

## LINEAR SYSTEMS OVER COMMUTATIVE RINGS: A SURVEY

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**Abstract.** An elementary presentation is given of some of the main motivations and known results on linear systems over rings, including questions of realization and control. The analogies and differences with the more standard case of systems over fields are emphasized throughout.

**Introduction.**

The theory of both discrete and continuous-time finite-dimensional constant linear systems with real or complex coefficients

$$\dot{x}(t) = Fx(t) + Gu(t), \quad y(t) = Hx(t) \quad (*)$$

and

$$x(t+1) = Fx(t) + Gu(t), \quad y(t) = Hx(t), \quad (**)$$

has been thoroughly explored in the last 15 years.

A conceptually important step was the realization that most of the « structural » properties of (\*) as well as of (\*\*) depend only on the triple of matrices  $(F, G, H)$ . Moreover, most algebraic results turned out to depend only on the fact that the entries of the matrices involved

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are « numbers » in a *field*. [The definitions of fields and other algebraic notions are reviewed in APPENDIX 2 at the end of this paper]. This set the stage for a more abstract study of linear systems, where all « numbers » appearing in the specification of  $(F, G, H)$  are taken from a fixed but arbitrary field (KALMAN, FALB and ARBIB [1968, Ch. 10]). Thus systems over the binary field, and in general over finite fields, became part of linear system theory. This provided a connection with the theory of linear sequential circuits and with coding theory; the BERLEKAMP-MASSEY decoding algorithm for *BCH* codes and FORNEY's study of convolution codes became a natural part of linear system theory.

The next step was to further generalize the concept of a linear system, to include the case in which coefficients belong to a *ring*. In a ring, addition and multiplication are defined, but (as opposed to a field) *division* is impossible in general. Examples of rings are the integers  $\mathbf{Z}$  or the polynomials in one or more variables, with their usual addition and multiplication. The work of ROUCHALEAU [1972] and ROUCHALEAU, WYMAN, and KALMAN [1972] constituted the first in-depth research into the properties of systems over rings. Many application areas (see Part 1 below) *require* working with systems over a ring.

Independently of the many examples discussed later, it should be clear that a strong motivation for the study of systems over rings lies in the hope of a better understanding of the meaning of « linearity » in system theory.

Our goal in writing this survey has been to stress the methods and to point out the difficulties. We leave many mathematical details aside; these can be found in the papers referenced at the appropriate points of the discussion. As much as possible, especially in Section 3, references are given to EILENBERG [1974, Chapter 16], where many properties of discrete-time systems are treated in detail. Open problems are mentioned as they appear, although undoubtedly many more will be easily raised by the reader.

We hope that the treatment is sufficiently intuitive, so that a superficial knowledge of the definitions (with a good knowledge of classical linear algebra) is enough to follow the main ideas. We shall always assume that all rings are *commutative*. (Systems over noncommutative rings have begun to be studied in general, but since the problems and methods are very different from those of the commutative case, they are not an appropriate topic for this paper).

This paper is divided into five parts. In the first part we give examples of the types of problems which have motivated much of the research in systems over rings. In the next part we abstract from the examples and define the systems to be studied. We then turn to the problem of realization of an input / output map; this is the problem which has attracted the largest research efforts until now. We give in part 4 a detailed illustration of the possible applications in control theory of some of the results and methods developed. We review there the topic of feedback control of systems over rings, and in particular, delay-differential systems. In the final section we briefly mention other areas which have been studied.

We would like to close this introduction with a few general comments. While systems over rings appear naturally in studying, say, systems over the integers, there are application areas, like delay-differential systems, where the theory discussed here is just one of the possible tools to be applied. In these latter areas, other methods have in fact been traditionally involved. Systems over rings, or any other algebraic approaches, have the advantage of leading in most cases to practical procedures for solving the given problems, instead of giving abstract existence proofs. Such abstract results are of course an indispensable part of any theory, but they will probably have limited practical impact until integrated with more algebraic methods.

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## 1. Some examples.

### A. DELAY - DIFFERENTIAL SYSTEMS.

A (retarded, linear, constant) *delay-differential system*, or *hereditary system*, is a system described by equations of the type

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{\alpha} F_i x(t-a_i) + \sum_{j=1}^{\beta} G_j u(t-b_j), \\ y(t) &= \sum_{k=1}^{\gamma} H_k x(t-c_k), \end{aligned} \tag{*}$$

where  $a_i, b_j, c_k$  are nonnegative real numbers, and where  $x(t)$  as well as the inputs  $u(t)$  and outputs  $y(t)$  are real vectors. The matrices  $F_i, G_j, H_k$  have real entries. Such systems appear naturally in many engineering applications and in fact in any situation in which transmission delays cannot be ignored.

A great deal of research is being carried on today on delay-differential systems. We shall now discuss a procedure which allows the study of certain kinds of problems (e. g., structural questions, like those dealing with the existence of representations as in (\*)) in the context of systems over appropriate rings.

Consider for example a system defined by

$$\begin{cases} \dot{x}_1(t) = 2x_1(t-1) + x_1(t) + x_2(t) + u(t), \\ \dot{x}_2(t) = x_1(t-1) - 3x_2(t-5) + u(t-1), \\ y(t) = x_1(t) - x_2(t-1). \end{cases} \quad (\text{a})$$

If we introduce the delay operator  $\sigma$  defined by

$$\sigma(x)(t) := x(t-1),$$

we can rewrite (a) in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2\sigma + 1 & 1 \\ \sigma & -3\sigma^5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \sigma \end{bmatrix} u$$

$$y = [1 - \sigma] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We see then that (a) can be expressed in a form very similar to the ordinary finite-dimensional constant linear systems of control theory, the only difference being that the matrices ( $F, G, H$ ) now have polynomial instead of real valued entries. When all the delays  $a_i, b_j, c_k$  in (\*) are integral multiples of a fixed delay  $\lambda$ , we can apply the same procedure as above, taking now for  $\sigma$  a shift of  $\lambda$  seconds. If, instead, the delays in (\*) are not commensurable, we need to define a finite set of delay operators  $\sigma_1, \dots, \sigma_r$  and then consider systems whose matrices have entries in the set of polynomials in  $\sigma_1, \dots, \sigma_r$ . This set is denoted by  $\mathbf{R}[\sigma_1, \dots, \sigma_r]$ .

As in the case of the integers, the set of polynomials  $\mathbf{R}[\sigma_1, \dots, \sigma_r]$  is not a field, although it is a ring if we define the addition and multiplication of polynomials in the usual way. Some authors (e. g. ANSELL [1964], NEWCOMB [1966], YOULA [1968]) introduce then

the field of rational functions in  $\sigma_1, \dots, \sigma_r$  in order to study networks with transmission lines, i. e. delay-differential systems. These methods have the disadvantage of introducing « ideal predictors », i. e. operators of the type

$$\sigma^{-1}(x)(t) = x(t+1),$$

which are not physically realizable. A more natural approach to the problem, that of applying the theory of systems over rings of polynomials, was first introduced by KAMEN [1975].

There is an important conceptual difference between the example of delay-differential systems and systems over an abstract ring, say, the integers. In the case of delay-differential systems only the entries of the matrices defining the systems belong to the ring in question; in the case of systems over the integers, the states, inputs and outputs are also in the corresponding ring. This difference will turn out to be irrelevant in the study of those problems whose solution depends only on the form of the defining matrices.

## B. OTHER EXAMPLES AND MOTIVATIONS.

When the theory of linear systems is applied in coding theory, the relevant systems have their coefficients in a finite field. It was shown by JOHNSTON [1973] that a larger class of codes, generalizing *BCH* codes, can be generated by considering instead *abelian group systems*, i. e. systems over the integers  $\mathbf{Z}$  (or over residue rings  $\mathbf{Z}_n$ ). Systems over  $\mathbf{Z}_n$  were also studied by MATLUK and GILL [1971], in the context of automata theory. Abelian group systems are also an important special case of the systems studied by BROCKETT and WILLSKY [1972].

The study of systems obtained from lumped approximations to linear partial differential equations can also be approached via systems over rings. The idea is in a sense to introduce an operator  $\sigma$  for each  $\frac{\partial}{\partial \tau_i}$  where  $\tau_i$  is a space variable. In fact, one studies in practice spatial discretizations of the above kind of equation, and  $\sigma$  becomes a shift operator. For instance, consider a heat equation

$$\frac{\partial x}{\partial t}(t, \tau) = \frac{\partial^2 x}{\partial \tau^2}(t, \tau) + u(t, \tau), \tau \in \mathbf{R}.$$

If we now discretize this equation by letting  $x_i, u_i$  denote temperatures at the integer point  $i$ , an approximation is given for example by the system

$$\dot{x}(t) = (\sigma - 2 + \sigma^{-1})x(t) + u(t), \quad (*)$$

where  $x, u$  are infinite column vectors with entries  $x_i, u_i$ , and where  $(\sigma x)_i = x_{i+1}$ . Therefore a system over the ring  $\mathbf{R}[\sigma, \sigma^{-1}]$  is obtained. For a wave equation, a second derivative of  $x$  would appear, and the system would be higher dimensional. If (\*) is written in matrix form  $\dot{x} = Fx + u$ , the infinite matrix  $F$  is of the Toeplitz type. Similarly, an equation evolving on the circle will induce a system whose matrices are circulant, or equivalently, a system over a group algebra  $\mathbf{R}[\mathbf{Z}_k]$ . In general, the rings appearing in this context will be *group algebras* (or suitable completions). [A group algebra is the set of all finite linear combinations  $\sum a_i g_i$  with all  $a_i$  in a field, say the reals or the complexes, and with the  $g_i$  taken out of a given group  $G$  (here  $G$  corresponds to the possible shift operators). The ring addition is coordinatewise (for each  $g_i$  add the  $a_i$ ) and the multiplication is the linear extension of the group multiplication among the  $g_i$ ]. The same framework applies to the study of (linear) cellular automata, i. e. an interconnection of equal subsystems operating in parallel. Different aspects of the above systems were studied in BROCKETT and WILLEMS [1971, 1974], JOHNSTON [1973] and SONTAG [1976]. An application to transport systems was given by WILLEMS [1971].

A general type of distributed (i. e., infinite-dimensional) systems which are « finitary » in the sense of being specifiable by finitely many parameters, like delay-differential or partial difference systems, can be fruitfully studied in many aspects via systems over appropriate rings of operators. The particular case of systems defined via rings of distributions is analyzed in KAMEN [1975], while a general approach can be found in SONTAG [1976, Part D].

There are problems which share the mathematical structure of systems over rings, whose interpretation is completely different from the above. The outstanding example of this situation is the study of multiplicities or « ambiguities » in the generation of languages by grammars or, equivalently, in the recognition of languages by automata. Such a study was begun by SCHÜTZENBERGER [1961], part of whose work is related to realization for systems over the integers  $\mathbf{Z}$ . An extended literature exists on these problems, and an excellent exposition can be found in the textbook by EILENBERG [1974]. Since in

language theory it is more natural to work with *semirings* rather than rings (subtraction is not allowed), and since other generalizations are made (alphabets of more than one letter), the questions posed are usually different from those asked about systems over rings. Exceptions are the FATOU-type problems (see Section 3 below) which appear also in this context; see for instance FLIESS [1972, 1974], SONTAG [1975] and SONTAG and ROUCHALEAU [1976]. These results can be interpreted through *bilinear* systems over rings, via the method of FLIESS [1973].

The study of time-varying control systems in the style of KAMEN [1976] is in many ways analogous and uses concepts similar to the study of constant systems over certain (noncommutative) rings. Realizations of multidimensional filters can also be obtained using tools of systems over rings, as in SONTAG [1976]. Even some areas of pure mathematics (the study of rational power series) are intimately related to our topic, beginning with the work of FATOU [1906]; we shall return to this later.

## 2. Systems and input/output maps.

We shall give a preliminary definition of the systems to be studied. The main illustrations will be: in the discrete-time case, systems over  $\mathbf{Z}$ ; in the continuous-time case, delay-differential systems. We shall see later (Section 3. E) that a larger class of systems must be considered in order to obtain a satisfactory theory.

Throughout this chapter,  $R$  denotes an arbitrary but fixed commutative ring. Notations are as explained in Appendix 2.

### A. A PRELIMINARY DEFINITION.

(2.1) DEFINITION. A (free) system  $\Sigma$  is given by a triple of matrices  $(F, G, H)$ , where  $F \in R^{n \times n}$ ,  $G \in R^{n \times m}$  and  $H \in R^{p \times n}$  for some integers  $n, m, p$ .  $n$  is called the rank of  $\Sigma$ . A scalar system  $\Sigma$  is a system for which  $m = p = 1$ .

The above definition of systems may seem in principle too abstract, since no explicit mention is made of dynamics, states, input and output functions. However, since we are interested in the simultaneous study of both discrete and continuous-time systems at an elementary level, we shall be content with informal interpretations. The

interested reader can find more complete definitions in SONTAG [1976, Part D], but this is not at all necessary in order to follow the reasonings in the present paper.

The main tool in the study of systems will be the *discrete-time* interpretation. Under this interpretation, (2.1) defines a system evolving according to:

$$\begin{aligned}x(t+1) &= Fx(t) + Gu(t), \\y(t) &= Hx(t),\end{aligned}\tag{2.2}$$

where  $t=0, 1, 2, \dots$ , and where the « states »  $x(t) \in R^n$ , the « inputs »  $u(t) \in R^m$  and the « outputs »  $y(t) \in R^p$  for all  $t$ . We also assume an initial state  $x(0)=0$ .

Given an input sequence  $u(0), u(1), u(2), \dots$ , we can solve (2.2) by recursion beginning with  $x(0) = 0$ . We then obtain sequences  $x(0), x(1), \dots$  and  $y(0), y(1), \dots$ . It is easy to show that  $y(0)=0$  and for  $t \geq 1$

$$y(t) = \sum_{i=0}^{t-1} HF^{t-i-1} Gu(i).$$

We see therefore that the *input/output map*  $f_{\Sigma} : (u(0), u(1), \dots) \rightarrow (y(0), y(1), \dots)$  is *completely determined* by the sequence of matrices  $(A_1, A_2, \dots)$ , where  $A_i := HF^{i-1}G \in R^p \times R^m$ . (In fact, it is not difficult to verify that *any* map which sends sequences  $u(0), u(1), \dots$  to sequences  $y(0), y(1), \dots$  and which satisfies natural « linearity », « causality » and « shift-invariance » conditions is determined by a suitable abstract sequence  $\{A_i\}$  so that the « convolution » formula  $y(t) = \sum A_{t-i} u(i)$  holds).

Another interpretation of (2.1), valid when  $R$  is a ring of real polynomials  $\mathbf{R}[\sigma_1, \dots, \sigma_r]$ , is in terms of *hereditary* systems. For simplicity, take  $r=1$ ; the general case is analogous. We shall interpret  $\sigma = \sigma_1$  as a delay of one second (the actual length of the delay is immaterial in this context; the important matter is to fix it throughout the reasoning). Then (2.1) corresponds to the system described by:

$$\begin{aligned}\dot{x}(t) &= F(\sigma)x(t) + G(\sigma)u(t), \\y(t) &= H(\sigma)x(t),\end{aligned}\tag{2.3}$$

where the functions  $x(\cdot), u(\cdot), y(\cdot)$  are defined for  $t \in \mathbf{R}$  with values in vector spaces  $\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^p$ . The matrices  $F = F(\sigma), G = G(\sigma),$

$H=H(\sigma)$  are matrices over  $R = \mathbf{R}[\sigma]$ , i. e. matrices of polynomials, and the interpretation of  $F(\sigma)x$ ,  $G(\sigma)u$ ,  $H(\sigma)x$  is that one given in Section 1. B. We shall not give here explicit spaces of time-functions for  $u(\cdot)$ ,  $x(\cdot)$ ,  $y(\cdot)$ , although we shall make the restriction that all admissible functions are zero in the past, i. e. for each such function  $\alpha$  there exists a  $t_0$  (which depends on  $\alpha$ ) such that  $\alpha(t)=0$  for all  $t < t_0$ . With such a restriction, given  $u(\cdot)$ , (2.3) has a unique solution  $x(\cdot)$ ,  $y(\cdot)$  (assuming the appropriate smoothness conditions on  $u(\cdot)$  are satisfied). There is no need to specify an initial state, since our restriction on admissible functions amounts to implicitly taking a zero initial state.

Let  $\widehat{v}(s)$  denote the Laplace transform of a time function  $v(t)$ . Since  $\widehat{\sigma v} = e^{-s}v$  and  $\widehat{v} = s\widehat{v}$ , (2.3) becomes, in terms of Laplace transforms,

$$\begin{aligned} s\widehat{x}(s) &= F(e^{-s})\widehat{x}(s) + G(e^{-s})\widehat{u}(s), \\ \widehat{y}(s) &= H(e^{-s})\widehat{x}(s), \end{aligned}$$

for a suitable class of transformable input functions. It follows that

$$\begin{aligned} \widehat{y}(s) &= H(e^{-s})(s - F(e^{-s}))^{-1}G(e^{-s})\widehat{u}(s), \\ &= W(s, e^{-s})\widehat{u}(s). \end{aligned}$$

Therefore, the input/output map  $f_{\Sigma}: u(\cdot) \mapsto y(\cdot)$  of the system  $\Sigma$  given by (2.3) is determined by the *transfer matrix* of  $\Sigma$ , a matrix  $W(s, e^{-s})$  rational in  $s$ ,  $e^{-s}$ . It is easy to see that  $W$  admits a (unique) expansion

$$W(s, e^{-s}) = \sum_{i=1}^{\infty} A_i(e^{-s})s^{-i}.$$

The sequence  $\{A_i(e^{-s})\}$  is obtained via the substitution  $\sigma \mapsto e^{-s}$  from a corresponding sequence of polynomial matrices  $\{A_i\}$ , where  $A_i = A_i(\sigma) \in \mathbf{R}[\sigma]^{p \times m}$ . In fact,  $A_i = HF^{i-1}G$  for all  $i$ . (It is possible to give a more direct characterization of those input/output maps  $u(\cdot) \mapsto y(\cdot)$  corresponding to delay-differential systems, via sets of operator equations. The characterization follows either from general facts about «finitary» systems or via systems defined by convolution equations as in KAMEN [1975]).

## B. STATEMENT OF THE REALIZATION PROBLEM.

The above considerations suggest the following abstract definition, for a fixed ring  $R$ :

(2.4) DEFINITION. An input/output map  $f$  is given by a sequence  $(A_1, A_2, \dots)$  where  $A_i \in R^{p \times m}$  for all  $i$ . A system  $\Sigma$  realizes  $f$  iff  $A_i = HF^{i-1}G$  for  $i=1, 2, \dots$ .

The justification for (2.4) lies in the fact that in both of the above interpretations (and in fact, in general) the sequence  $\{HF^iG\}$  completely characterizes the input/output behavior of  $\Sigma$ . In particular interpretations, the realization problem might be initially given in terms of different but equivalent data (kernels, operator equations, etc.); we shall assume here that a reduction to the present form has already been made.

The study of existence, and the construction, of realizations are mathematically nontrivial. This difficulty, added to the basic system-theoretic significance of these problems, accounts for the fact that realization theory is the most developed area of system theory over rings at present.

The problem of realization is a purely algebraic one of finding factorizations  $A_i = HF^{i-1}G$  of a given sequence of matrices. We are free, however, to use our intuition via an interpretation of our choice. In particular, we shall use discrete-time systems (2.2) over the ring  $R$ .

### 3. Realization of an input/output map.

As in Section 2,  $R$  is again an arbitrary but fixed commutative ring.

#### A. AN ABSTRACT GENERAL CRITERION.

Let  $f=(A_1, A_2, \dots)$  be an input/output map. The first question which arises is: *What finiteness conditions must be imposed upon  $f$  so that  $f$  is realizable (i. e., so that there exists a system  $\Sigma$  which realizes  $f$ )?*

We can easily see that *some* condition is needed. Indeed, assume that  $f$  is realizable,  $A_i = HF^{i-1}G$ , for all  $i$ . By the Cayley-Hamilton Theorem (valid over all commutative rings), there exist  $\alpha_0, \dots, \alpha_{n-1}$  in  $R$  so that

$$F^n = \sum_{i=0}^{n-1} \alpha_i F^i.$$

Therefore,  $F^{n+k} = \sum \alpha_i F^{i+k}$  for all  $k \geq 0$ . It follows that  $f$  is *recurrent*, i. e. there exists a recursion

$$A_{n+k} = \sum_{i=0}^{n-1} \alpha_i A_{i+k}, \quad \text{for all } k \geq 0. \tag{3.1}$$

Somewhat less trivial is the fact that recurrency is *sufficient* for realizability:

(3.2) THEOREM. (KALMAN, FALB and ARBIB [1969, Chapter 10, Lemma 11.7]) *An input /output map  $f$  is realizable if and only if it is recurrent.*

Indeed, if  $f$  satisfies (3.1) for some  $\alpha_0, \dots, \alpha_{n-1}$  then the system  $\Sigma=(F, G, H)$  realizes  $f$ , where

$$F = \begin{bmatrix} \overline{0} & 0 & 0 & \cdots & 0 & \alpha_0 I \\ I & 0 & 0 & \cdots & 0 & \alpha_1 I \\ 0 & I & 0 & \cdots & 0 & \alpha_2 I \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{n-2} I \\ 0 & 0 & 0 & \cdots & I & \alpha_{n-1} I \end{bmatrix}, \quad G = \begin{bmatrix} I \\ \vdots \\ 0 \end{bmatrix}, \quad H = [A_1, A_2, \dots, A_n].$$

(All block matrices above are  $rn \times m$ ).

An alternative way of viewing recurrence is via the use of *formal power series*. To each input/output map  $f$  we can naturally associate a formal matrix power series  $W(z^{-1}) = A_1 z^{-1} + A_2 z^{-2} + \dots$  in the symbol  $z^{-1}$ . Since each  $A_i$  is a  $p \times m$  matrix, the above series can be also represented as a matrix  $(w_{ij}(z^{-1}))$ ,  $i=1, \dots, p$ ,  $j=1, \dots, m$ , where each  $w_{ij}(z^{-1})$  is a power series in  $z^{-1}$  with scalar coefficients. A series with scalar coefficients is called *rational* whenever it is the expansion in powers of  $z^{-1}$  of a quotient  $p(z)/q(z)$ , where  $p(z), q(z)$  are polynomials on  $z$  with coefficients in  $R$  and where  $q(z)$  has leading coefficient equal to one and degree greater than that of  $p(z)$ . An arbitrary series  $W(z^{-1})$  is *rational* if all  $w_{ij}(z^{-1})$  are rational (in other words, when  $W(z^{-1})$  is the expansion of a suitable rational matrix). A proof of the following result can be found in EILENBERG [1974, Ch. 16, Prop. 3.2 and Prop. 9.1].

(3.3) PROPOSITION.  *$f$  is recurrent if and only if its associated power series  $W$  is rational.*

### B. A MORE CONCRETE CRITERION.

We have seen in (3.2) that realizability is equivalent to recurrency. An important problem, however, is to give « computable » criteria allowing to decide whether a given  $f$  is recurrent.

Given a (possibly infinite) matrix  $M$ , define the *rank* of  $M$  as the smallest  $l$  such that all minors of  $M$  of order greater than  $l$  are zero. In particular,  $M$  has infinite rank when there exist nonzero minors of arbitrary order. The *Hankel* or *behavior matrix* of an input/output map  $f$  is the doubly infinite matrix given by, in block form,

$$\mathbf{B}(f) := \begin{bmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}.$$

In the case of systems over *fields*, a well-known criterion (see KALMAN [1968, Chapter 8]) is:

(3.4)  $f$  is realizable if and only if  $\mathbf{B}(f)$  has finite rank.

Thus (3.4) is a natural candidate for generalization. We first make a technical restriction on  $R$ . For the rest of this section and until Section E,  $R$  will be an integral domain (see APPENDIX 2). This does rule out some rings of interest, but still allows the consideration of systems over integers or delay-differential systems. Moreover, many of the results easily generalize to a larger class of rings (see Section F below). The advantage in restricting  $R$  to be an integral domain lies in the existence of the *quotient field*  $Q = Q(R)$ .

Let  $f = (A_1, A_2, \dots)$  be an input/output map. Since  $R$  is a subset of  $Q$ , the sequence  $(A_1, A_2, \dots)$  can also be viewed as a sequence of matrices over  $Q$ , i. e. as an input/output map over the field  $Q$ . As (3.4) is true for fields, the statement «  $\mathbf{B}(f)$  regarded as a matrix over  $Q$  has finite rank » is equivalent to «  $f$  is realizable over  $Q$  ». Thus, for a domain  $R$ , (3.4) can be expressed equivalently by the conclusion of the following (conjectured)

(3.4. a) THEOREM.  $f$  is realizable over  $R$  if and only if  $f$  is realizable over  $Q$ .

A realization over  $R$  is also a realization over  $Q$ , so (3.4. a) reduces to: « If the sequence of  $R$ -matrices  $\{A_i\}$  factors as  $A_i = HF^{i-1}G$ ,

where  $F, G, H$  are matrices with entries in  $Q$ , then there exist  $R$ -matrices  $\widehat{F}, \widehat{G}, \widehat{H}$  (possibly of higher dimensions) such that  $A_i = \widehat{H}\widehat{F}^{-1}\widehat{G}$ . Recalling the discussion at the end of Section A, (3.4) can be expressed in still another way, in the scalar case:

(3.4.b) If  $w(z^{-1}) = \sum a_i z^{-i}$ ,  $a_i \in R$ , is the expansion of  $p(z)/q(z)$  with  $p(z), q(z)$  polynomials with coefficients in  $Q$ , there exist  $\widehat{p}(z), \widehat{q}(z)$  polynomials over  $R$  such that  $\widehat{q}(z)$  has leading coefficient equal to 1 and  $w(z^{-1}) = \widehat{p}(z)/\widehat{q}(z)$ .

In the general case, (3.4. b) is still the relevant criterion since a matrix is rational iff all its entries are rational.

The problem of finding domain  $R$  over which (3.4. b) holds true is associated with the name of FATOU, who in 1906 proved that the ring of integers  $\mathbf{Z}$  is such a domain. In a system-theoretic context, ROUCHALEAU, WYMAN and KALMAN [1972] were the first to prove that every Noetherian integral domain (see APPENDIX 2) satisfies (3.4. a). (See also EILENBERG [1974], Chapter 16, Theorem 12.1] and JOHNSTON [1973], Section 6.2]). We give in Appendix 1 of this paper a new proof of this result. A complete characterization of domains satisfying the equivalent statements (3.4. a) - (3.4. b), implying in particular that not every  $R$  satisfies them, was found by CAHEN and CHABERT [1972].

An interesting interpretation of (3.4. a) in terms of a generalized notion of stability of systems was given by ROUCHALEAU and WYMAN [1974].

Realization questions aside, results like (3.4. a) are clearly highly significant from a system-theoretic viewpoint, in that they allow the reduction of many problems on systems over rings to corresponding problems in the better-known area of systems over fields.

### C. CONSTRUCTING REALIZATIONS.

The criterion stated in (3.4) is essentially an existential argument and does not provide an efficient method of finding an explicit realization, or, equivalently, finding a recurrence. For a more restricted class of integral domains, the problem can be again reduced to the study of systems over fields. This reduction will obviously be achieved if the following statement holds true:

(3.5) Let  $f$  be an input/output map over  $R$ . Assume that  $f = (A_1, A_2, \dots)$  is realizable over  $Q$  and that  $\alpha_0, \dots, \alpha_{n-1}$  is a recursion over  $Q$  with  $n$  minimal. Then  $\alpha_i \in R$  for all  $i$ .

Observe the strength of (3.5): it claims that, no matter what calculations are carried out over the field  $Q$  in order to obtain a minimal recursion, the coefficients will belong to  $R$ . It is then surprising that (3.5) turns out to be true for a large class of integral domains, including in particular  $\mathbf{Z}$  and  $\mathbf{R}[\sigma_1, \dots, \sigma_r]$ . It is not difficult to prove that it is enough to study (3.5) for scalar input/output maps, so the problem can be stated in terms of rational power series. In the latter form, the ring of integers was shown to satisfy (3.5) by FATOU [1906]. A direct proof of (3.5) was given by ROUCHALEAU, WYMAN, and KALMAN [1972] for Noetherian domains which are *integrally closed* (see Appendix 2). In Appendix 1 we indicate how (3.4. a) plus the assumption  $R = \text{integrally closed}$  imply (3.5). An abstract characterization of rings which satisfy (3.5) was given by CHABERT [1972]; see EILENBERG [1974, Chapter 16, Theorem 12.2]. The class of rings which satisfy (3.5) is a proper subclass of those rings for which (3.4) is true. For many system-theoretic purposes it is enough to know that  $R$  is *integrally closed whenever  $R$  is a unique factorization domain* (rings for which every element can be factored in an essentially unique way as a product of irreducibles).

#### D. MINIMALITY.

Knowledge of a recurrence, even a minimal one, is not in general enough to construct a realization  $\Sigma = (F, G, H)$  of minimal rank, since the construction in (3.2) has a high degree of redundancy. An important special case, however, is that in which  $m=1$  [or  $p=1$ ]. In that situation, the realization in (3.2) [or a suitable dual] has rank  $n$ , where  $n$  is the rank of  $\mathbf{B}(f)$ , and is therefore minimal (because it is minimal as a realization over  $Q$ , as follows from well-known results over fields).

If the restrictions on  $m$  and  $p$  are removed, the statement

(3.6) Assume that  $\text{rank } \mathbf{B}(f) = n$ . Then  $f$  has a realization of rank  $n$ .

is true for principal-ideal domains, hence for  $\mathbf{Z}$ ,  $\mathbf{R}[\sigma]$  (see Appendix 1).

Noetherian integral domains which satisfy (3.6) for all input/output maps  $f$  are completely characterized in ROUCHALEAU and SONTAG [1976]. In this paper, the problem of deciding for which (Noetherian)

domains (3.6) holds is shown equivalent to one of the most important questions in commutative algebra and algebraic geometry. In particular, a result of SHESHADRI translates into « (3.6) holds for all input/output maps over polynomial rings  $k[\sigma_1, \sigma_2]$ ,  $k$  any field ». On the other hand, the input/output map  $f$  over  $\mathbf{R}[\sigma_1, \sigma_2, \sigma_3]$  given by (with  $m=3$ ,  $p=2$ )

$$A_1: = \begin{bmatrix} \sigma_1 & \sigma_3 & 0 \\ \sigma_2 & 0 & -\sigma_3 \\ 0 & -\sigma_2 & -\sigma_1 \end{bmatrix}, \quad A_2: = A_3: = \dots = 0,$$

is shown to have no realizations of rank  $2 = \dim \mathbf{B}(f)$ . These results and example serve to finally clarify the questions with respect to synthesis of networks posed in NEWCOMB [1966].

The question of classifying realizations of minimal rank is very difficult. Under the natural notion of isomorphism (natural at least under the discrete-time interpretation), two systems  $\Sigma_1, \Sigma_2$  of rank  $n_1, n_2$  are *isomorphic* if and only if  $n_1 = n_2 = n$  and there exists  $T \in R^{n \times n}$  such that  $TG_1 = G_2$ ,  $F_2 T = TF_1$ ,  $H_2 T = H_1$ , and  $T^{-1} \in R^{n \times n}$ . If  $R$  is not a field, it is false that all minimal realizations of the same input/output map are necessarily isomorphic. Indeed, if  $a \in R$ ,  $a \neq 0$ , and  $a^{-1} \notin R$ , then  $\Sigma_1: = (1, a, 1)$  and  $\Sigma_2: = (1, 1, a)$  are nonisomorphic minimal realizations of  $f = (a, a, a, \dots)$ . In fact, there may exist infinitely many mutually nonisomorphic minimal realizations: if  $R = \mathbf{R}[\sigma]$ , then for each real number  $\lambda$  the system

$$\Sigma_\lambda: = \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sigma & 0 \\ -\lambda & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \lambda & \sigma \end{bmatrix} \right),$$

is a minimal realization of  $f = (\sigma I, \sigma I, \dots)$ , and  $\Sigma_\lambda$  is not isomorphic to  $\Sigma_\mu$  if  $\lambda \neq \mu$ .

However, over  $R =$  principal-ideal domain, a characterization can be given of the set of isomorphism classes of minimal realizations of a given input/output map in terms of a lattice of submodules of a finitely generated torsion  $R$ -module (SONTAG [1977 a]). So, in particular, for systems over the integers  $\mathbf{Z}$  there can exist only *finitely many* nonisomorphic minimal realizations of a fixed map.

E. REACHABLE, OBSERVABLE AND CANONICAL SYSTEMS.

When discussing realizations of minimal rank we mentioned the problem of redundancy. The question arises then as to whether there exist realizations that contain no information which is not implied

from the input/output behavior of the systems under consideration. From a general system-theoretic viewpoint, such realizations are characterized by properties of « reachability » and « observability ». These properties also appear naturally in relation to problems of controllability, filtering, etc.

The present setup, because of its generality, is not the appropriate one in which to define such concepts with such precision and generality that they will specialize to meaningful notions of reachability and observability under all possible interpretations. For instance, function-space reachability will be different from ring reachability for delay-differential systems. We shall therefore make all definitions using as a model the discrete-time case. The reader interested primarily in other types of systems (e. g., delay-differential) can view these notions simply as (very useful) *tools* in the study of systems over rings. We again let  $R$  denote an arbitrary commutative ring.

A discrete-time  $R$ -system  $\Sigma$ , described as in (2.2), is *reachable*, or *accessible* (from the zero state), when for each  $x \in R^n$  there exist  $t \geq 0$  and  $u(0), \dots, u(t-1)$  such that, solving (2.2),  $x(t) = x$ . The system  $\Sigma$  is *observable*, or *reduced*, if no two different states produce the same input/output map when considered as initial states for  $\Sigma$ . It can be proved that, as in the case of systems over fields,

(3.7)  $\Sigma$  is reachable if and only if every  $x \in R^n$  is an  $R$ -linear combination of the columns of  $G, FG, \dots, F^{n-1}G$ ;

(3.8)  $\Sigma$  is observable if and only if there exists no  $x \in R^n$  such that  $Hx = HFx = \dots = HF^{n-1}x = 0$  and  $x \neq 0$ .

Based on the above discrete-time intuition, we shall call an (abstract) system  $\Sigma = (F, G, H)$  *reachable* if (3.7) is valid, and *observable* if (3.8) holds.  $\Sigma$  is *canonical* when it is both reachable and observable.

Assume that the discrete-time system  $\Sigma$  is observable but not reachable. The natural choice for a canonical realization is then the « subsystem »  $\Sigma_r$  of  $\Sigma$ , obtained by restricting the state space to the set of reachable states  $X_r$ . In order to define  $\Sigma_r$  as a system, i. e. as a triple of matrices, we need to find matrix representations for the restrictions of the linear maps  $F, G, H$  to  $X_r$ . This involves the previous step of naturally representing elements of  $X_r$  as vectors in  $R^s$ , for some  $s$ . This representation should be uniquely defined; otherwise two vectors in  $R^s$  which « code » the same state would introduce unobservability in  $\Sigma_r$ .

For example, if  $R = \mathbf{R}[\sigma_1, \sigma_2]$ ,  $\Sigma = (1, (\sigma_1, \sigma_2), 1)$ , then the reachable states are given by all combinations  $\sigma_1 p_1(\sigma_1, \sigma_2) + \sigma_2 p_2(\sigma_1, \sigma_2)$ , where  $p_1, p_2$  are arbitrary polynomials in  $\sigma_1, \sigma_2$ . The obvious parametrization by pairs  $(p_1, p_2)$  is redundant since, for instance,  $(\sigma_2, \sigma_1)$  and  $(2\sigma_2, 0)$  both stand for  $2\sigma_1\sigma_2$ . It can be proved that in fact no « linear » bijective coding exists.

The example is typical of the fundamental difficulty involved in working with rings instead of fields: the nonexistence, in general, of *bases*. Therefore the above procedure to find a canonical realization via  $\Sigma$ , may fail to work. Actually, the problem is deeper, since *in general no canonical realizations exist* (in the sense of Definition (2.1)). The above reasoning, however, points to the possible solution: relax the definition of system so that more general « linear » (but still in some sense « finite ») objects than those of type  $R^n$  are allowed as state-spaces.

Algebra provides such general objects:  $R$ -modules, or, more precisely, finitely generated  $R$ -modules. We then generalize:

(3.9) DEFINITION. A system  $\Sigma$  is given by  $(X, F, G, H)$ , where  $X$  is a finitely generated  $R$ -module and  $G: R^m \rightarrow X, F: X \rightarrow X, H: X \rightarrow R^p$  are  $R$ -linear maps.  $\Sigma$  realizes  $f = (A_1, A_2, \dots)$  if  $A_i = H \circ F^{i-1} \circ G, i = 1, 2, \dots$ .

By « system » we shall refer henceforth to (3.9), and call the special case of those in (2.1), for which  $X = R^n$ , « free » systems. If the input/output map  $f$  of a system is defined via the sequence of linear transformations (or matrices)  $\{H \circ F^i \circ G\}$ , it is easy to prove that  $f$  can also be realized by a free system. This means that the phrases « realizable » and « realizable by a free system » are equivalent. Non-free systems have an obvious discrete-time interpretation, but we shall not attempt to give here a delay-differential interpretation.

Reachability and observability can be defined for systems again via (3.7) and (3.8) (replace « columns of  $G$  » by  $G(e_1), \dots, G(e_m)$ , etc.). In the larger class of systems (3.9) one can immediately generalize KALMAN's module approach to linear systems, and prove that (see EILENBERG [1974, Ch. 16, Section 5], where « minimal » is used for our « canonical »):

(3.10) THEOREM. Let  $f$  be a realizable input/output map. Then there exists a canonical system  $\Sigma$  which realizes  $f$ . If  $\widehat{\Sigma}$  is another canonical realization of  $f$ , then  $\Sigma$  is isomorphic to  $\widehat{\Sigma}$ , i. e. there exists an

invertible  $R$ -linear map  $T: X \rightarrow \widehat{X}$  such that  $T \circ G = \widehat{G}$ ,  $\widehat{F} \circ T = T \circ F$  and  $\widehat{H} \circ T = H$ .

In general canonical realizations are not free. Exceptions are principal-ideal domains, so that over  $\mathbf{Z}$  or polynomial rings in one variable  $\mathbf{R}[\sigma]$  there is no need in principle to consider nonfree systems.

If a canonical realization is free then it can be proven to be of minimal rank among free realizations. The converse is not true. For example (with  $R = \mathbf{Z}$ ),  $f = (2, 2, \dots)$  admits the minimal free realization  $(1, 2, 1)$  which is not canonical: indeed,  $(1, 2, 1)$  is not reachable, since (3.7) would require that every element of  $R^1$  (i. e. every integer) be an  $R$ -linear combination (i. e. an integer multiple) of 2. A canonical realization of  $f$  is given instead by  $(1, 1, 2)$ .

The canonical realization (unique up to isomorphism) can be defined directly from the input/output map  $f$ . For this purpose, let  $X_f :=$  the  $R$ -module consisting of all  $R$ -linear combinations of the columns of the Hankel matrix  $\mathbf{B}(f)$ ,  $F :=$  the  $R$ -linear map induced in  $X_f$  by the shift of columns,  $G(a_1, \dots, a_m) = a_1 b_1 + \dots + a_m b_m$  (where  $b_i :=$   $i$ -th column of  $\mathbf{B}(f)$ ) and  $H$  is defined on columns by « reading out » their first  $p$  rows. Then  $(X_f, F, G, H)$  is easily seen to be a canonical realization. (This construction was used by ROUCHALEAU [1972] and FLIESS [1972, 1974] to obtain various realization results). It follows from the Cayley-Hamilton Theorem that if  $X_f$  has  $s$  generators then  $X_f$  is already equal to the  $R$ -module generated by the first  $s$  block columns of  $\mathbf{B}(f)$ . In fact, by block symmetry, only first  $s$  block rows need be considered. We shall use later a corollary of this construction:

(3.11) LEMMA.  $f$  is realizable if and only if  $X_f$  is a finitely generated  $R$ -module.

We remark that further generalizations of the concept of system are possible by dropping the requirement that input and output spaces be free; see EILENBERG's treatment of discrete-time systems.

#### F. « LOCAL » METHODS.

When  $R$  is not an integral domain, the arguments in Sections B to E break down, since no field containing  $R$  can exist. The standard way to avoid such a limitation in commutative algebra is to look for embeddings of  $R$  in suitable products of fields.

For example, consider  $R := \mathbf{R}[\sigma]/(\sigma^k - 1)$ , the ring of real polynomials in  $\sigma$  of degree less than  $k$  added and multiplied subject to the relation  $\sigma^k = 1$ . The ring  $R$  is easily seen to be isomorphic to a direct product of various copies of the fields  $\mathbf{R}$  and  $\mathbf{C}$ ; there is one copy of  $\mathbf{R}$  for each real  $k$ -th root of unity and one copy of  $\mathbf{C}$  for each pair of complex conjugate roots. Therefore systems over  $R$  can be studied via the simultaneous study of a set of real and complex systems. In fact,  $R$  is the ring treated by BROCKETT and WILLEMS [1971, 1974], and the above decomposition underlies their whole method.

In general, however, it is impossible to obtain a complete decomposition of  $R$  into a product of fields, as in the above example. However, under the hypothesis that  $R$  be a *reduced* ring, an *inclusion* in such a product is possible. (A ring is reduced when it has no nonzero nilpotents, i. e. no elements for which  $a^n = 0$ . For example  $\mathbf{Z}_6$  is reduced but  $\mathbf{Z}_4$  is not, since  $2^2 = 0$  modulo 4). Given an input/output map  $f$  over  $R$ , let  $f_\lambda$  be the maps corresponding to the projections on the different factors of the product in which  $R$  is embedded. Just as in the case of integral domains (where the « product » had but one factor) each  $f_\lambda$  can be considered as a map over an appropriate field. The analog of (3.4) can be studied in this context, where now each  $\mathbf{B}(f_\lambda)$  is required to be of finite rank. For Noetherian reduced rings it is immediate to prove (based on the integral domain case) that the modified form of (3.4) is then true. Even the stronger statement (3.5) easily generalizes for the appropriate types of reduced rings (see ROUCHALEAU [1972]).

Another useful tool in drawing conclusions on systems over rings using the case of fields is the following, valid also (*mutatis mutandis*) for nonfree systems:

(3.12) LEMMA. *Let  $\Sigma = (F, G, H)$  be a free system. For each maximal ideal  $M$  of  $R$  define the  $R/M$ -system  $\Sigma(M) := (F(M), G(M), H(M))$ , where  $F(M), \dots$  is obtained by applying the canonical projection  $R \rightarrow R/M$  to each entry of  $F, \dots$ . Then,  $\Sigma$  is reachable if and only if all  $\Sigma(M)$  are reachable.*

PROOF. Reachability of  $\Sigma$  [resp.  $\Sigma(M)$ ] is equivalent to proving that the map given by the matrix  $(G, FG, \dots, F^{n-1}G)$  [resp.  $(G(M), \dots, F(M)^{n-1}G(M))$ ] is surjective. The equivalence is then immediate from BOURBAKI [Algèbre Commutative, II. 3.3, Prop. 11].  $\square$

We shall see the above lemma used in Part 4. This lemma falls into the category of «local methods» of commutative algebra. Note that (3.12) is valid without any restrictions on  $R$ .

#### 4. State-Feedback and Regulation.

The «separation property» of finite dimensional linear systems, which permits regulation via the independent designs of a state-feedback and a state estimator, extends immediately to systems over rings. This motivates a study into the possibility of «pole-shifting» and the existence of «observers» for systems over rings.

##### A. STATE - FEEDBACK.

One of the main algebraic results in the area of finite-dimensional linear systems over fields deals with the possibility of arbitrarily modifying the characteristic polynomial of a reachable system. This was proved for systems over infinite fields by WONHAM [1967] and for arbitrary fields by HEYMANN [1968] and KALMAN [1968].

We now ask whether a similar result holds true for free systems over rings. In other words, we want to characterize those pairs of matrices  $(F, G)$ ,  $F \in R^{m \times n}$ ,  $G \in R^{n \times m}$ , which are *coefficient-assignable* in the sense that

(4.1) *Given an arbitrary  $p(z) \in R[z]$  monic of degree  $n$ , there exists  $K \in R^{m \times n}$  such that  $\det(zI - F + GK) = p(z)$ .*

Closely related is the problem of finding those  $(F, G)$  which are only *pole-assignable*, i. e.

(4.2) *Given arbitrary  $a_1, \dots, a_n \in R$ , there exists  $K \in R^{m \times n}$  such that  $\det(zI - F + GK) = (z - a_1) \dots (z - a_n)$ .*

Since in general a polynomial  $p(z) \in R[z]$  does not have all its roots in  $R$ , (4.1) is a stronger statement than (4.2) in which existence of  $K$  is required for only some polynomials.

For example, consider a delay-differential system as in (2.3). The characteristic polynomial  $\chi_F(z) = \det(zI - F(\sigma))$  can be viewed either as a polynomial in  $z$  whose coefficients are polynomials in  $\sigma$ , i. e.  $\chi_F \in R[z] = (\mathbf{R}[\sigma])[z]$ , or equivalently, as a real polynomial in the two variables  $z, \sigma$ . It is well-known that stability properties of a delay system are determined by the values of  $s$  which make  $\chi_F$ , viewed as a

polynomial in  $s$ ,  $e^{-s} = \sigma$ , vanish. In particular, stabilization with arbitrary convergence rates is closely related to both (4.1) and (4.2). Observe the feedback law  $K$  is allowed to be a matrix of polynomials in  $\sigma$ .

#### B. REACHABILITY IS NECESSARY.

In the case of systems over fields, it is easy to verify that (4.2) implies that  $(F, G)$  is reachable in the sense of (3.7) ( $H$  is not relevant here). Indeed, it follows from the decomposition of the state-space of  $\Sigma$  into the reachable part and a complement, that for nonreachable systems there exists a fixed polynomial of nonzero degree which divides  $\chi_F - G_K$ , for all  $K$ , and this clearly contradicts (4.2) (KALMAN [1968, Corollary 5.9]). This proof fails for rings, since as the example over the integers  $(F, G) = (1, 2)$  shows, the reachable set does not in general admit a complementary submodule. The following result, valid for arbitrary  $R$ , shows the use of local methods:

(4.3) PROPOSITION. *If  $(F, G)$  is pole-assignable then  $(F, G)$  is reachable.*

PROOF. With the notations of (3.12), it is not difficult to verify that if  $(F, G)$  satisfies (4.2) then for each  $M$ ,  $(F(M), G(M))$  is pole-assignable over the field  $R/M$ . The previously mentioned result for systems over fields implies that  $(F(M), G(M))$  is reachable, for all  $M$ . It follows from (3.12) that  $(F, G)$  must be reachable.  $\square$

#### C. THE CONVERSE.

It is much more difficult to prove the converse of (4.3), i. e. that reachability implies (4.1), or at least (4.2). In fact, these problems are still unsolved for arbitrary rings. In the particular case  $m=1$ , however, the problem is trivial, since reachability of  $(F, g)$  implies that  $\{g, \dots, F^{n-1}g\}$  is a basis of  $R^n$ . Therefore  $(F, g)$  can be transformed into the « control canonical form » and the proof of (4.1) can be completed as in the case  $R = \text{field}$  (see, for instance, KALMAN [1968, Corollary 5.9]). This suggests trying to reduce the general case to the case  $m=1$ . In the case of fields this reduction is carried out for reachable  $(F, G)$  using the following consequence of a lemma of HEYMANN [1968]:

(4.4) *There exist  $u \in R^m$  and  $K \in R^{m \times n}$  such that  $(F - GK, Gu)$  is reachable.*

Only under strong restrictions on  $R$  can (4.4) be generalized:

(4.5) THEOREM. *Assume that  $R$  has only finitely many maximal ideals. Then (4.4), and hence (4.1), are true for arbitrary reachable  $(F, G)$ .*

PROOF. It follows from BOURBAKI [Algèbre Commutative, II. 3.2, Corollary 2] that it is enough to verify (4.4) for the ring  $R/\text{rad } R$  where  $\text{rad } R$  is the *radical* of  $R$ . Since  $R/\text{rad } R$  is a product of fields, the problem is reduced to the already solved case of fields.  $\square$

The above theorem applies to some rings of system-theoretic interest. Every *finite* ring obviously satisfies (4.5), so that the result has been proved for the most general case of linear sequential circuits. This generalizes the corresponding result for finite fields studied in some detail in MITTER and FOULKES [1971].

Another broad class of rings to which (4.5) can be applied are finite-dimensional algebras over fields, in particular group algebras on finite abelian groups. The result applies to classes of « cellular » or « partial difference » systems, as in (1. C), when the original equation evolves in a bounded space (so the group is finite). In this case, it can be proved that ring-theoretic reachability is equivalent to reachability of the equation. A  $K$  in  $R^{m \times n}$  is interpreted as a state-feedback law with the same regularities (e. g., a circulant matrix) as the matrices in the equation. For  $u$  in  $R^m$ ,  $Gu$  corresponds to a regular combination of the controls obtained in a similar fashion. Thus (4.4) insures that a reachable cellular system may be modified, through locally defined  $K$  and  $u$ , so that control is possible with scalar inputs at each point. The interpretations of (4.1) and (4.2) are somewhat more delicate, and the reader is referred to BROCKETT and WILLEMS [1974] for a detailed treatment of the case  $R = \mathbf{R}[\sigma]/(\sigma^k - 1)$ .

Unfortunately, (4.4) is not true in general. Call a ring  $R$  *suitable* iff reachability of  $(F, G)$  implies (4.4). The following lemma will imply that many innocent-looking  $R$  are not suitable:

(4.6) LEMMA. *Assume that  $R$  is a suitable principal-ideal domain. Let  $a, b$  be two relatively prime elements of  $R$ . Then there exist  $\alpha, \beta$  in  $R$  such that  $\beta$  is a unit and*

$$b \alpha^2 \equiv \beta \pmod{a}.$$

PROOF. Define

$$F := \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}, G := \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}.$$

Clearly  $(F, G)$  is reachable, because by assumption that  $(a, b)$  are relatively prime, the identity is a linear combination of  $a$  and  $b$ .

Since  $R$  is suitable, there exists a  $2 \times 2$  matrix  $K$  and a vector  $u$  such that  $(F - GK, Gu)$  is reachable. Then the matrix with columns  $Gu, (F - GK)Gu$  must have a unit determinant  $\beta$ . Writing

$$K = [k_{ij}], \quad u = \begin{bmatrix} \alpha \\ \gamma \end{bmatrix},$$

this determinant becomes  $\beta = ah + b\alpha^2$ , where  $h = -\alpha^2 k_{21} - 2\alpha\gamma k_{22} + \gamma\alpha k_{11} - a\gamma k_{12}$ .  $\square$

Now consider the particular case  $R := \mathbf{R}[x]$ ,  $a := x^2 - 1$ ,  $b := x$ . If  $R$  is suitable, by the lemma there would exist polynomials  $h, \alpha$ , and a nonzero real number  $\beta$  such that

$$h(x^2 - 1) = \alpha^2 x - \beta.$$

Specializing  $x = 1$  and  $x = -1$  in the above equality,

$$1 \cdot \alpha^2 = (-1) \cdot \alpha^2 = \beta \neq 0.$$

This last equation gives a contradiction, since  $1 \cdot \alpha^2$  and  $(-1) \cdot \alpha^2$  are both elements of  $\mathbf{R}$ .

A similar negative result holds for  $R := \mathbf{Z}$ ,  $a := 5$ ,  $b := 2$ ; in this case a check for  $\alpha = 0, \dots, 4$  shows that  $2\alpha^2 \equiv \pm 1 \pmod{5}$  is unsatisfiable over  $\mathbf{Z}$ .

Therefore *neither  $\mathbf{R}[x]$  nor  $\mathbf{Z}$  are suitable rings*. This answers a question raised by MORSE [1974, Example in Section 3].

A direct approach to (4.1) might still be possible, but the problem remains open. However, MORSE [1974] gave a constructive proof of the following result:

(4.7) THEOREM. *Assume that  $R$  is the polynomial ring  $\mathbf{R}[\sigma]$ . Then any reachable pair  $(F, G)$  satisfies (4.2).*

The above result can be applied in the study of stabilization of delay-differential systems. The main drawback in this application is the fact that the requirement that  $(F, G)$  be reachable is very restrictive; for instance, in the important special case  $m = 1$ ,  $(F, g)$  is reachable only when  $\det(g, Fg, \dots, F^{n-1}g)$  is a nonzero constant. Nonetheless it still illustrates the power of the present method, since a condition like (3.7) does not appear naturally in the study of delay systems while it suggests itself immediately in the context of systems over rings.

We sketch now another approach to the stabilization of delay-differential systems, based on systems over rings. Let  $\mathbf{R}_s[\sigma]$  denote the set of those rational functions  $p(\sigma)/q(\sigma)$  in  $\sigma$ , with real coefficients, which satisfy the condition

$$q(\sigma) \text{ has no zeroes in } \{z \in \mathbf{C}, |z| \leq 1\}.$$

It is easy to verify that  $\mathbf{R}_s[\sigma]$  is a ring with the usual operations on rational functions. Moreover,  $\mathbf{R}_s[\sigma]$  is a principal ideal domain: the g. c. d. of two elements  $p(\sigma)/q(\sigma)$  and  $\widehat{p}(\sigma)/\widehat{q}(\sigma)$  can be found by calculating the g. c. d. of  $p(\sigma) \cdot \widehat{q}(\sigma)$  and  $\widehat{p}(\sigma) \cdot q(\sigma)$  and then dividing the result by  $q(\sigma) \cdot \widehat{q}(\sigma)$ .

The polynomial ring  $\mathbf{R}[\sigma]$  is a subring of  $\mathbf{R}_s[\sigma]$ . Therefore a given pair  $(F, G)$  of  $\mathbf{R}[\sigma]$ -matrices can be also viewed as a pair of  $\mathbf{R}_s[\sigma]$ -matrices. It is not difficult to see that (4.7) depends only on  $\mathbf{R}[\sigma]$  being a principal ideal domain. In fact, MORSE's algorithm can be easily extended to the case of  $\mathbf{R}_s[\sigma]$ -matrices. Therefore if  $(F, G)$  is a pair reachable over  $\mathbf{R}_s[\sigma]$  there exists a matrix  $K$ , also over  $\mathbf{R}_s[\sigma]$ , such that  $\det(zI - F + GK) = p(\sigma, z)$ , where  $p(e^{-s}, s)$  has all its zeroes with real part less than some negative number.

We now interpret the meaning of our construction. Since the entries of  $K$  are rational functions  $p(\sigma)/q(\sigma)$  with  $q(0) \neq 0$ , the transformation  $\sigma \mapsto \sigma^{-1}$  and multiplication of  $p(\sigma^{-1})$  and  $q(\sigma^{-1})$  by a suitable power of  $\sigma$  lets us view  $K$  as a transfer function of a continuous-time linear system built from delays rather than integrators. The restriction on the zeroes of  $q$  translates into stability of the system described by the transfer function  $K$ . Let

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) + Ju(t) \end{aligned} \quad (*)$$

be a minimal realization of  $K$ . This system is stable because of minimality and stability of  $K$ . Consider the closed-loop system obtained by controlling  $(F, G)$  with  $(*)$ , i. e. the system

$$\begin{aligned} \dot{x}(t) &= (F(\sigma) - G(\sigma)J)x(t) - G(\sigma)Hv(t) \\ v(t+1) &= \widehat{F}v(t) + \widehat{G}x(t). \end{aligned} \quad (**)$$

Then  $(**)$  can be represented as a neutral delay-differential equation

on  $x(\cdot)$ , with characteristic equation

$$\Delta(s) = q(e^{-s})^n p(e^{-s}, s),$$

where  $q(\sigma)$  is a common denominator for  $K=K(\sigma)$ . A zero of  $\Delta(s)$  is either a zero of  $q(e^{-s})$  or of  $p(e^{-s}, s)$ . Since  $q(\sigma)$  has no roots with  $|\sigma| \leq 1$ , it follows that all roots of  $\Delta(s)$  have real parts bounded from above by a negative number. Therefore (\*\*) is uniformly asymptotically stable (1).

It is clear that the same reasoning would have applied if  $\sigma$  represented a delay of  $\lambda \neq 1$  seconds. It is not difficult to prove that the stabilization procedure we have just sketched is insensitive to small variations in the plant and controller parameters.

The  $\mathbf{R}_s[\sigma]$ -reachability condition under which the above algorithm applies can be equivalently expressed as follows. Let

$$\left[ \begin{array}{cccc|cccc} \psi_1 & 0 & \dots & 0 & 0 & \dots & 0 & \\ 0 & \psi'_2 & \dots & \cdot & \cdot & & & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \psi_n & 0 & \dots & 0 & \end{array} \right]$$

be the Smith form of the polynomial matrix  $(G, FG, \dots, F^{n-1}G)$ . Then  $(F, G)$  is reachable over  $\mathbf{R}_s[\sigma]$  provided that the  $\psi_i$  have no roots  $\sigma$  with  $|\sigma| \leq 1$ . By contrast, application of (4.7) would only be possible if all  $\psi_i$  are scalar. Note, however, that we have paid a price in weakening the reachability criteria, in that the feedback law is more complex than in the polynomial case. The necessity of including dynamic compensation, in what from the viewpoint of systems over rings is a state-feedback problem, is better understood once it is realized that  $x(t)$  does not represent the true state of equation (2.3) but rather a projection of the state. We are solving an output stabilization problem in disguise.

Comparison with the conditions given by PANDOLFI [1974] and BHAT and KOIVO [1976] shows that  $\mathbf{R}_s[\sigma]$ -reachability is sufficient but not necessary for stabilization. The comparison is immediate once we

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(1) Professor R. E. KALMAN has pointed out to us the striking similarity between our constructions using  $\mathbf{R}_s[\sigma]$  and some aspects of BODE's theory of feedback amplifiers. This remains a topic for future investigation.

observe that, over any ring  $R$ , reachability is equivalent to right-invertibility of  $(zI - F, G)$ . In particular,  $(F(\sigma), G(\sigma))$  is  $\mathbf{R}_\sigma[\sigma]$ -reachable when

$$\text{rank}(\mu I - F(e^{-\lambda}), G(e^{-\lambda})) = n$$

for all  $\lambda$  with nonnegative real part and for all  $\mu$ . The necessary conditions in the above references require only full rank for  $\lambda = \mu$ . The procedure discussed here, on the other hand, is completely algorithmic, based on simple matrix calculations, and requires no functional-analytic tools.

#### D. STATE - RECONSTRUCTION.

When the state variables are not directly available, the design of regulators is achieved with the use of Kalman filters or «observers». In other words, a system must be designed which must estimate the unknown state using the knowledge of inputs and outputs of the original system. When trying to apply this design method to the case of delay systems (using observers which are themselves delay systems), the arguments of the classical case are easily generalized and the condition  $(F', H') = \text{pole assignable}$  appears naturally (see a discussion of the case over the real field in WONHAM [1974, Chapter 3]). A similar situation holds for discrete-time systems over  $\mathbf{Z}$ .

The above examples suggest the study in general of the problem of finding conditions under which a given input/output map will admit a free realization  $(F, G, H)$  for which both of the following conditions hold: (i)  $(F, G)$  is reachable and (ii)  $(F', H')$  is reachable (i. e., the «dual system»  $(F', H', G')$  is reachable). These realizations are called *split* in SONTAG [1977 b]. A particular case of a result proved there is the following

(4.8) THEOREM. *Let  $R$  be a principal-ideal domain. Suppose  $f$  is an input/output map for which  $\text{rank } \mathbf{B}(f) = n$ . Then the following statements are equivalent:*

- (i)  *$f$  admits a split realization.*
- (ii) *The greatest common divisor of all the  $n$ -minors of*

$$\begin{bmatrix} A_1 & \cdots & A_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_n & \cdots & A_{2n-1} \end{bmatrix}$$

*is a unit.*

For instance, if  $R=\mathbf{Z}$ , then  $f=(2, 2, \dots)$  admits no split realizations. Indeed, the canonical realization  $(1, 1, 2)$  does not satisfy  $(1, 2)=\text{reachable}$ .

When  $m=p=1$ , condition (ii) in (4.7) can be easily expressed in terms of the « transfer function »  $p/q$  of  $f$ , where  $q$  is the lowest denominator of  $f$  over the quotient field  $Q$ . Let  $R(p, q)$  be the *resultant* of  $p, q$  (see LANG [1965, p. 135]). Then

(4.9)  $f$  admits a split realization iff  $R(p, q)$  is a unit.

If an input/output map satisfies (4.8), the design of output compensators proceeds just as in the case of fields.

## 5. Other Topics.

Although the problems in Sections 3 and 4, especially the former, are those that have attracted most of the past research, results have been obtained also in other areas. Some of these results are direct generalizations (with essentially the same proofs) of facts known over fields. In other cases, the results are known only for special rings, and in fact may not even be expressed in terms of systems over rings. It is of course not possible to review here all such topics. We shall limit ourselves to making brief observations on three examples: system decomposition, optimal control, and generalizations of known algebraic conditions.

Results on series or parallel decompositions à la Krohn-Rhodes correspond to decompositions of (state) modules, over  $R$  or  $R[z]$  respectively, as explained in EILENBERG [1974, Ch. 16, Section 7]. The latter purely algebraic problems are in general very difficult, except when very strong finiteness conditions are assumed on  $R$  (or on the state modules). Under such restrictive conditions a rather good theory exists and the results can be applied to the system-theoretic problem. In particular, decompositions can be obtained when the state-space is a finite abelian group (MATLUK and GILL [1971], JOHNSTON [1973]).

The related topic of obtaining « canonical forms » for system matrices is essentially unexplored, except in the scalar case, as mentioned in Section 4. C.

The quadratic optimization problem for discretized partial differential equations was studied by BROCKETT and WILLEMS [1971, 1974], WILLEMS [1971]. Their main result is that if the matrices defining the criterion correspond to operators in the ring over which

the system is defined (i. e. are circulant or Toeplitz), then an optimal feedback law can be implemented using operators on the same ring (i. e. using a feedback matrix of the circulant or Toeplitz type). Moreover, a spectral factorization solution can be obtained via a factorization of polynomials with coefficients on the ring.

Many algebraic criteria for finite-dimensional linear systems over fields can be easily generalized, giving in some cases new results and, in others, more concise expressions and elegant proofs of already known facts. Consider for example the question of pointwise or « Euclidean » reachability for delay-differential systems, i. e. the possibility of reaching every value  $x(t) \in \mathbb{R}^n$ , given zero initial conditions. The ring reachability condition (3.7) is unrelated to this problem, as shown by the trivial example  $\dot{x}(t) = u(t-1)$ . This system is clearly pointwise reachable, but  $(F, G) = (0, \sigma)$  does not satisfy (3.7).

A more concrete definition of systems over rings is possible, however (SONTAG [1976, Section 7]). In this context a general result can be proved, which implies in particular that a delay-differential system  $\dot{x} = Fx + Gu$ , where both  $F$  and  $G$  are matrices of polynomials in  $\sigma_1, \dots, \sigma_r$  (the  $\sigma_i$  denoting noncommensurable delays), is pointwise reachable if and only if

*there exists no real constant vector  $V$  such that  $V'(G, FG, \dots, F^{n-1}G) = 0$ .* (\*)

Since a polynomial is zero if and only if each coefficient is, (\*) can also be expressed (in a very involved way) as a rank condition on real matrices. In the latter form, some very special cases ( $r=1$ , only one delay on  $F$ , etc.) of this result were known before; see KIRILLOVA and ČURAKOVA [1967]. By reading the various proofs (and noticing the triviality of (\*) once that the appropriate rings are introduced), the advantages of working with systems over rings, at least for this problem, will be clear.

## APPENDIX 1

We give here proofs of (3.4) for Noetherian integral domains, of (3.6) for principal ideal domains, and of (3.5) for integrally closed domains which satisfy (3.4).

It is not too difficult to prove that *if  $M$  is a finitely generated  $R$ -module and  $R$  is Noetherian then every submodule of  $M$  is also finitely generated.* (Since an ideal of  $R$  is in particular a submodule of the finitely generated  $R$ -module  $R$ , this last property is *equivalent* to the definition of Noetherian integral domain). *Similarly, if  $M$  admits a system of  $n$  generators and  $R$  is a principal-ideal domain then every submodule of  $M$  can be generated by at most  $n$  elements* (see LANG [1965, pages 144 and 387]).

Let  $L$  be an infinite matrix over  $R$  and let  $N$  be the  $R$ -module consisting of all  $R$ -linear combinations of the columns of  $L$  (seen as infinite vectors). Assume that  $\text{rank } L = n < \infty$ . Then there exist columns  $v_1, \dots, v_n$  which generate the  $Q$ -vector space spanned by the columns of  $L$ , where  $Q$  is the quotient field of  $R$ . Any other column  $v$  can then be written as  $\sum_i (d_i/d) v_i$ , where  $d$  is the determinant of an  $n \times n$  matrix of full rank obtained from the columns  $v_1, \dots, v_n$  by a choice of suitable rows, and where  $d_i$  is the determinant of the matrix obtained by replacing the column  $v_i$  by the column  $v$  (Cramer's rule). Observe that by definition  $d_i \in R$  for all  $i$ .

Denote  $w_i := \frac{1}{d} v_i$ , and let  $M$  be the  $R$ -module generated by  $w_1, \dots, w_n$ . The previous discussion shows that every column  $v$  is an  $R$ -linear combination of  $w_1, \dots, w_n$ . Therefore  $N$  is a submodule of  $M$ . It follows that  $N$  is finitely generated if  $R$  is Noetherian, and  $N$  admits a basis of  $n$  generators if  $R$  is a principal-ideal domain.

The above reasoning, along with (3.11), and applied to  $\mathbf{B}(f)$ , proves (3.4) for Noetherian rings and (3.6) for principal-ideal domains. Explicit algorithms for the principal-ideal domain case can be easily given, based on algorithms for finding bases of modules. See for instance ROUCHALEAU and SONTAG [1977] or KAMEN [1975].

Assume now that  $R$  is integrally closed and satisfies (3.4). We outline a proof of (3.5) along the lines of ROUCHALEAU [1972]. Let  $f$  be realizable over  $Q$ . By (3.4), there exists a realization  $(F, G, H)$  over  $R$ . Consider the set  $I(f)$  of all those polynomials  $z^n + a_{n-1} z^{n-1} + \dots + a_0$  in  $Q[z]$  for which  $a_0, \dots, a_{n-1}$  is a recursion for  $f$ . It is easily verified that  $I(f)$  is a principal ideal, whose generator is the polynomial  $m(z)$  corresponding to the minimal recursion for  $f$ . By the Cayley-Hamilton theorem the characteristic polynomial  $\chi_F(z)$  is in  $I(f)$ . So  $m(z)$  divides  $\chi_F(z)$ . By BOURBAKI [Algèbre Commutative, V. 1, 3, Prop. 11],  $m(z)$  is in  $R[z]$ , as wanted.

## APPENDIX 2

We review here some elementary algebraic notions. The reader may find details in any modern algebra book; for instance MACLANE and BIRKHOFF [1967].

A *ring*  $R$  (with identity) is a set, together with two binary operations  $(x, y) \mapsto x+y$  and  $(x, y) \mapsto xy$ , and two distinguished elements  $0, 1$ , such that:

a)  $(R, +, 0)$  is an abelian group (i. e. addition is commutative, associative,  $0$  is the neutral element and every element  $x$  has an additive inverse  $-x$ );

b)  $(xy)z = x(yz)$  for all  $x, y, z$ ;

c)  $1x = x1 = x$  for all  $x$ ;

d)  $x(y+z) = xy+xz$  and  $(y+z)x = yx+zx$  for all  $x, y, z$ .

Example: the integers  $\mathbf{Z}$  with their usual addition and multiplication.

$R$  is *commutative* when  $xy=yx$  for all  $x, y$ . All rings considered in this paper are commutative.  $I \subseteq R$  is an *ideal* of  $R$  iff (i)  $I$  is closed under addition, and (ii) for any  $r$  in  $R$  and  $a$  in  $I$ ,  $ra$  is in  $I$  (e. g., the even integers form an ideal in  $\mathbf{Z}$ ).

A commutative ring  $R$  is

— an *integral domain* iff  $xy=0$  implies  $x=0$  or  $y=0$  (e. g.,  $\mathbf{Z}$  is an integral domain; instead, the ring of residues modulo 4 is *not* an integral domain since  $2 \cdot 2=0$  modulo 4, but  $2 \neq 0$ );

— a *field* iff for each  $x \neq 0$  there exists a  $y$  in  $R$  such that  $xy=1$  (then  $y$  is unique and is denoted  $x^{-1}$ );

— *Noetherian* iff every ideal  $I$  of  $R$  is finitely generated, i. e. there exist  $a_1, \dots, a_n$  in  $I$  ( $n$  depending on  $I$ ) such that every  $b$  in  $I$  admits an expression  $b=r_1 a_1 + \dots + r_n a_n$  for some  $r_i$  in  $R$ . (« Noetherian » is a very weak restriction on *commutative* rings — but very strong otherwise —; in fact every commutative ring which has appeared until now in system theory is Noetherian);

— a *principal-ideal domain* iff every ideal  $I$  is principal ( $n=1$  above, for all  $I$ ); this is equivalent to requiring Noetherian + every pair of elements  $a_1, a_2$  has a greatest common divisor which can be expressed as an  $R$ -combination of  $a_1, a_2$ .

Let  $R$  be an integral domain. Consider the set of all fractions  $r/s, r, s$  in  $R, s \neq 0$ , subject to the equivalence relation:  $r/s = r_1/s_1$  if  $rs_1 = r_1s$ . When such fractions are added and multiplied in the obvious way, one obtains a field  $Q(R)$ , the *quotient field* of  $R$ , which contains  $R$  when the latter is identified with the fractions  $r/1$ . For instance,  $Q(\mathbf{Z}) =$  rational numbers; if  $R = k[X_1, \dots, X_s]$  is the ring of all polynomials in the indeterminates  $X_1, \dots, X_s$  with coefficients in a field  $k$ , then  $Q(R)$  is the field of all rational functions in  $X_1, \dots, X_s$ .

A map  $f: R \rightarrow S$  between two rings  $R, S$  is a *ring-homomorphism* iff  $f$  preserves addition and multiplication and  $f(1) = 1$ . Let  $I$  be an ideal of  $R$ ; the *residue ring*  $R/I$  is the set of equivalence classes  $[r], r$  in  $R$ , where  $[r] = [s]$  iff  $r - s$  belongs to  $I$ , with operations  $[r] \pm [s] = [r \pm s]$ , and  $[0], [1]$  as neutral element and identity. The *canonical map*  $R \rightarrow R/I: r \mapsto [r]$  is a ring homomorphism. An ideal is *maximal* iff  $R/I$  is a field.

The integral domain  $R$  is *integrally closed* iff any  $q$  in  $Q(R)$  which satisfies an equation

$$z^n + a_{n-1}z^{n-1} + \dots + a_0 = 0, \text{ all } a_i \text{ in } R,$$

is already in  $R$ . Examples:  $\mathbf{Z}, k[X_1, \dots, X_s]$  for any field  $k$  and integer  $s$ .

An  $R$ -*module*  $M$  is a vector space in which the scalars belong to a ring. So  $M$  is an abelian group equipped with an operation  $R \times M \rightarrow M: (r, m) \mapsto rm$  which satisfies, for all  $r, r_1, r_2$  in  $R$  and  $m, m_1, m_2$  in  $M$ ,

- a)  $1m = m;$
- b)  $r(m_1 + m_2) = rm_1 + rm_2;$
- c)  $(r_1 + r_2)m = r_1m + r_2m;$
- d)  $(r_1 r_2)m = r_1(r_2m).$

Example: an ideal of  $R$  is an  $R$ -module. The elements  $m_1, \dots, m_s$  of  $M$  *generate*  $M$  iff every  $m$  in  $M$  can be expressed as  $m = r_1m_1 + \dots + r_sm_s$  for some  $r_i$  in  $R$ . If such  $m_1, \dots, m_s$  exist for some  $s$ ,  $M$  is

*finitely generated*. A (finite) *basis* for  $M$  is a generating set  $m_1, \dots, m_s$ , such that, for each  $m_i$ , the  $r_i$  are *unique*.  $M$  is (finitely generated) *free* if  $M$  has a basis. A map  $L: M \rightarrow \widehat{M}$  between two  $R$ -modules is ( $R$ )-*linear* or an  $R$ -*homomorphism* iff  $L(r_1 m_1 + r_2 m_2) = r_1 L(m_1) + r_2 L(m_2)$  for all  $r_i$  in  $R$ ,  $m_i$  in  $M$ . An invertible linear map is an *isomorphism*.

A *free* module  $M$  is always isomorphic to a module of the type  $R^s$ , where  $R^s :=$  column vectors over  $R$  with coordinatewise addition and with  $r(r_1, \dots, r_s)' := (rr_1, \dots, rr_s)'$ . Let  $R^{t \times s}$  denote the set of  $t \times s$  matrices with entries in  $R$ . A linear map  $L: R^s \rightarrow R^t$  can be represented uniquely by a matrix  $L$  in  $R^{t \times s}$  whose columns are given by  $L(e_1), \dots, L(e_s)$ , where  $e_i$  is the vector whose only nonzero entry is a 1 in the  $i$ -th position. Conversely, each matrix in  $R^{t \times s}$  defines a linear map  $R^s \rightarrow R^t$ . We shall therefore speak interchangeably about linear maps  $R^s \rightarrow R^t$  and matrices in  $R^{t \times s}$ .

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