Changing Supply Functions in Input/State Stable Systems

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Abstract— **We consider the problem of characterizing possible supply functions for a given dissipative nonlinear system, and provide a result that allows some freedom in the modification of such functions.**

Keywords— **nonlinear stability, ISS, dissipation, input/output stability**

I. Introduction

The "input to state stability" (ISS) property has been recently introduced in nonlinear systems analysis ([4]), and, together with close variants, has already found some uses in feedback design ([2], [3]; see also [5] for an expository introduction). It provides one natural framework in which to formulate notions of stability with respect to input perturbations. In this note, we explore certain questions associated to the ISS property.

It was shown in [6] that this property can be equivalently characterized in terms of a dissipation inequality (in the style of the work in [7], [1]). More precisely, consider a general nonlinear system evolving in Euclidean space $\mathbb{R}^n$ and with inputs taking values in $\mathbb{R}^m$:

$$\dot{x} \;=\; f(x,u)\;. \tag{1}$$

(We assume that $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is locally Lipschitz, and $f(0,0)$=0.) Given the above-cited equivalences, for the purposes of this paper we simply define the system (1) to be ISS if there is some smooth (infinitely differentiable), positive definite ($V(x)$>0 for $x\neq 0$, $V(0)$=0) and proper (that is, radially unbounded) function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ (a "storage function for the system") and there are two class $\mathcal{K}_\infty$ functions α and γ, so that

$$\dot{V}(x,u) \;:=\; \nabla V(x)\cdot f(x,u) \;\leq\; \gamma(|u|)-\alpha(|x|) \tag{2}$$

for all $x\in\mathbb{R}^n$ and all $u\in\mathbb{R}^m$. (We are using $|\cdot|$ to indicate Euclidean norms in the respective space; recall that the class $\mathcal{K}_\infty$ consists of all functions $\gamma:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ which are continuous, strictly increasing, and satisfy $\gamma(0)=0$ and $\gamma(s)\to+\infty$ as $s\to+\infty$.) In other words, along each trajectory of (1) there holds the estimate $dV(x(t))/dt\leq\gamma(|u(t)|)-\alpha(|x(t)|)$.

The combination of the functions γ and α serves as one characterization of the "input to state gain" of the system. For instance, when $\gamma(r)=g^2r^2$ and $\alpha(r)=r^2$, existence of a storage function as in (2) implies that the zero-initial-state L^2 gain of the system is bounded by g. (Note that it is the combination of the two functions that matters; in this example, using a scalar multiple of V provides a new equation (2) with $\gamma(r)=cg^2r^2$ and $\alpha(r)=cr^2$, for any constant $c>0$; there is no intrinsic reason to prefer $c=1$ over other choices.)

Definition I.1: A pair of class-$\mathcal{K}_\infty$ functions (γ,α) is a *supply pair* for the system (1) if there is some storage function V so that (2) holds. □

The following problem seems natural:

Given a system, characterize its possible supply pairs.

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One may ask, in particular, for which functions γ there is some α so that (γ,α) is a supply pair for the given system, and viceversa, for which α there is a suitable γ. It is easy to see that some restrictions are necessary. To illustrate, assume the system is one-dimensional and has the form $\dot{x}=f(x)+u$. If $\dot{V}(x,u)\leq\gamma(|u|)-\alpha(|x|)$ then, in particular for $u=0$ one has $V'(x)f(x)\leq-\alpha(|x|)$, from which the bound $\alpha(|x|)/|f(x)|\leq|V'(x)|$ results for all $x\neq 0$. Since V must have a local minimum at zero, and so $V'(0)=0$, it must be the case that

$$\alpha(|x|)=o(f(x)) \;\text{ as } x\to 0$$

and hence α is severely restricted for small x. Our main result, in informal terms, will be that if (γ,α) is a supply pair, then one can arbitrarily modify α for *large* arguments, and a similar conclusion applies to γ and small arguments. We now state the results precisely. In the rest of this note, a system (1) is assumed to be fixed.

Theorem 1: Assume that (γ,α) is a supply pair. Suppose that $\tilde{\gamma}$ is a $\mathcal{K}_\infty$ function so that $\gamma(r)=O(\tilde{\gamma}(r))$ as $r\to\infty$. Then there exists a $\tilde{\alpha}\in\mathcal{K}_\infty$ so that $(\tilde{\gamma},\tilde{\alpha})$ is a supply pair.

Theorem 2: Assume that (γ,α) is a supply pair. Suppose that $\tilde{\alpha}$ is a $\mathcal{K}_\infty$ function so that $\tilde{\alpha}(r)=O(\alpha(r))$ as $r\to 0^+$. Then there exists a $\tilde{\gamma}\in\mathcal{K}_\infty$ so that $(\tilde{\gamma},\tilde{\alpha})$ is a supply pair.

These theorems will be proved in the next section. Properness and positive definiteness of a storage function V are equivalent to the existence of class-$\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ so that, for all x,

$$\underline{\alpha}(|x|)\leq V(x)\leq\overline{\alpha}(|x|)\;.$$

The constructions will show how to build the new supply pair using only $\underline{\alpha}$ and $\overline{\alpha}$ (as well as the original pair) but not V itself.

Before closing the introduction, we note the following interesting fact:

Corollary Assume that two ISS systems are given. Then there are $\mathcal{K}_\infty$ functions $\tilde{\gamma}_1$, $\tilde{\alpha}_2$, and $\tilde{\alpha}_1$, so that $((1/2)\tilde{\alpha}_2,\tilde{\alpha}_1)$ is a supply pair for the first system and $(\tilde{\gamma}_2,\tilde{\alpha}_2)$ is a supply pair for the second.

Proof: Start with (γ_1,α_1), (γ_2,α_2). Apply Theorem 2 to the second pair, with $\tilde{\alpha}_2=\alpha_2$ near 0 and $=\gamma_1$ for large s. This provides a $\tilde{\gamma}_2$. Now define $\tilde{\gamma}_1:=(1/2)\tilde{\alpha}_2$, and apply Theorem 1 to obtain $\tilde{\alpha}_1$. ■

This applies in particular to the following situation, illustrated in Figure 1.

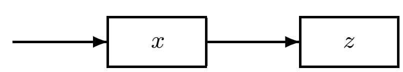

Fig. 1. Cascade

Consider the system in cascade form

$$\begin{aligned}\dot{z} &= f(z,x)\\ \dot{x} &= g(x,u)\end{aligned}$$

where $f(0,0)=g(0,0)=0$, the second equation is ISS, and the first equation is ISS when x is seen as an input. Then the composite system is ISS. This can be shown in many ways (cf. [4]), but a proof based on the above Corollary is particularly elegant. Indeed, assume one has found storage functions V_1 and V_2 so that V_1 satisfies a dissipation estimate

$$\nabla V_1(z)\cdot f(z,x) \;\leq\; (1/2)\tilde{\alpha}_2(|x|)-\tilde{\alpha}_1(|z|)$$

for the first subsystem, while V_2 is a storage function for the x-subsystem so that

$$\nabla V_2(x)\cdot g(x,u) \;\leq\; \tilde{\gamma}_2(|u|)-\tilde{\alpha}_2(|x|)\,.$$

Then $V := V_1(z) + V_2(x)$ is a storage function for the composite system, since

$$\dot{V}\Big((x,z),u\Big) \;\leq\; \tilde{\gamma}_2(|u|) - (1/2)\tilde{\alpha}_2(|x|) - (\tilde{\alpha}_1|z|)\;,$$

which, by means of elementary manipulations, can be transformed into a dissipation inequality of the form considered here.

II. Proofs.

Assume that (γ,α) is a supply pair for the given system, with corresponding storage function V. For both theorems, we will define a new storage function by means of the formula

$$W \;:=\; \rho \circ V \tag{3}$$

where ρ is a $\mathcal{K}_\infty$ function defined in turn by an integral of the form

$$\rho(s) \;:=\; \int_0^s q(t)\,dt$$

and where q is a suitably chosen function in $\mathcal{SN}$, the class of all smooth nondecreasing functions $[0,\infty) \to [0,\infty)$ which satisfy $q(t) > 0$ for $t > 0$. (In this manner, W is automatically smooth, proper, and positive definite.) From the definition (3) we will then have that

$$\dot{W}(x,u) = \rho'(V(x))\dot{V}(x,u) \leq q(V(x))\,[\gamma(|u|) - \alpha(|x|)]\;. \tag{4}$$

We claim that the right-hand side of (4) is bounded by

$$q(\theta(|u|))\gamma(|u|) \;-\; (1/2)q(V(x))\alpha(|x|)\;, \tag{5}$$

where $\theta \in \mathcal{K}_\infty$ is defined as

$$\theta \;:=\; \overline{\alpha} \circ \alpha^{-1} \circ (2\gamma)\,.$$

To show this, we consider separately two cases:

1. $\gamma(|u|) \leq (1/2)\alpha(|x|)$: In this case, the right-hand side of (4) is bounded already by the term $-(1/2)q(V(x))\alpha(|x|)$.
2. $(1/2)\alpha(|x|) \leq \gamma(|u|)$: Now $V(x) \leq \overline{\alpha}(|x|) \leq \theta(|u|)$, so the right-hand side of (4) is bounded by $q(\theta(|u|))\gamma(|u|) - q(V(x))\alpha(|x|)$.

Observe that one can in turn bound (5) by

$$q(\theta(|u|))\gamma(|u|) \;-\; (1/2)q(\underline{\alpha}(|x|))\alpha(|x|)\,. \tag{6}$$

Thus, the theorems will be proved if one shows that, under the assumptions of Theorem 1, there are a $q \in \mathcal{SN}$ and an $\tilde{\alpha} \in \mathcal{K}_\infty$ so that

$$q(\theta(r))\gamma(r) - (1/2)q(\underline{\alpha}(s))\alpha(s) \;\;\leq\;\; \tilde{\gamma}(r) - \tilde{\alpha}(s) \quad \forall\, r,s \geq 0\,, \tag{7}$$

and analogously for Theorem 2.

We first observe these two trivial facts:

Lemma 1. Assume that the functions $\beta, \tilde{\beta} \in \mathcal{K}_\infty$ are such that $\beta(r) = O(\tilde{\beta}(r))$ as $r \to +\infty$. Then there exists a function $q \in \mathcal{SN}$ so that

$$q(r)\beta(r) \;\leq\; \tilde{\beta}(r)$$

for all $r \in [0,\infty)$.

Lemma 2. Assume that the functions $\tilde{\beta}, \tilde{\beta} \in \mathcal{K}_\infty$ are such that $\tilde{\beta}(s) = O(\beta(s))$ as $s \to 0^+$. Then there exists a function $q \in \mathcal{SN}$ so that

$$\tilde{\beta}(s) \;\leq\; q(s)\beta(s)$$

for all $s \in [0,\infty)$.

To prove Lemma 1, it is sufficient to note that $\tilde{\beta}(r)/\beta(r)$ is well-defined and continuous for $r > 0$, and it is bounded below by a positive number on any interval of the form $[r_0, +\infty)$, $r_0 > 0$. Thus $\tilde{q}(r) := \inf_{r' \geq r} \tilde{\beta}(r')/\beta(r')$ for $r > 0$, is nondecreasing and positive. Now any $q \in \mathcal{SN}$ which satisfies $q(r) < \tilde{q}(r)$ for all $r > 0$ is as desired.

Similarly, Lemma 2 is established by noting that $\tilde{\beta}(s)/\beta(s)$ is well-defined and continuous for $s > 0$, and it is bounded above on any interval of the form $(0, s_0]$, $s_0 > 0$. Thus $\tilde{q}(s) := \sup_{0<s'\leq s} \tilde{\beta}(s')/\beta(s')$ for $s > 0$ is a nondecreasing well-defined function. Any $q \in \mathcal{SN}$ which satisfies $q(s) > \tilde{q}(s)$ for all $s > 0$ is as wanted for Lemma 2 (the inequality at $s = 0$ follows by continuity).

We now return to proving (7). Assume that $\gamma(r) = O(\tilde{\gamma}(r))$ as $r \to +\infty$. Define $\tilde{\alpha}(s) := (1/2)q(\underline{\alpha}(s))\alpha(s)$, and note that this is a $\mathcal{K}_\infty$ function because $\alpha \in \mathcal{K}_\infty$ and $q \in \mathcal{SN}$. Let $\beta := \gamma \circ \theta^{-1}$ and $\tilde{\beta} := \tilde{\gamma} \circ \theta^{-1}$; these satisfy the hypotheses of Lemma 1 because $\theta \in \mathcal{K}_\infty$. If q is as in the conclusion of the Lemma, then (7) holds.

If instead we know that $\tilde{\alpha}(r) = O(\alpha(r))$ as $r \to 0^+$, we similarly apply Lemma 2, with $\beta := (1/2)(\alpha \circ \underline{\alpha}^{-1})$ and $\tilde{\beta} := \tilde{\alpha} \circ \underline{\alpha}^{-1}$, and we let $\tilde{\gamma}(r) := q(\theta(r))\gamma(r)$.

References

[1] Hill, D.J., and P. Moylan, "Dissipative dynamical systems: Basic input-output and state properties," *J. Franklin Institute* **5**(1980): 327-357.

[2] Jiang, Z.-P., A. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math of Control, Signals, and Systems* **7**(1994): 104-130.

[3] Praly, L., and Z.-P. Jiang, "Stabilization by output feedback for systems with ISS inverse dynamics," *Systems & Control Letters* **21**(1993): 19-34.

[4] Sontag, E.D., "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control* **AC-34**(1989): 435-443.

[5] Sontag, E.D., "State-space and i/o stability for nonlinear systems," *Feedback Control, Nonlinear Systems, and Complexity* (B.A. Francis and A.R. Tannenbaum, eds.), Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1995, pp. 215-235.

[6] Sontag, E.D., and Y. Wang, Sontag, E.D., and Y. Wang, "On characterizations of the input-to-state stability property," *Systems and Control Letters* **24** (1995): 351-359. (Preliminary version in "Notions equivalent to input-to-state stability," in *Proc. IEEE Conf. Decision and Control, Orlando, Dec. 1994*, IEEE Publications, 1994, pp. 3438-3447.)

[7] Willems, J.C., "Mechanisms for the stability and instability in feedback systems," *Proc. IEEE* **64** (1976): 24-35.