$  be mapped to the Kronecker delta,  $\delta_{ij}$ . Similarly  $\rho(Y) = \min(r, n)$ . Thus if  $R$  is a Sylvester domain then by (10)  $r \geq \min(m, r) + \min(r, n)$ , or equivalently,  $r \geq m + n$ . Similarly, if every left full matrix is

left regular then  $r > n$ , and by symmetry if every full matrix is regular then  $r > \max(m, n)$ .

To obtain the reverse implications we use the methods of [6]. Let  $S$  be the  $k$ -algebra with generators  $e, f, g, x, y$  and relations saying that  $e, f, g$  are mutually orthogonal idempotents summing to 1, and  $exf = x, fyg = y, xy = 0$ . As  $k$ -space,  $S$  is five-dimensional, and as ring can be viewed as the image, modulo the square of the radical, of the ring of  $3 \times 3$  upper triangular matrices over  $k$ . The  $k$ -subalgebra  $K$  of  $S$  generated by  $e, f, g$  is isomorphic to  $k \times k \times k$ , and  $S$  is obtained by universally adjoining to  $K$  maps  $x: Ke \rightarrow Kf, y: Kf \rightarrow Kg$ , whose composite is specified to be zero.

Let  $M(k)$  denote the full ring of  $(m+r+n) \times (m+r+n)$  matrices over  $k$ . There is an injective  $k$ -algebra homomorphism  $K \rightarrow M(k)$  that sends

$$e \text{ to } \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad g \text{ to } \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}.$$

These two partitions are different, and  $f = 1 - e - g$  is mapped to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There is a homomorphism from  $M(k) \amalg_K S$  to the ring  $M(R)$ , of all  $(m+r+n) \times (m+r+n)$  matrices over  $R$ , that extends the natural map  $M(k) \rightarrow M(R)$  and sends  $x, y$  to

$$\begin{pmatrix} 0 & X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. It is not difficult to verify directly from the universal properties that this is an isomorphism. Alternatively, this can be derived as a special case of the very general result [6, Theorem 3.4].

Now as in [6] we will use the coproduct results to obtain information on  $M(R)$ , and Morita equivalence will then give us information on  $R$ .

Let us sketch the elementary data on  $S$  that we will be using. The finitely generated indecomposable projective left  $S$ -modules are, up to isomorphism,  $Se, Sf, Sg$ . The nonzero  $\text{Hom}_S$  sets between these are  $eSe = ke, eSf = kx, fSf = kf, fSg = ky, gSg = kg$ , all of which are one-dimensional over  $k$ . Observe that

$$0 \rightarrow Se \xrightarrow{x} Sf \xrightarrow{y} Sg$$

is exact. Now let  $\beta: P \rightarrow Q$  be a homomorphism of finitely generated projective left  $S$ -modules, and choose decompositions

$$u: P \cong \bigoplus_{i=1}^b P_i, \quad v: Q \cong \bigoplus_{j=1}^c Q_j$$

into indecomposables. The map  $u^{-1}\beta v: \bigoplus P_i \rightarrow \bigoplus Q_j$  may be viewed as a  $b \times c$  matrix whose  $(i, j)$ th entry,  $\beta_{ij}$ , lies in  $\text{Hom}_S(P_i, Q_j)$ . From our description of the  $\text{Hom}_S$  sets for  $Se, Sf, Sg$  we see that if some column has two nonzero entries, say  $\beta_{ij}, \beta_{i'j}$ , then these both lie in  $ke$  or  $kx \cup kf$  or  $ky \cup kg$ , so one factors over the other. Say there exists  $\gamma \in \text{Hom}_S(P_{i'}, P_i)$  such that  $\gamma\beta_{ij} = \beta_{i'j}$ . The automorphism

$$\begin{pmatrix} 1_i & 0 \\ -\gamma & 1_{i'} \end{pmatrix}$$

of  $P_i \oplus P_{i'}$  induces a row operation on  $u^{-1}\beta v$ , namely subtracting the composite of  $\gamma$  with the  $i$ th row from the  $i'$ th row, that converts  $\beta_{i'j}$  to 0. Since similar statements hold with "row" and "column" interchanged, we can transform  $u^{-1}\beta v$  by automorphisms of  $\bigoplus P_i, \bigoplus Q_j$  until no row or column has two nonzero entries. By interchanging the  $P_i$  or the  $Q_j$  we may further assume that the nonzero entries are arranged along the diagonal. In particular if  $P' \xrightarrow{\alpha} P \xrightarrow{\beta} P''$  are homomorphisms of finitely generated projective left  $S$ -modules such that  $\alpha\beta = 0$  then they can be extended to a commutative diagram

$$\begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ \downarrow & & \downarrow & & \uparrow \\ P_0 \oplus (Se)^a & \longrightarrow & P_0 \oplus (Sf)^a \oplus P_1 & \longrightarrow & (Sg)^a \oplus P_1 \end{array}$$

where  $a$  is an integer and the maps in the bottom row are the obvious ones composing to zero.

Any homomorphisms  $P' \xrightarrow{\alpha} P \xrightarrow{\beta} P''$  of finitely generated projective left  $M(k)$ -modules such that  $\alpha\beta = 0$  can be extended to a commutative diagram

$$\begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ \downarrow & & \downarrow & & \uparrow \\ P_0 & \longrightarrow & P_0 \oplus P_1 & \longrightarrow & P_1 \end{array}$$

where the maps in the bottom row are clear.

Thus by Lemma 16 any homomorphisms  $P' \xrightarrow{\alpha} P \xrightarrow{\beta} P''$  of finitely generated projective left  $M(R)$ -modules such that  $\alpha\beta = 0$  can be extended to a commutative diagram

$$\begin{array}{ccccc} P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\ \downarrow & & \downarrow & & \uparrow \\ P_0 \oplus (M(R)e)^a & \longrightarrow & P_0 \oplus (M(R)f)^a \oplus P_1 & \longrightarrow & (M(R)g)^a \oplus P_1 \end{array}$$

where the maps in the bottom row are clear,  $a$  some integer.



Now by Morita equivalence this says that any homomorphisms  $P' \xrightarrow{\alpha} P \xrightarrow{\beta} P''$  of finitely generated projective left  $R$ -modules such that  $\alpha\beta = 0$  can be extended to a commutative diagram

$$\begin{array}{ccccc}
 P' & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & P'' \\
 \downarrow & & \downarrow & & \uparrow \\
 P_0 \oplus R^{ma} & \longrightarrow & P_0 \oplus R^{ra} \oplus P_1 & \longrightarrow & R^{na} \oplus P_1
 \end{array}$$

where the maps in the bottom row are clear,  $a$  some integer. Now observe that  $K \rightarrow S$  induces a bijection on finitely generated projectives over the respective rings, and hence the same is true of  $M(k) \rightarrow M(R)$  by [5, Corollary 2.11]. This implies that  $R$  is projective-free. So we have proved that for any matrices  $A, B$  over  $R$  if  $AB = 0$  then there exist factorizations

$$A = A' \begin{pmatrix} I_0 & & & 0 \\ & X & & \\ & & \ddots & \\ & & & X \\ 0 & & & & 0 \end{pmatrix} U^{-1}, \quad B = U \begin{pmatrix} 0 & & & 0 \\ & Y & & \\ & & \ddots & \\ & & & Y \\ 0 & & & & I_1 \end{pmatrix} B'$$

where  $U$  is a square matrix. If there are  $a$  occurrences of  $X, Y$  then

$$\rho(A) \leq \rho(I_0) + a\rho(X), \quad \rho(B) \leq a\rho(Y) + \rho(I_1).$$

Now if  $r \geq m + n$  then

$$\rho(A) + \rho(B) \leq \rho(I_0) + ar + \rho(I_1)$$

and (10) is satisfied. Thus  $R(m, r, n)$  is a Sylvester domain if and only if  $r \geq m + n$ .

The other results follow similarly.

A more careful analysis shows that  $R(m, r, n)$  has left and right global dimension 2 and is left and right coherent. We do not know if all the flat modules are directed unions of free submodules, except in the cases where they are spacial by Theorem 5.

Notice that the ring  $R(2, 3, 2)$  has all the good module theoretic properties that could be desired; since every full matrix is regular, the flat modules are all spacial, and it is known that the left and right global dimension is 2. And yet  $R(2, 3, 2)$  is not a Sylvester domain.

Notice also that over  $R(3, 3, 2)$  every full matrix is left regular but not necessarily right regular, which shows further that the conditions of Theorem 5 are not left-right symmetric.

For any positive integer  $r$  let us call a nonzero ring an  $r$ -Sylvester domain if for  $n = 1, \dots, r$  whenever  $A, B$  are matrices with  $n$  columns and rows respectively, if  $AB = 0$  then  $n \geq \rho(A) + \rho(B)$ . Thus the 1-Sylvester domains are the nonzero rings

without proper zerodivisors. Among these the 2-Sylvester domains are characterized by the property that the intersection of any two principal right (or left) ideals is locally-cyclic, that is, the directed union of the principal right (left) ideals it contains. An integrally closed integral domain with this "Riesz interpolation property" is called a *Schreier ring*, cf. [4, pp. 73, 150].

For any positive integer  $r$  the class of  $r$ -Sylvester domains is contained in the class of  $(r+1)$ -Sylvester domains, and these classes are not equal since, by the proof of Theorem 18,  $R(m, r+1, n)$  is always an  $r$ -Sylvester domain, but is not an  $(r+1)$ -Sylvester domain if  $m+n > r+1$ .

## 7. Open problems

1. Over a Sylvester domain is every (countably-generated) projective module free?
2. Is every local integral domain that has weak global dimension 2 a Sylvester domain? More generally, is every projective-free integral domain that has weak global dimension 2 a Sylvester domain?
3. Let  $B$  be a Bezout domain. Is every free  $B$ -algebra a Sylvester domain? Is this even true for  $B$  a valuation ring? Is every free power series  $B$ -algebra a Sylvester domain? Is this even true for a principal ideal domain? Or a valuation ring? What is the weak global dimension of a free power series  $B$ -algebra?
4. (Bergman, after Lewin) Let  $k$  be a field. Is the tensor product over  $k$  of two free  $k$ -algebras a Sylvester domain? More generally, if in a free  $k$ -algebra,  $k\langle X \rangle$ , relations are imposed saying that certain pairs of elements of  $X$  commute but these pairs are chosen so that no three elements of  $X$  commute pairwise then the resulting ring is known to have left and right global dimension at most 2. Is it a Sylvester domain?

## Appendix I. Related results of G.M. Bergman

In Section 2 it was observed that for a ring to satisfy Sylvester's law of nullity, its finitely generated projective modules have to be free. This appendix presents a weaker law of nullity that applies to a wider class of rings, whose finitely generated projectives are not quite so restricted. The theory, propounded by G.M. Bergman in a 1971 letter to P.M. Cohn, applies for example to the class of (left or right) semihereditary rings, and gives information about homomorphisms to skew fields in terms of the finitely generated projective modules.

Bergman's original arguments were phrased in the language of dependence relations, and have in some ways benefitted from being transcribed here into matrix

language. For example what here is the rather simple “law of nullity” was in Bergman’s notation a rather complicated matroid like condition.

Let  $R$  be a ring. The set of isomorphism classes,  $[P]$ , of finitely generated projective left  $R$ -modules,  $P$ , is a commutative semigroup,  $S_{\oplus}(R)$ , under the operation  $[P]+[Q]=[P\oplus Q]$ . By a *projective rank function*  $p$  on  $R$  we mean a homomorphism  $p: S_{\oplus}(R) \rightarrow \mathbb{N}$  of semigroups-with-distinguished-element, where  $\mathbb{N}$  denotes the additive semigroup of non-negative integers. Thus  $p$  is a semigroup homomorphism such that  $p([R])=1$ ; this is a retraction of the canonical homomorphism  $\mathbb{N} \rightarrow S_{\oplus}(R)$  of semigroups-with-distinguished-element. The existence of a projective rank function on a ring already restricts the ring to some extent; for example it implies that it has IBN, and is not a proper matrix ring.

Recall that every finitely generated projective left  $R$ -module is the homomorphic image of an idempotent matrix (viewed as an endomorphism of a finitely generated free left  $R$ -module) and that the image of every idempotent matrix is a finitely generated projective left  $R$ -module. Further two idempotent matrices  $E, F$  (not necessarily of the same order) have isomorphic images if and only if there are factorizations  $E = AB, F = BA$ . (By replacing  $A$  with  $ABA$  and  $B$  with  $BAB$  we may always assume  $EAF = A, FBE = B$ .) In this event let us say that  $E$  and  $F$  are *isomorphic*.

There is a natural bijective correspondence between projective rank functions on  $R$  and functions  $r$  that assign to each matrix  $A$  over  $R$  a non-negative integer  $rA$ , called the *r-rank* of  $A$ , such that the following hold:

(30) If  $E, F$  are isomorphic idempotent matrices then  $rE = rF$ .

(31) If  $E, F$  are idempotent matrices then  $r\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} = rE + rF$ .

(32)  $r(1) = 1$ .

(33) For any matrix  $A, rA = \min\{rE \mid A = BEC, E \text{ idempotent}\}$ .

In fact from the connection between finitely generated projective left modules and idempotent matrices, it is clear that (30)–(32) are precisely the conditions for  $r$  to determine a projective rank function. Thus we must show that (33) does not impose any further conditions on the value of  $r$  on idempotent matrices. Although this is obvious in terms of projective modules, let us verify it in terms of matrices. Suppose that  $F$  is an idempotent matrix and that  $F = BEC$  with  $E$  idempotent. Without loss of generality  $FBE = B$ , and  $ECF = C$  so  $F = BC, E - CB$  is idempotent and

$$E = (C \quad E - CB) \begin{pmatrix} B \\ E - CB \end{pmatrix}$$

is isomorphic to

$$\begin{pmatrix} B \\ E - CB \end{pmatrix} (C \quad E - CB) = \begin{pmatrix} F & 0 \\ 0 & E - CB \end{pmatrix}.$$

So by (30) and (31),  $rE \geq rF$ . Hence for idempotent matrices  $A$ , (33) is a consequence of (30), (31) so (33) extends  $r$  from the set of idempotent matrices to the set of all matrices. For this reason, let us call  $r$  an *inner projective rank function*.

For example, a projective-free ring has a unique projective rank function, and the corresponding inner projective rank function is the inner rank. Let us now show that conversely, if the inner rank is an inner projective rank function then the ring must be projective-free. This will follow immediately from the following result, shown to us by J. Lewin.

**Proposition 19** (Lewin). *The inner rank of an idempotent matrix is the minimum number of generators of the image.*

**Proof.** Let  $E$  be an idempotent matrix. For any factorization  $E = AB$ ,  $BABA$  is an idempotent matrix isomorphic to  $E$ . If the number of columns of  $A$  is chosen as small as possible then this is the inner rank of  $E$ , and is also the smallest order of an idempotent matrix isomorphic to  $E$ , that is, the minimum number of generators of the image of  $E$ .

**Corollary 20.** *Over a nonzero ring  $R$ , inner rank is an inner projective rank function if and only if  $R$  is projective-free.*

**Proof.** We have seen one direction of this. To see the other direction suppose that the inner rank is an inner projective rank function. Then by Proposition 19 “minimum number of generators” is additive on finitely generated projectives. In particular  $R^n$  has minimum number of generators  $n$ , and if  $P$  is projective with minimum number of generators  $n$  then  $R^n \cong P \oplus Q$  where  $Q$  has minimum number of generators 0, so  $P \cong R^n$  for a unique  $n$ , and  $R$  is projective-free.

This says that inner projective rank functions would generalize inner rank, if we had only defined inner rank on projective-free rings.

(A related result, not difficult to show, is that a ring is projective-free if and only if it is nonzero and every full idempotent matrix is an identity matrix.)

Let us say that an inner projective rank function  $r$  satisfies the law of nullity if

(34) for any matrices  $A, B$  if  $AB = 0$  then  $rA + rB \leq rE$  for every idempotent matrix  $E$  such that  $AE = A, EB = B$ .

To see this is equivalent to Sylvester’s law of nullity for projective-free rings, observe that (34) certainly implies (10) when  $r$  is the inner rank. Conversely, suppose (10) holds. In the situation of (34), we know by projective-freeness that  $E = XY, YX = I_n$  for some  $n$ , and by Proposition 19,  $n = \rho(E)$ . Further

$$\rho(A) + \rho(B) = \rho(AXY) + \rho(XYB) \leq \rho(AX) + \rho(YB)$$

and now by (10) this is at most  $n$  so  $\rho(A) + \rho(B) \leq \rho(E)$ , as desired. Thus (34) is a generalization of Sylvester's law of nullity.

A rich source of examples can be arrived at through the following definition. A ring  $R$  is said to be *weakly semihereditary* if for any matrices  $A, B$  if  $AB = 0$  then there exists an idempotent matrix  $E$  such that  $AE = A, EB = 0$ . For example a (left or right) semihereditary ring is weakly semihereditary, and thus so is any left or right hereditary ring.

Now if a ring  $R$  is projective-free then any idempotent matrix  $E$  is of the form

$$U^{-1} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} U$$

for some invertible square matrix  $U$ . (To see this write  $E = XY, YX = I, I - E = WZ, ZW = I'$  and take

$$U = (EX \quad W - EW), \quad U^{-1} = \begin{pmatrix} YE \\ Z - ZE \end{pmatrix}.$$

Cf. [8, Proposition 0.2.6].) Thus projective-free weakly semihereditary rings satisfy (13) and so are precisely the semifirs. And indeed weakly semihereditary rings seem to be the analogue for the law of nullity of what semifirs were for Sylvester's law of nullity.

**Proposition 21.** *If  $r$  is an inner projective rank function on a weakly semihereditary ring then  $r$  satisfies the law of nullity.*

**Proof.** Suppose  $AB = 0$  so  $AE = A, EB = b$ , for some idempotent,  $E$ . Then  $A(B - I - E) = 0$  so by the weakly semihereditary property there exists an idempotent matrix  $F$  such that  $AF = A, F(B - I - E) = 0$ . Hence  $EFE$  is idempotent and

$$rA + rB = r(AEFE) + r((E - EFE)B) \leq r(EFE) + r(E - EFE) \leq rE$$

where the last inequality holds by (31). This verifies (34).

For an inner projective rank function  $r$ , let us call an  $n \times n$  matrix,  $A$ ,  $r$ -full if  $rA = n$ .

**Theorem 22.** *Let  $r$  be an inner projective rank function over a ring  $R$ , and let  $P$  be the set of all square matrices over  $R$  that are not  $r$ -full. If  $r$  satisfies the law of nullity then  $P$  is a minimal prime matrix ideal,  $R_P$  is a skew field, and  $R \rightarrow R_P$  is rank-preserving in the sense that the  $r$ -rank of each matrix over  $R$  equals the inner rank of the image over  $R_P$ . In particular the kernel of  $R \rightarrow R_P$  is generated by the entries of the idempotent matrices of  $r$ -rank 0.*

**Proof.** We need the following:

(35) For any matrices  $A, B$   $r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = rA + rB$ .

(36) For any matrices  $A, B, C$  which all have the same number of rows, if  $r(A \ B) = r(A \ C) = rA$  then  $r(A \ B \ C) = rA$ .

The proofs of these are exactly as for (11), (12) and will be omitted.

The conditions (1)–(4) for  $P$  to be a prime matrix ideal are readily verified using (35), (36), and the “transpose” of (36).

For any  $m \times n$  matrix  $A$  over  $R$ , if  $q$  is the greatest integer such that  $A$  has a  $q \times q$   $r$ -full submatrix then every  $q + 1 \times q + 1$  submatrix has  $r$ -rank at most  $q$ . So by induction and (36), every  $q + 1 \times n$  submatrix has  $r$ -rank at most  $q$ . Now by symmetry we can add rows and find that every  $m \times n$  submatrix has  $r$ -rank at most  $q$ , that is,  $rA = q$ . In other words, if  $rA = q$  then every  $q + 1 \times q + 1$  submatrix is not  $r$ -full, and some  $q \times q$  submatrix is  $r$ -full. Hence over  $R_P/P$  every  $q + 1 \times q + 1$  submatrix of  $A$  is singular, and some  $q \times q$  submatrix is nonsingular. This shows that  $R \rightarrow R_P/P$  is rank-preserving. In particular, the kernel of  $R \rightarrow R_P/P$  consists of all  $1 \times 1$  matrices of  $r$ -rank 0, and by (33) this is the ideal generated by the entries of the idempotent matrices  $E$  with  $rE = 0$ .

Now  $R_P$  is local, so projective-free, so  $R_P \rightarrow R_P/P$  preserves the unique projective rank functions. Hence also  $R \rightarrow R_P$  preserves projective rank functions. In particular  $R \rightarrow R_P$  cannot increase the inner projective rank of any matrix, so every element of  $P$ , being non  $r$ -full, is mapped to a non-full matrix in  $R_P$ . Now by Proposition 4,  $R_P$  is a skew field; it is then obvious that  $P$  is a minimal prime matrix ideal, which completes the proof.

The following is immediate from Proposition 21 and Theorem 22.

**Theorem 23.** *Let  $R$  be a weakly semihereditary ring. Then there are bijective correspondences between the following (possibly empty) sets:*

- (i) *Projective rank functions  $p$  on  $R$ .*
- (ii) *Inner projective rank functions  $r$  on  $R$ .*
- (iii) *Minimal prime matrix ideals  $P$  over  $R$ .*
- (iv) *Skew fields  $F$  obtained from  $R$  by inverting matrices.*

*The correspondences for the last three are given by:*

$$r \mapsto P = \{\text{all square non } r\text{-full matrices}\},$$

$$P \mapsto F = R_P,$$

$$F \mapsto r = \text{pullback of the inner rank along } R \rightarrow F.$$

*Further, for any projective rank function  $p$  on  $R$  the corresponding skew field  $F$  will contain  $R$  if and only if  $p$  is nonzero on nonzero projectives.*

One consequence of the theorem is that a weakly semihereditary ring has a *homomorphism* to a skew field if and only if  $S_{\oplus}(R)$  has a retraction to  $\mathbb{N}$ , and has an *embedding* in a skew field if and only if  $S_{\oplus}(R) - \{0\}$  has a retraction to  $\mathbb{N} - \{0\}$ .

For any ring  $R$  every prime matrix ideal  $P$  gives rise to a homomorphism to a skew field  $R \rightarrow R_P/P$ , and every homomorphism to a skew field  $R \rightarrow F$  gives rise to a projective rank function  $p$  on  $R$ . Thus it can be deduced that over a weakly semihereditary ring, every prime matrix ideal contains exactly one minimal prime matrix ideal. In this connection, notice that it is clear from Theorem 23 that a weakly semihereditary ring has a unique minimal prime matrix ideal if and only if it has a unique projective rank function; the corresponding condition for the skew fields is that the ring has a “universal” map to a skew field in a particular category, cf. [8, Section 7.2].

Although Bergman’s letter also treats arbitrary semigroup homomorphisms  $S_{\oplus}(R) \rightarrow \mathbb{N}$  and ring homomorphisms from  $R$  to full matrix rings over skew fields, this would take us rather far afield so we decided, with regret, not to include it.

## Appendix II. Free radical rings

Let  $R$  be a commutative ring and  $A$  an  $R$ -algebra-without-1. There is associated with  $A$  a homomorphism to an  $R$ -algebra (with 1),  $A \rightarrow A(1) = R \oplus A$ , that is universal with this property. Also associated with  $A$  is a homomorphism to a (Jacobson) radical  $R$ -algebra-without-1,  $A \rightarrow \omega(A)$ . The existence of such a map can be seen by considering the variety of rings-without-1 with a quasi-inverse. In [9, Theorem 1] it was shown that  $A \rightarrow \omega(A)$  is universal with the property that every square matrix over  $A$  becomes quasi-invertible. Thus  $A(1) \rightarrow (\omega(A))(1)$  is the universal  $\Sigma$ -inverting map, where  $\Sigma$  is the set of all square matrices over  $A(1)$  which become invertible in  $A(1)/A = R$ .

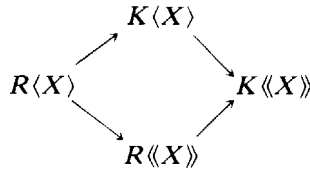
In this situation we will be able to apply the following result.

**Theorem 24.** *For a principal ideal domain  $R$  and set  $X$ , the  $X$ -adic completion  $R\langle X \rangle \rightarrow R\langle\langle X \rangle\rangle$  is rank-preserving. If  $\Sigma$  denotes the set of all matrices over  $R\langle X \rangle$  inverted by this homomorphism, and  $R \rightarrow R_{\Sigma}$  is the universal  $\Sigma$ -inverting map then  $R_{\Sigma} \rightarrow R\langle\langle X \rangle\rangle$  is an embedding.*

The (faithful) image of  $R_{\Sigma}$  in  $R\langle\langle X \rangle\rangle$  is denoted  $R^{\text{rat}}\langle\langle X \rangle\rangle$ , called the  $R$ -algebra of rational power series in  $X$ .

**Proof.** By [11, p. 416] this result holds in the case where  $R$  is a field, so holds for the field of fractions  $K$  of an arbitrary principal ideal domain  $R$ . Thus in the

commutative diagram



the upper path is rank-preserving by Theorem 13 and the result just quoted from [11]. Hence the first arrow in the lower path is rank-preserving, which proves the first part of the statement of the theorem. To see the second part, apply Proposition 4 with  $R$  and  $S$  of that proposition being  $R\langle X \rangle$ , and with  $T$  being  $R\langle\langle X \rangle\rangle$ . Trivially (14) holds, we have just verified (16), and (15) is immediate from the fact that a square matrix over  $R\langle X \rangle$  is in  $\Sigma$  if and only if the determinant of its constant term is a unit in  $R$ .

**Theorem 25.** *For a principal ideal domain  $R$  and set  $X$ , the free radical  $R$ -algebra-without-1 on  $X$  is  $XR^{\text{rat}}\langle\langle X \rangle\rangle$ . In particular, the free radical ring-without-1 on  $X$  is  $X\mathbb{Z}^{\text{rat}}\langle\langle X \rangle\rangle$ .*

**Proof.** By universal properties the free radical  $R$ -algebra-without-1 is  $\omega(XR\langle X \rangle)$  since  $XR\langle X \rangle$  is the free  $R$ -algebra-without-1 on  $X$ . The remarks preceding Theorem 24 together with Theorem 24 itself now show that  $XR^{\text{rat}}\langle\langle X \rangle\rangle$  is the free radical  $R$ -algebra-without-1 on  $X$ .

Theorem 25 was known for  $R$  a field, [9]. It should be noted that [9, Theorem 5] alleges that for any integral domain  $R$  the free radical  $R$ -algebra-without-1 on  $X$  is contained in  $R\langle\langle X \rangle\rangle$ , but the argument given is not valid. The error does not affect any results other than [9, Theorem 5].

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