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Forward completeness, unboundedness observability, and their Lyapunov characterizations ☆

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Abstract

A finite-dimensional continuous-time system is forward complete if solutions exist globally, for positive time. This paper shows that forward completeness can be characterized in a necessary and sufficient manner by means of smooth scalar growth inequalities. Moreover, a version of this fact is also proved for systems with inputs, and a generalization is also provided for systems with outputs and a notion (unboundedness observability) of relative completeness. We apply these results to obtain a bound on reachable states in terms of energy-like estimates of inputs. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider general nonlinear systems of the type

$$\dot{x} = f(x,u), \qquad y = h(x) \tag{1}$$

with states x in $\mathbb{R}^n$, inputs u taking values in $\mathbb{R}^m$, and outputs y in $\mathbb{R}^p$, but *our results will be novel even for classical differential equations*, *that is*, *in the cases when controls do not appear in the system description and there is no output map.*

We assume that the maps $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ and $h:\mathbb{R}^n\to\mathbb{R}^m$ are locally Lipschitz continuous. (Later we remark that the assumption on h can be relaxed to continuity.) We use the symbol $|\cdot|$ for Euclidean norms in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$, and $\|\cdot\|_\infty$ for essential supremum. By an *input signal* or *control* for (1) we mean any measurable locally essentially bounded function of time, $u(\cdot):\mathbb{R}\to\mathbb{R}^m$. For any such u and any $\xi\in\mathbb{R}^n$, there exists a unique maximally extended solution of the initial value problem:

$$\dot{x} = f(x,u), \quad x(0) = \xi. \tag{2}$$

Such a solution is defined over some open interval $(\sigma^{\min}_{\xi,u},\sigma^{\max}_{\xi,u})$ where $\sigma^{\min}_{\xi,u} < 0 < \sigma^{\max}_{\xi,u}$ and is denoted as $x(\cdot,\xi,u)$. We also write $y(t,\xi,u) := h(x(t,\xi,u))$ for all ξ, u, and each $t\in(\sigma^{\min}_{\xi,u},\sigma^{\max}_{\xi,u})$.

A system is called *forward complete* if for every initial condition ξ and every input signal u, the corresponding solution is defined for all $t\geqslant 0$, i.e. $\sigma^{\max}_{\xi,u} = +\infty$.

A strictly weaker property is that of unboundedness observability, introduced in [4–6]. System (1) has the *unboundedness observability property* (or just "UO") if, for each state ξ and control u such that

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$\sigma = \sigma^{\max}_{\xi,u} < \infty$, necessarily

$$\limsup_{t \nearrow \sigma} |y(t,\xi,u)| = +\infty. \tag{3}$$

In other words, it is possible to "observe" any unboundedness of the state. The contrapositive statement of this property says that, if $\sup_{t\in[0,T)}|y(t)| < \infty$ then $x(T)$ is defined.

Remark 1.1. Notice that for systems with a bounded output function (for example, if $h \equiv 0$), the property of unboundedness observability is equivalent to forward completeness. Notice that in general UO does not imply forward completeness; for instance, every system is UO taking as the output the whole state x.

Our main result is the following Lyapunov characterization of unboundedness observability.

Theorem 1. *System* (1) *has the unboundedness observability property if and only if there exist a proper and smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

$$DV(x)f(x,u) \leqslant V(x) + \sigma_1(|u|) + \sigma_2(|h(x)|),$$
$$\forall x \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^m \tag{4}$$

holds for some σ_1, σ_2 *of class* $\mathscr{K}_\infty$.

We will study also systems for which all inputs are required to take values in a fixed compact set, typically the unit ball in $\mathbb{R}^m$. Such inputs can be interpreted in our context as "disturbances" and we use therefore the notation "d" instead of "u" for them. In general, if $\mathscr{D} \subset \mathbb{R}^m$ is compact, we denote by $\mathscr{M}_\mathscr{D}$ the set of all measurable functions $d : \mathbb{R} \to \mathscr{D}$. Consider a system

$$\dot{x} = g(x,d) \tag{5}$$

with inputs in $\mathscr{M}_\mathscr{D}$.

The following Lyapunov characterization of forward completeness is a particular consequence of Theorem 1, but it is of interest in itself (and also, will be proved first, as part of the proof of Theorem 1).

Theorem 2. *System* (5) *is forward complete if and only if there exists a proper and smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that the following exponential growth condition is verified*:

$$DV(x)g(x,d) \leqslant V(x), \quad \forall x \in \mathbb{R}^n, \quad \forall d \in \mathscr{D}. \tag{6}$$

It would appear that even the special case when there are no disturbances d is a new result. Most of the literature in differential equations is in fact concerned with sufficient conditions for the global existence of solutions (see [1–3,10]), with the remarkable exception of [8], where a converse Lyapunov condition for forward completeness of time-varying systems without inputs is proved. Nevertheless, in that paper, only a continuous function is obtained, and in addition, due to the particular construction and the unusual notion of radial unboundedness adopted, the resulting Lyapunov function turns out to be time-dependent even in the special case of autonomous systems.

Besides the main results, this paper contains several intermediate estimates for completeness and unboundedness observability, which may be of independent interest. Finally, as an application, we obtain a bound on reachable states in terms of energy-like estimates of inputs (cf. Corollary 2.13).

2. Proofs

The proofs are organized as follows. We first provide growth estimates for solutions, then construct V for forward completeness, and finally apply this to the general problem by means of a combination of a "small gain" (for handling inputs) and an "output injection" (for outputs) trick.

2.1. Bounded reachable sets

The result which we prove first will be a critical step in our constructions; it shows that the set of states reachable from any compact set, in bounded time and using bounded controls, is bounded, provided that the outputs remain bounded. When there are no outputs ($h \equiv 0$), this fact amounts to the statement that reachable states from compact sets, in bounded time and using bounded controls are bounded, a fact which was proved in [7]; we shall prove the result by a reduction to that special case. (Note that when there are also no controls and we have just a differential equation $\dot{x} = f(x)$, the statement is an easy consequence of continuous dependence of solutions on initial conditions.)

For each nonnegative real numbers μ, η, ρ, T, and τ, and each state $\xi \in \mathbb{R}^n$, we let:

$$\mathscr{U}(\xi,\mu,\eta,\tau) := \{u \mid \|u\|_\infty \leqslant \mu, \sigma^{\max}_{\xi,u} \geqslant \tau, \text{ and } |y(t,\xi,u)| \leqslant \eta \ \forall t \in [0,\tau]\},$$

$$R(\xi,\mu,\eta,\tau) := \{x(\tau,\xi,u) \mid u \in \mathscr{U}(\xi,\mu,\eta,\tau)\}$$

and

$$\mathscr{R}^{\leqslant T}(\rho,\mu,\eta) := \bigcup_{|\xi|\leqslant\rho,\,\tau\in[0,T]} R(\xi,\mu,\eta,\tau).$$

Note that a state ζ belongs to the reachable set $\mathscr{R}^{\leqslant T}(\rho,\mu,\eta)$ if and only if there is some state ξ with $|\xi|\leqslant\rho$, some time $\tau\leqslant T$, and some input u bounded by μ such that $\zeta = x(\tau,\xi,u)$, where the solution $x(\cdot,\xi,u)$ is defined on the interval $[0,\tau]$ and has $|y(t,\xi,u)|\leqslant\eta$ for all $t\in[0,\tau]$.

Observe that $\mathscr{U}(\xi,\mu,\eta,\tau)$ increases with each of μ and τ, so $R(\xi,\mu,\eta,\tau)$ does, too. Therefore, the sets $\mathscr{R}^{\leqslant T}(\rho,\mu,\eta)$ are also increasing, and thus the function

$$\gamma(T,\rho,\mu,\eta) := \sup\{|\zeta| \mid \zeta\in\mathscr{R}^{\leqslant T}(\rho,\mu,\eta)\}$$

(possibly taking infinite values) is nondecreasing separately on each of the variables T,ρ,μ,η.

Lemma 2.1. *If system* (1) *is* UO, *then* $\gamma(T,\rho,\mu,\eta) < \infty$ *for all* T,ρ,μ,η.

Proof. The idea of the proof is this: since we are interested in sets of states which can be reached with output bounded by η, the dynamics of the system in the part of the state space where the outputs become larger than η do not affect the value of γ; thus, we modify the dynamics for those states, using a procedure motivated by the standard "output injection" construction in control theory. The modified system will be forward complete, and previously known results will be then applicable. Take any T,ρ,μ,η.

We start by picking any smooth function $\mathbb{R}\to[0,1]$ with the following properties:

$$\phi_\eta(r) = \begin{cases} 1 & \text{if } r\leqslant\eta,\\ 0 & \text{iff } r\geqslant\eta+1.\end{cases}$$

Next, we introduce the following auxiliary system:

$$\dot{x} = f(x,u)\phi_\eta(|h(x)|), \qquad y = h(x). \tag{7}$$

Observe that the function $f(x,u)\phi_\eta(|h(x)|)$ is still locally Lipschitz because h is such. The set $\mathscr{R}^{\leqslant T}(\rho,\mu,\eta)$ for this new system is equal to the respective one defined for the original system. So, if we prove that system (7) is forward complete, then Proposition 5.1 in [7] will give that $\mathscr{R}^{\leqslant T}(\rho,\mu,\eta)$ is bounded, since that reference states that the reachable sets for forward complete systems (in bounded time, starting from a compact set, and using bounded controls) are bounded.

Suppose that system (7) would not be forward complete, and pick an initial condition ξ and an input v such that the maximal solution of $\dot{z}=f(z,v)\phi_\eta(|h(z)|)$ with $z(0)=\xi$ has

$$|z(s)|\to\infty \quad \text{as } s\nearrow S<\infty. \tag{8}$$

We claim that $|h(z(s))| < \eta+1$ for all s. If this were not the case, then there would be some $s_0\in[0,S)$ so that $\zeta_0 := z(s_0)$ has $|h(\zeta_0)|\geqslant\eta+1$. But then $\hat{z}\equiv\zeta_0$ is a solution of the same equation (because ζ_0 is an equilibrium, since $\phi_\eta(|h(\zeta_0)|)=0$), and hence by uniqueness we have that $\hat{z}=z$, and thus z is bounded, contradicting (8). We conclude that $\phi_\eta(|h(z(s))|)>0$ for all $s\in[0,S)$. So, the function

$$\varphi(s) := \int_0^s \phi_\eta(|h(z(\tau))|)\,\mathrm{d}\tau$$

is strictly increasing, and maps $[0,S)$ onto an interval $[0,T)$ (with, in fact, $T\leqslant S$, because $\phi_\eta\leqslant 1$ everywhere). We let $x(t) := z(\varphi^{-1}(t))$ for all $t\in[0,T)$. This is an absolutely continuous function, and it satisfies $\dot{x}=f(x,u)$ on $[0,T)$, where u is the input $u(t)=v(\varphi^{-1}(t))$. Note that $x(0)=\xi$ and (8) says that $x(t)\to\infty$ as $t\nearrow T$, so $T=\sigma^{\max}_{\xi,u}$. The unboundedness observability property says then that $y(t,\xi,u)$ is unbounded on $[0,T)$. But $y(t,\xi,u)=h(x(t))=h(z(s))$, where $s=\varphi^{-1}(t)$, and we already proved that this last expression always has norm $\leqslant\eta+1$, so we arrived at a contradiction. □

2.2. Estimates for states

Lemma 2.2. *System* (1) *has the* UO *property if and only if there exist* $\mathscr{K}$ *functions* $\chi_1,\chi_2,\chi_3,\chi_4$ *and a constant* c *such that*

$$|x(t,\xi,u)| \leqslant \chi_1(t)+\chi_2(|\xi|)+\chi_3(\|u_{[0,t]}\|_\infty) + \chi_4(\|y_{[0,t]}\|_\infty)+c \tag{9}$$

holds for all $\xi\in\mathbb{R}^n$, *all input signals* u, *and all* $t\in[0,\sigma^{\max}_{\xi,u})$.

In order to keep notations simple, if the initial state ξ and input u are clear from the context, we use the convention that when we write "y", or "$y_{[0,t]}$" as above, we mean the output function $y(\cdot,\xi,u)$, or its restriction to the interval $[0,t]$, respectively.

Proof. Let us assume that (9) holds, and let us take any state ξ and input signal u. If it were the case that $|y(t,\xi,u)|$ remains bounded, say by L, then $|x(t,\xi,u)|\leqslant\chi_1(\sigma^{\max}_{\xi,u})+\chi_2(|\xi|)+\chi_3(M)+\chi_4(L)<\infty$ for all $t\in[0,\sigma^{\max}_{\xi,u})$, where $M := \|u_{[0,t)}\|_\infty$, and this contradicts the definition of $\sigma^{\max}_{\xi,u}$ (see e.g. [9, exercise C.3.14]). This proves the sufficiency part of the Lemma.

To prove the converse implication, assume the system is UO. By Lemma 2.1, $\gamma(t,\rho,\mu,\eta)<\infty$ for all t,ρ,μ,η. Pick any $\xi\in\mathbb{R}^n$, input signal u, and $t\in[0,\sigma^{\max}_{\xi,u})$, and let $\rho:=|\xi|$, $\mu:=\|u_{[0,t]}\|_\infty$, and $\eta:=\|y_{[0,t]}\|_\infty$. Then $x(t,\xi,u)\in\mathcal{R}^{\leqslant t}(\rho,\mu,\eta)$, so

$$|x(t,\xi,u)|\leqslant\gamma(t,\rho,\mu,\eta)\leqslant\chi(t)+\chi(\rho)+\chi(\mu)+\chi(\eta),$$

where $\chi(r):=\gamma(r,r,r,r)$. The function $\chi:\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$ is nondecreasing, because γ is nondecreasing in each variable, as remarked earlier. Thus, there exist a function $\tilde{\chi}\in\mathcal{K}_\infty$ and a constant c_0 such that $\chi(r)\leqslant\tilde{\chi}(r)+c$ for all r, and therefore (9) is valid with all $\chi_i=\tilde{\chi}$ and $c=4c_0$. $\square$

By Remark 1.1, we have the following corollary of Lemma 2.2.

Corollary 2.3. *System* (1) *is forward complete if and only if there exist* $\mathcal{K}$ *functions* χ_1,χ_2,χ_3 *and a constant* c *such that*

$$|x(t,\xi,u)|\leqslant\chi_1(t)+\chi_2(|\xi|)+\chi_3(\|u_{[0,t)}\|_\infty)+c \tag{10}$$

holds for all $\xi\in\mathbb{R}^n$, *all input signals* u, *and all* $t\in[0,\sigma^{\max}_{\xi,u})$. (*And thus,* $\sigma^{\max}_{\xi,u}=+\infty$.)

Consider now systems (5) with inputs in $\mathcal{M}_\mathcal{D}$. Taking $\chi_3(D)+c$ instead of c, where D is an upper bound on the elements of $\mathcal{D}$, Corollary 2.3, implies:

Corollary 2.4. *A system* (5) *with inputs in* $\mathcal{M}_\mathcal{D}$ *is forward complete if and only if there exist functions* χ_1,χ_2 *of class* $\mathcal{K}$ *and a constant* c *such that*

$$|x(t,\xi,d)|\leqslant\chi_1(t)+\chi_2(|\xi|)+c \tag{11}$$

holds for all $\xi\in\mathbb{R}^n$, *all* $d(\cdot)\in\mathcal{M}_\mathcal{D}$, *and all* $t\in[0,\sigma^{\max}_{\xi,d})$. (*And thus,* $\sigma^{\max}_{\xi,d}=+\infty$.)

Finally, the following result provides a "relative forward completeness" characterization of unboundedness observability, and is stated here for ease of future reference.

Corollary 2.5. *System* (5) *with inputs in* $\mathcal{M}_\mathcal{D}$, *and with an output function* $y=h(x)$, *has the* UO-*property if and only if there exist class* $\mathcal{K}_\infty$ *functions* ρ,χ_1,χ_2 *such that the following implication holds for all* $\xi\in\mathbb{R}^n$ *and all* $T\in[0,\sigma^{\max}_{\xi,d})$:

$$|h(x(t,\xi))|\leqslant\rho(|x(t,\xi)|)\quad\forall t\in[0,T]$$
$$\Rightarrow |x(t,\xi)|\leqslant\chi_1(t)+\chi_2(|\xi|)+c \tag{12}$$

for all $t\in[0,T]$.

Proof. One direction of the result follows simply considering separately the two cases $\|y\|_\infty\leqslant\rho(\|x\|_\infty)$ and $\|y\|_\infty>\rho(\|x\|_\infty)$. For the converse implication, we first absorb $\chi_3(D)$ into c, as before; then, by a standard small-gain argument, it is enough to let $\rho(r)=\chi_4^{-1}(r/2)$ in (9) having assumed without loss of generality $\chi_4\in\mathcal{K}_\infty$ (otherwise just take $\tilde{\chi}_4(r)=\chi_4(r)+r$). $\square$

2.3. Margins

We start by choosing a fixed smooth function θ: $\mathbb{R}_{\geqslant 0}\to[0,1]$ such that

$$\theta(r)=\begin{cases}1 & \text{if } r\leqslant 0,\\ 0 & \text{iff } r\geqslant 1.\end{cases}$$

Given system (1) and a function $\rho\in\mathcal{K}_\infty$, we introduce the following auxilliary system, with inputs in $\mathcal{M}_\mathcal{D}$, where $\mathcal{D}$ is the closed unit ball in $\mathbb{R}^m$:

$$\dot{x}=g(x,d)=f(x,d\rho(|x|))\theta(|h(x)|-\rho(|x|)). \tag{13}$$

Observe that $g(x,d)$ is still locally Lipschitz, because h is. Since every state ζ with $|h(\zeta)|\geqslant\rho(|\zeta|)+1$ is an equilibrium state for the system (13), it follows, by uniqueness of solutions, that for each initial state with $|h(\xi)|<\rho(|\xi|)+1$ and each $d\in\mathcal{M}_\mathcal{D}$, the trajectory $x(t,\xi,d)$ never enters the set where $|h(x)|\geqslant\rho(|x|)+1$.

Lemma 2.6. *Suppose that system* (1) *has the unboundedness observability property. Then, there exists a function* $\rho\in\mathcal{K}_\infty$ (*called a* "*margin*") *such that the system* (13) *is forward complete.*

Proof. We define

$$\rho(r)=\min\{\chi_3^{-1}(r/4),\chi_4^{-1}(r/4)/2\}$$

in terms of comparison functions as in Lemma 2.2, assuming without loss of generality that they are of class $\mathcal{K}_\infty$. Let us pick any state ξ and input $d\in\mathcal{M}_\mathcal{D}$, and consider the maximal solution z of $\dot{z}=g(z,d)$ with $z(0)=\xi$, defined on $[0,S)$. We want to show that $S=+\infty$. If $|h(\xi)|\geqslant\rho(|\xi|)+1$ then, as pointed out above, $z\equiv\xi$, so indeed $S=+\infty$. So we may suppose that the trajectory has $\theta(|h(z(s))|)-\rho(|z(s)|)>0$ for all s. As in the proof of Lemma 2.1, we consider the time reparametrization $\varphi : [0,S) \to [0,T)$ defined by

$$\varphi(s) := \int_0^s \theta(|h(z(\tau)|) - \rho(|z(\tau)|)\,ds.$$

Note that $T \leqslant S$ because $\theta \leqslant 1$ everywhere, and that φ is strictly increasing.

Letting $x(t) := z(\varphi^{-1}(t))$, we have that $\dot{x} = f(x,u)$, where we are defining the input $u(t) = d(\varphi^{-1}(t))\,\rho(|x(t)|)$, and $x(0) = \xi$. Thus, applying the estimate in Lemma 2.6 and the supremum of $x(t)$ over an interval $[0,t]$ with $t < T$, we obtain, for all such t:

$$\|x_{[0,t]}\|_\infty \leqslant \chi_1(t) + \chi_2(|\xi|) + \chi_3(\|u_{[0,t]}\|_\infty) + \chi_4(\|y_{[0,t]}\|_\infty) + c. \qquad (14)$$

On the other hand, $\|u_{[0,t]}\|_\infty \leqslant \rho(\|x_{[0,t]})\|_\infty)$, so $\chi_3(\|u_{[0,t]})\|_\infty) \leqslant \|x_{[0,t]})\|_\infty/4$ because of the choice of ρ, and, since $|h(x(t))| - \rho(|x(t)|) < 1$ for all t,

$$\chi_4(\|y_{[0,t]})\|_\infty) \leqslant \chi_4\left(\frac{1}{2}\chi_4^{-1}\left(\frac{\|x_{[0,t]})\|_\infty}{4}\right) + 1\right) \leqslant \frac{\|x_{[0,t]})\|_\infty}{4} + \chi_4(2)$$

and we conclude from (14) that

$$\|x_{[0,t]})\|_\infty \leqslant 2\chi_1(t) + 2\chi_2(|\xi|) + 2c + 2\chi_4(2)$$

for all $t \in [0, \sigma^{\max}_{\xi,u})$, so it follows that the trajectory is bounded by $2\chi_1(T) + 2\chi_2(|\xi|) + 2c + 2\chi_4(2)$, which is a contradiction if $T < \infty$. So the system is indeed complete. □

2.4. *Lyapunov functions for forward completeness*

Given a forward complete system (5), we associate to it the following function $W : \mathbb{R}_{\geqslant} 0 \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$:

$$W(t,\xi) := \inf_{-t \leqslant \tau \leqslant 0,\, d \in \mathscr{M}_{\mathscr{D}}} |x(\tau,\xi,d)|. \qquad (15)$$

As the system is not assumed to be backward complete, the solution $x(t,\xi,d)$ might fail to exist for some $\tau < 0$ and d; in that case, we make the convention that $|x(\tau,\xi,d)| = +\infty$ when defining the infimum. We have

$$0 \leqslant W(t,\xi) \leqslant |\xi|, \quad \forall t \geqslant 0, \ \forall \xi \in \mathbb{R}^n.$$

Conversely, W is bounded below by ξ in the following sense. Let $\chi_1, \chi_2 \in \mathscr{K}$ and c be as in (11).

Lemma 2.7. *For each $t \geqslant 0$ and $\xi \in \mathbb{R}^n$, $|\xi| \leqslant \chi_1(t) + \chi_2(W(t,\xi)) + c$.*

Proof. Pick any $\varepsilon > 0$. There is some $\tau \in [-t, 0]$ and some $d \in \mathscr{M}_D$ such that $\zeta := x(\tau,\xi,d)$ satisfies

$$|\zeta| \leqslant W(t,\xi) + \varepsilon.$$

Letting $\tilde{d}(s) := d(s+\tau)$, we have $\xi = x(-\tau,\zeta,\tilde{d})$, so forward completeness gives that

$$|\xi| \leqslant \chi_1(-\tau) + \chi_2(|\zeta|) + c \leqslant \chi_1(t) + \chi_2(W(t,\xi) + \varepsilon) + c$$

and taking limits as $\varepsilon \searrow 0$ gives the conclusion. □

The lemma tells us that $W(t,x)$ is radially unbounded, in the sense that $W(t,x) \to +\infty$ as $x \to +\infty$, uniformly in t for t in any compact subset of $\mathbb{R}_{\geqslant 0}$. This latter property was first introduced in [8], with the name "mild radial unboundedness".

Lemma 2.8. *The function $W(t,x) : \mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is locally Lipschitz continuous.*

Proof. We pick an $M > 0$, and our goal is to find an $L = L_M > 0$ so that

$$|W(t,\xi) - W(\bar{t},\zeta)| \leqslant L(|\xi - \zeta| + |t - \bar{t}|)$$

holds for all $|\xi|, |\zeta| \leqslant M$ and all $0 \leqslant t$, $\bar{t} \leqslant M$. We let

$$K = K_M := \chi_1(M) + \chi_2(M+1) + c,$$

and, in general, use $\mathscr{B}_r \subseteq \mathbb{R}^n$ be the ball of radius r centered at the origin. Note that

$$\zeta \in \mathscr{B}_{M+1} \Rightarrow x(\tau,\zeta,d) \in \mathscr{B}_K \qquad (16)$$

for all $d \in \mathscr{M}_D$ and all $\tau \in [0,M]$.

Claim. *If*

$$\dot{x} = \tilde{g}(x,d) \qquad (17)$$

is any other system with the property that $g(x,d) = \tilde{g}(x,d)$ for all $x \in \mathscr{B}_K$, then, if we let $\tilde{W}$ be the function analogous to W, but defined for this other system, it holds that $W(t,\xi) = \tilde{W}(t,\xi)$ for all $\xi \in \mathscr{B}_M$ and all $t \in [0,M]$.

Proof. Pick any $\xi \in \mathscr{B}_M$ and $t \in [0,M]$, and any $\varepsilon \in (0,1)$. By definition of W, there is some $s \in [-t,0]$ and some input d such that

$$|x(s,\xi,d)| \leqslant W(t,\xi) + \varepsilon.$$

Since $W(t,\xi) \leqslant |\xi| \leqslant M$ and $\varepsilon < 1$, we have $\zeta := x(s,\xi,d) \in \mathscr{B}_{M+1}$. Thus (16) gives us $x(\tau,\xi,d) \in \mathscr{B}_K$ for each $\tau \in [s,0]$. Since the equations coincide for states in $\mathscr{B}_K$, $x(s,\xi,d) = \tilde{x}(s,\xi,d)$, where we are using "$\tilde{x}$" for the solutions of (17). Therefore, $\tilde{W}(t,\xi)\leqslant|\tilde{x}(s,\xi,d)|\leqslant W(t,\xi)+\varepsilon$, and taking $\varepsilon\searrow 0$ we conclude that $\tilde{W}(t,\xi)\leqslant W(t,\xi)$.

Conversely, pick any ξ,t,ε as before, and some $s\in[-t,0]$ and d such that $|\zeta|\leqslant\tilde{W}(t,\xi)+\varepsilon\leqslant M+1$, where $\zeta:=\tilde{x}(s,\xi,d)$. Property (16) says that $x(\tau,\zeta,\tilde{d})\in\mathscr{B}_K$ for each $\tau\in[0,-s]$, where $\tilde{d}(a)=d(a+s)$. Define $\xi':=x(-s,\zeta,\tilde{d})\in\mathscr{B}_K$. Since $x(\ell,\zeta,\tilde{d})\in\mathscr{B}_K$ for all intermediate times ℓ, and the equations coincide on $\mathscr{B}_K$, we have that

$$\xi'=x(-s,\zeta,\tilde{d})=\tilde{x}(-s,\zeta,\tilde{d})=\xi.$$

Therefore $\zeta=x(s,\xi,d)$, from which it follows by definition of $W(t,\xi)$ that $W(t,\xi)\leqslant|x(s,\xi,d)|\leqslant\tilde{W}(t,\xi)+\varepsilon$, and, once more letting $\varepsilon\searrow 0$, we have that also the reverse inequality $W(t,\xi)\leqslant\tilde{W}(t,\xi)$ holds, establishing the claim. □

We may apply the conclusion of this claim to the new system

$$\dot{z}=\tilde{g}(z,d)=\varphi(z)g(z,d),$$

where $\varphi:\mathbb{R}^n\to[0,1]$ is smooth and has $\varphi(z)=1$ when $z\in\mathscr{B}_K$ and $\varphi(z)=0$ if $z\notin B_{K+1}$. This system is complete (in negative as well as positive time), because the right-hand side vanishes outside a ball. To prove a Lispchitz estimate for W restricted to $x\in\mathscr{B}_M$ and $t\leqslant M$ it is enough, in view of the claim, to study this system. Thus, from now on we assume, without loss of generality, that the given *system* (5) *is complete.*

Pick any $\xi,\zeta\in\mathscr{B}_M$ and any positive $t,\bar{t}$ such that $0\leqslant t,\ \bar{t}\leqslant M$, and let $\varepsilon\in(0,1)$. By definition of W, we have that there are some $d_{\varepsilon,\zeta,t}$ and some $\tau_{\varepsilon,\zeta,t}\in[-t,0]$ such that

$$|x(\tau_{\varepsilon,\zeta,t},\zeta,d_{\varepsilon,\zeta,t})|\leqslant W(t,\zeta)+\varepsilon. \tag{18}$$

Taking increments of W yields:

$$W(t,\xi)-W(\bar{t},\zeta)=[W(t,\xi)-W(t,\zeta)] +[W(t,\zeta)-W(\bar{t},\zeta)]. \tag{19}$$

The first term in (19) can be estimated by virtue of (18) according to:

$$W(t,\xi)-W(t,\zeta) \leqslant \inf_{\tau\in[-t,0]}|x(\tau,\xi,d_{\varepsilon,\zeta,t})|-|x(\tau_{\varepsilon,\zeta,t},\zeta,d_{\varepsilon,\zeta,t})|+\varepsilon. \tag{20}$$

Since the system is backwards complete, there is some $K=K(M)$ with the property that $x(s,\xi,d)\in\mathscr{B}_K$ for all $s\in[-M,0]$ and all $\xi\in\mathscr{B}_M$ (just reverse time and apply the forward completeness results). By Lipschitz continuity of f, we have that there is some G such that

$$|f(x,d)-f(y,d)|\leqslant G|x-y|$$

for all $x,y\in\mathscr{B}_K$. So, by Gronwall's lemma:

$$\begin{aligned}
&\inf_{\tau\in[-t,0]}|x(\tau,\xi,d_{\varepsilon,\zeta,t})|-|(\tau_{\varepsilon,\zeta,t},\zeta,d_{\varepsilon,\zeta,t})|+\varepsilon\\
&\quad\leqslant|x(\tau_{\varepsilon,\zeta,t},\xi,d_{\varepsilon,\zeta,t})|-|x(\tau_{\varepsilon,\zeta,t},\zeta,d_{\varepsilon,\zeta,t})|+\varepsilon\\
&\quad\leqslant|x(\tau_{\varepsilon,\zeta,t},\xi,d_{\varepsilon,\zeta,t})-x(\tau_{\varepsilon,\zeta,t},\zeta,d_{\varepsilon,\zeta,t})|+\varepsilon\\
&\quad\leqslant e^{GM}|\xi-\zeta|+\varepsilon.
\end{aligned} \tag{21}$$

From (20) and letting $\varepsilon\searrow 0$, we conclude that have

$$W(t,\xi)-W(t,\zeta)\leqslant e^{GM}|\xi-\zeta|. \tag{22}$$

Similarly, there are $\tau_{\varepsilon,\zeta,\bar{t}}\in[-\bar{t},0]$ and $d_{\varepsilon,\zeta,\bar{t}}$ so that, assuming without loss of generality $\bar{t}>t$ the second contribution in (19) can be estimated as follows:

$$\begin{aligned}
&W(t,\zeta)-W(\bar{t},\zeta)\\
&\quad\leqslant\inf_{\tau\in[-t,0]}|x(\tau,\zeta,d_{\varepsilon,\zeta,\bar{t}})|-|x(\tau_{\varepsilon,\zeta,\bar{t}},\zeta,d_{\varepsilon,\zeta,\bar{t}})|+\varepsilon\\
&\quad\leqslant\begin{cases}\varepsilon & \text{if } \tau_{\varepsilon,\zeta,\bar{t}}\in[-t,0],\\ |x(t,\zeta,d_{\varepsilon,\zeta,\bar{t}}) & \\ \quad -x(\tau_{\varepsilon,\zeta,\bar{t}},\zeta,d_{\varepsilon,\zeta,\bar{t}})|+\varepsilon & \text{if } \tau_{\varepsilon,\zeta,\bar{t}}\in[-\bar{t},-t],\end{cases}\\
&\quad\leqslant R|t-\bar{t}|+\varepsilon,
\end{aligned} \tag{23}$$

where R is an upper bound for $|f(x,d)|$ when $x\in\mathscr{B}_K$ (for instance take $R=GM$). Letting $\varepsilon\searrow 0$ and combining the inequalities, we obtain

$$W(t,\xi)-W(\bar{t},\zeta)\leqslant e^{GM}(|\xi-\zeta|+|t-\bar{t}|).$$

By a symmetric argument, we can find a similar estimate for $W(\bar{t},\zeta)-W(t,\xi)$; thus, we conclude that W is locally Lipschitz. □

Recall that, from Lemma 2.7, we know that $|\xi|\leqslant\chi_1(t)+\chi_2(W(t,\xi))+c$ for each $t\geqslant 0$ and $\xi\in\mathbb{R}^n$. We assume, without loss of generality, that $\chi_1,\chi_2\in\mathscr{K}_\infty$. This gives that

$$\chi_1^{-1}\left(\frac{|\xi|}{2}\right)\leqslant t+\chi_1^{-1}(\chi_2(W(t,\xi))+c)$$

and thus, taking exponentials of both sides,

$$\gamma(|\xi|)e^{-t/2}\leqslant\alpha(W(t,\xi)) \tag{24}$$

for all ξ, t, where

$$\gamma(r) := \exp\left[\frac{\chi_1^{-1}(r/2)}{2}\right]$$

and

$$\alpha(r) := \exp\left[\frac{\chi_1^{-1}(\chi_2(r)+c)}{2}\right].$$

Note that γ and α are both strictly increasing and continuous, and $\gamma(0) = \exp[\chi_1^{-1}(0)/2] = 1 > 0$. Without loss of generality (just replacing α by a suitable upper bound), we may also assume that α is locally Lipschitz. Consider now the function $U(\xi)$ defined by the formula:

$$U(\xi) := \inf_{t\geqslant 0} \alpha(W(t,\xi))\mathrm{e}^t.$$

Since, by Eq. (24), $U(\xi) \geqslant \inf_{t\geqslant 0}\gamma(|\xi|)\mathrm{e}^{t/2}$, and also $U(\xi) \leqslant \alpha(W(0,\xi))$, we have that

$$\gamma(|\xi|) \leqslant U(\xi) \leqslant \alpha(|\xi|). \tag{25}$$

for all ξ, and, in particular, U is proper (radially unbounded).

Lemma 2.9. *The function U is locally Lipschitz.*

Proof. Pick any $M > 0$. Because of (25), there is some $T > 0$ such that

$$U(\xi) = \min_{t\in[0,T]} \alpha(W(t,\xi))\mathrm{e}^t$$

for all $\xi \in \mathscr{B}_M$. So, for all $\xi, \zeta \in \mathscr{B}_M$, we have

$$\begin{aligned} U(\xi) - U(\zeta) &= \min_{t\in[0,T_M]} \alpha(W(t,\xi))\mathrm{e}^t - \alpha(W(\tau_\zeta,\zeta))\mathrm{e}^{\tau_\zeta} \\ &\leqslant \alpha(W(\tau_\zeta,\xi))\mathrm{e}^{\tau_\zeta} - \alpha(W(\tau_\zeta,\zeta))\mathrm{e}^{\tau_\zeta} \\ &\leqslant C|\xi-\zeta|, \end{aligned}$$

where we let $C := K\mathrm{e}^{\mathrm{T}}M$, where K is a Lipschitz constant for $\alpha(W(\cdot,\cdot))$ and by continuity

$$\tau_\zeta := \arg\min_{t\in[0,T_M]} \alpha(W(t,\zeta))\mathrm{e}^t.$$

By symmetry, U is indeed Lipschitz.

We consider the upper Dini derivatives along trajectories:

$$\dot U(\xi,d) = \limsup_{h\to 0^+} \frac{U(x(h,\xi,d)) - U(\xi)}{h}. \tag{26}$$

As a consequence of definition (15), W is non-increasing along trajectories of (5), in the following sense:

$$W(t+h, x(h,\xi,d)) \leqslant W(t,\xi), \quad \forall d, \forall h\geqslant 0, \forall \xi. \tag{27}$$

Then, by definition of U we obtain

$$\begin{aligned} \dot U(\xi,d) &= \limsup_{h\to 0^+} \frac{1}{h}\left\{\inf_{t\geqslant 0}[\alpha(W(t,x(h,\xi,d)))\mathrm{e}^t] - \inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t]\right\} \\ &\leqslant \limsup_{h\to 0^+} \frac{1}{h}\left\{\inf_{t\geqslant h}[\alpha(W(t,x(h,\xi,d)))\mathrm{e}^t] - \inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t]\right\} \\ &\leqslant \limsup_{h\to 0^+} \frac{1}{h}\left\{\inf_{t\geqslant h}[\alpha(W(t-h,\xi))\mathrm{e}^t] - \inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t]\right\} \\ &= \limsup_{h\to 0^+} \frac{1}{h}\left\{\mathrm{e}^h\inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t] - \inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t]\right\} \\ &\leqslant \limsup_{h\to 0^+} \frac{\mathrm{e}^h-1}{h}\inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t] \\ &= \inf_{t\geqslant 0}[\alpha(W(t,\xi))\mathrm{e}^t] = U(\xi) \end{aligned} \tag{28}$$

where the second inequality follows by (27).

Proof of Theorem 2. By Theorem B.1 in [7], there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies the following properties:

$$\begin{aligned} &|V(\xi) - U(\xi)| < U(\xi)/2, \quad \forall \xi \in \mathbb{R}^n, \\ &DV(\xi)f(\xi,d) \leqslant U(\xi) + U(\xi)/2, \\ &\quad \forall \xi \in \mathbb{R}^n,\ \forall d \in \mathscr{D}. \end{aligned} \tag{29}$$

It follows by (29) that $V(\xi) \geqslant U(\xi)/2$, and hence V is proper. Further,

$$DV(\xi)f(\xi,d) \leqslant 3V(\xi), \quad \forall \xi\in\mathbb{R}^n,\ \forall d\in\mathscr{D}. \tag{30}$$

Notice that, $V(\xi) \geqslant U(\xi)/2 \geqslant \gamma(|\xi|)/2 \geqslant \gamma(0)/2 > 0$; as a consequence, $V(\xi)^{1/3}$ is a smooth Lyapunov function satisfying (6). □

2.5. Proof of Theorem 1

Sufficiency is obvious, since the differential inequality for $V(x(t))$ along trajectories is linear on $V(x(t))$. To show necessity, we use, by Lemma 2.6 and

Proposition 2, that there exists a proper and smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$DV(x)f(x,\rho(|x|)d)\theta(|h(x)| - \rho(|x|)) \leqslant V(x)$$

holds for each $x \in \mathbb{R}^n$ and each $d \in \mathscr{D}$. Hence,

$$|h(x)| \leqslant \rho(|x|) \quad \text{and} \quad |u| \leqslant \rho(|x|) \Rightarrow DV(x)f(x,u) \leqslant V(x).$$

Then, letting σ_1 and σ_2 be defined by

$$\sigma_1(r) = r + \max_{|x| \leqslant \rho^{-1}(|u|), |u| \leqslant r} |DV(x)f(x,u)|, \tag{31}$$

$$\sigma_2(r) = r + \max_{|x| \leqslant \rho^{-1}(|h(x)|), |h(x)| \leqslant r} |DV(x)f(x,u)|, \tag{32}$$

(the additive r's are just to insure that the maps are strictly increasing) we obtain

$$DV(x)f(x,u) \leqslant V(x) + \sigma_1(|u|) + \sigma_2(|h(x)|) \tag{33}$$

thus finishing the proof. □

Remark 2.10. Notice that, as far as the sufficiency part of Theorem 1 is concerned, it is enough to check for satisfaction of (4) outside a ball of arbitrarily large radius (basically we only need the inequality to be satisfied in a neighborhood of ∞). Hence, unboundedness observability is equivalent to the existence of a proper and smooth function $V(x)$ such that, for some σ_1, σ_2 of class $\mathscr{K}_\infty$, there is some $M > 0$ so that

$$DV(x)f(x,u) \leqslant V(x) + \sigma_1(|u|) + \sigma_2(|h(x)|) \tag{34}$$

holds for all $|x| \geqslant M$ and $u \in \mathbb{R}^m$.

2.6. Some restatements and consequences

There are several interesting ways to restate our results, and also some consequences worth pointing out. (As a matter of fact, trying to prove these consequences was the motivation behind this work.) As a corollary of the previous theorem and by virtue of Remark 1.1, we have the following Lyapunov characterization of forward completeness:

Corollary 2.11. *System* (1) *is forward complete if and only if there exists a smooth and proper function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *and such that*

$$DV(x)f(x,u) \leqslant V(x) + \sigma(|u|), \quad \forall x \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^m$$

holds for some σ *of class* $\mathscr{K}_\infty$.

It is an easy consequence of Theorem 1, taking as a function $W(x) = \log(1 + V(x))$, that the following Lyapunov characterizations of unboundedness observability and forward completeness are also true:

Corollary 2.12. *System* (1) *is forward complete if and only if there exists a smooth and proper function* $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

$$DW(x)f(x,u) \leqslant 1 + \sigma(|u|), \quad \forall x \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^m, \tag{35}$$

for some σ *of class* $\mathscr{K}_\infty$. *Similarly, system* (1) *has the unboundedness observability property if and only if there exists a smooth and proper function* $W : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

$$DW(x)f(x,u) \leqslant 1 + \sigma_1(|u|) + \sigma_2(|h(x)|), \quad \forall x \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^m \tag{36}$$

holds for some σ_1, σ_2 *in* $\mathscr{K}_\infty$.

By properness of the function W in Corollary 2.12, we know that there exists a class $\mathscr{K}_\infty$ function α such that $\alpha(|x|) \leqslant W(x)$ for all $x \in \mathbb{R}^n$. It is straightforward from (35) and (36) that the following inequalities are equivalent, respectively, to forward completeness and unboundedness observabililty:

$$\alpha(|x(t,\xi,u)|) \leqslant W(x(t,\xi,u)) \leqslant W(\xi) + t + \int_0^t \sigma(|u(s)|)\,\mathrm{d}s,$$

$$\alpha(|x(t,\xi,u)|) \leqslant W(x(t,\xi,u)) \leqslant W(\xi) + t + \int_0^t \sigma_1(|u(s)|)\,\mathrm{d}s + \int_0^t \sigma_2(|y(s)|)\,\mathrm{d}s.$$

Hence, recalling that the inverse of a $\mathscr{K}_\infty$ function is still of class $\mathscr{K}_\infty$ and exploiting continuity of W, we have proved the following result.

Corollary 2.13. *System* (1) *is forward complete if and only if there exist functions* $\chi_1, \chi_2, \chi_3, \sigma$ *of class* $\mathscr{K}_\infty$, *and a constant* $c \geqslant 0$, *such that*

$$|x(t,\xi,u)| \leqslant \chi_1(t) + \chi_2(|\xi|) + \chi_3\left(\int_0^t \sigma(|u(s)|)\,\mathrm{d}s\right) + c$$

holds for all $t > 0$, *all* $\xi \in \mathbb{R}^n$, *and all input signals u. Similarly, system* (1) *has the unboundedness observability property, if and only if there exist functions*

$\chi_1, \chi_2, \chi_3, \chi_4, \sigma_1, \sigma_2$ of class $\mathcal{K}_\infty$ and a positive constant $c \geqslant 0$ such that

$$|x(t,\xi,u)| \leqslant \chi_1(t) + \chi_2(|\xi|) + \chi_3\left(\int_0^t \sigma_1(|u(s)|)\,\mathrm{d}s\right) + \chi_4\left(\int_0^t \sigma_2(|y(s)|)\,\mathrm{d}s\right) + c$$

holds for all $t > 0$, all $\xi \in \mathbb{R}^n$, and all input signals u.

Remark 2.14. Notice that, while the two estimates in Corollary 2.13 imply respectively (10) and (9) (with possibly different comparison functions), it was not obvious that the converse implications should also be true.

Remark 2.15. The assumption that h is locally Lipschitz can be relaxed to simply continuity, while preserving all the results given. Indeed, suppose that h is continuous. Pick any locally Lipschitz function $h_0 : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that $|h(x)| < h_0(x) < |h(x)| + 1$ for all $x \in \mathbb{R}^n$ (such functions always exist; in fact, one could even pick h_0 smooth). Clearly, if the original system is UO, then the system $\dot{x} = f(x,u)$ with output $y = h_0(x)$ also is. Applying the various results to this new system then gives the desired results for the original one.

Remark 2.16. One may wonder if it is possible to always pick $\sigma = \mathrm{Id}$, thus reducing the energy σ to an L_1 norm of the input. This can not be achieved in general, as illustrated by the forward complete system $\dot{x} = u^3$. Choosing as an input sequence $u_n(t) = n$ when $t \in [0, 1/n]$ and equal to zero elsewhere, we have that $\int_0^1 |u_n(t)|\,\mathrm{d}t \leqslant 1$; on the other hand, there is no uniform bound for $|x(1,0,u_n)|$.

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