Uniformly Universal Inputs

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Dedicated to Alberto Isidori on his 65th birthday

Summary. A result is presented showing the existence of inputs universal for observability, uniformly with respect to the class of all continuous-time analytic systems. This represents an ultimate generalization of a 1977 theorem, for bilinear systems, due to Alberto Isidori and Osvaldo Grasselli.

1 Introduction

One of the key concepts in control theory is that of a *universal input* for observability and parameter identification. Informally stated, an input u_0 is universal (for a given system) provided that the following property holds: if two internal states x_1 and x_2 are in principle distinguishable by any possible input/output experiment, then x_1 and x_2 can be distinguished by forcing the system with this particular input u_0 (and observing the corresponding output function). *Universal input theorem(s)* for distinguishability show that such inputs indeed do exist, and, furthermore, show that "generic" (in an appropriate technical sense) inputs have this property. Viewing unknown parameters as constant states, one may re-interpret the universal input property as one regarding parameter identifiability instead of observability.

In the seminal 1977 paper [5], Alberto Isidori (together with Osvaldo Grasselli) provided the first general result on existence of universal inputs for a wide class of nonlinear systems (bilinear systems). Motivated by this work [8] provided analogous results for discrete time systems as well as continuous-time analytic systems with compact state spaces, and this was extended to arbitrary continuous-time analytic systems in [14]. (See also the related work in [7]

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for linear automata.) A different proof of the result in [14] was given in [21], where implications to the study of a nonlinear analog of "transfer functions" were discussed as well.

In the present paper, we provide an ultimate extension of the theorems for analytic continuous-time systems, showing that there are inputs that are universal with respect to *all* finite dimensional analytic systems, and, moreover, the set of such inputs is generic. A preliminary version of our result was presented at the 1994 IEEE Conference on Decision and Control [20] (see also [11]).

Besides their intrinsic theoretical appeal, universal input theorems help provide a rationale for systems identification when using information provided by "random" or unknown inputs. For example, in [16] universal inputs were used to justify the "dependent input" approach to the identification of molecular-biological systems, for which high complexity and the lack of sufficient quantitative measurements prevent the use of arbitrary test signals. The approach in [16], applied to measurements of nitrogen uptake fluxes in baker's yeast (*Saccharomyces cerevisiae*), was to view unmodeled dynamics (possibly due to mutations in the yeast strains being used) as generating fictitious "dependent inputs". In another direction, universal input theorems provide a basis for certain numerical methods for path planning in nonlinear systems, see for example [11, 10, 12].

2 Analytic Input/Output Operators

We first review some standard notions regarding analytic input/output operators. Let m be a fixed nonnegative integer. By an input we mean a Lebesgue measurable, essentially bounded function $u : [0, T] \to \mathbb{R}^m$ for some $T > 0$.

Consider a set $\Theta = \{X_0, X_1, \ldots, X_m\}$, whose elements will be thought as $m+1$ non-commuting variables. We use Θ^* to denote the free monoid generated by Θ, where the neutral element of Θ^* is the empty word, and the product is concatenation. We define $\mathbb{R}[\Theta]$ to be the $\mathbb{R}$-algebra generated by Θ^*, that is, the set of all polynomials in the X_i's. By a *power series* in the variables $X_0, X_1, \ldots, X_m$ we mean a formal power series:

$$c = \sum_{w \in \Theta^*} \langle c,\ w \rangle w\,,$$

where $\langle c,\ w\rangle \in \mathbb{R}$ for each $w \in \Theta^*$. We use $\mathbb{R}[[\Theta]]$ to denote the set of all power series in the X_i's. This is a vector space with "+" defined coefficient-wise. There is a linear duality between $\mathbb{R}[[\Theta]]$ and $\mathbb{R}[\Theta]$ provided by:

$$\langle c,\ d\rangle = \sum_{w \in \Theta^*} \langle c,\ w\rangle \langle d,\ w\rangle \tag{1}$$

for any $c \in \mathbb{R}[[\Theta]]$ and $d \in \mathbb{R}[\Theta]$.

A series $c \in \mathbb{R}[[\Theta]]$ is a *convergent* series if there is a positive (radius of convergence) ρ and a constant M such that

$$|\langle c,\ w\rangle| \;\leq\; M\rho^l l!, \qquad \forall |w| = l\,, \tag{2}$$

where $|w|$ denotes the length of w, i.e., $|w| = l$ if $w = X_{i_1}X_{i_2}\cdots X_{i_l}$.

Let $L^m_{e,\infty}$ denote the set of measurable, locally essentially bounded functions $u : [0,\,\infty) \to \mathbb{R}^m$. For each $u \in L^m_{e,\infty}$ and $S_0 \in \mathbb{R}[[\Theta]]$, consider the initial value problem

$$\dot{S}(t) \;=\; \left(X_0 + \sum_{i=1}^{m} X_i u_i\right) S(t), \quad S(0) = S_0 \tag{3}$$

seen as a differential equation over $\mathbb{R}[[\Theta]]$. A solution is an absolutely continuous curve, where derivative is understood coefficient-wise. For any locally essentially bounded $u(\,\cdot\,)$, by the Peano-Baker formula, there is always a solution in $\mathbb{R}[[\Theta]]$ whose coefficients are iterated integrals of u. Furthermore, one can prove the uniqueness of the solutions successively by induction. In particular, the solution $C[u]$ with $C[u](0) = S_0 = 1$ defines the *generating* (or "Chen-Fliess") series of u (cf. [1, 2, 14]). Explicitly, For each u, the generating series $C[u]$ is given by

$$C[u](t) \;=\; \sum_w V_w[u](t)w\,,$$

where $V_w[u]$ is given recursively by $V_\phi[u](t) \;=\; 1$, and

$$V_{X_i w}[u](t) \;=\; \int_0^t u_i(s)V_w[u](s)\,ds\,, \qquad \forall w \in \Theta^*\,, \tag{4}$$

where $u_0 \equiv 1$. We say that a pair (T, r) of positive real numbers with $r \geq 1$ is *admissible* for a convergent series c if for some M and ρ as in (2) the following inequality holds:

$$Tr\rho(m+1) < 1.$$

For each pair (T, r) (where $r \geq 1$) that is admissible for a convergent series c, the series c defines an i/o operator $F_c^{T,r}$ on the set

$$\mathcal{V}_T(r) := \{u|\ u : [0,\,T] \to \mathbb{R}^m,\ \|u\|_\infty \leq r\}$$

by means of the following formula:

$$F_c[u](t) \;=\; \langle c,\ C[u](t)\rangle \;=\; \sum_w \langle c,\ w\rangle V_w[u](t)\,. \tag{5}$$

It is known (c.f. [6]) that the series in (5) converges uniformly on $[0,\,T]$.

Note that, for every convergent series c, and for every two pairs (T_1, r_1) and (T_2, r_2) that are admissible for c, the functions $F_c^{T_1,r_1}$ and $F_c^{T_2,r_2}$ coincide on $V_r(T)$, where $T = \min\{T_1, T_2\}$ and $r = \min\{r_1, r_2\}$. Therefore, one may define a mapping F_c on the union of the sets $V_T(r)$ for all pairs (T, r) that are admissible for c, as an extension of the maps $F_c^{T,r}$. Such operators defined by convergent series have been extensively studied, c.f. [3, 6, 18, 19].

3 Uniformly Universal Inputs

In this section we study the distinguishability of operators by analytic input functions.

Let c and d be two convergent series. We say that c and d are *distinguishable by an input function* $u : [0, T_0] \to \mathbb{R}^m$, denoted by $c \not\sim_u d$, if for every $T \in (0, T_0]$ for which $(T, \max\{\|u\|_\infty, 1\})$ is admissible for both c and d, it holds that

$$F_c[u] \neq F_d[u]$$

as functions defined on $[0, T]$. Note here that "$c \not\sim_u d$" is stronger than merely requiring $F_c[u](t) \neq F_d[u](t)$ as functions over some interval. In our context, we require that $F_c[u] \neq F_d[u]$ as functions over *every* interval $[0, T]$ for which $(T, \max\{\|u\|_\infty, 1\})$ is admissible for both c and d.

An input u is called a *uniformly universal input* if $c \not\sim_u d$ for any convergent series c and d such that $c \neq d$. Note that an input u is a uniformly universal input if and only if u distinguishes c from 0 whenever $c \neq 0$.

For each $T > 0$, we consider $C^\infty[0, T]$, the set of all smooth functions from $[0, T]$ to $\mathbb{R}^m$, a topological space endowed with the Whitney topology. We will say that a subset S of a topological space is *generic* if S contains a countable intersection of open dense sets. Since $C^\infty[0, T]$ is a Baire space (cf.[4]), a generic subset of $C^\infty[0, T]$ is dense.

Let Ω^T denote the set of all uniformly universal inputs defined on $[0, T]$. The following is the main result.

Theorem 1. *For any fixed $T > 0$, the set Ω^T of uniformly universal inputs is a generic subset of $C^\infty[0, T]$.*

Theorem 1 asserts the existence of smooth uniformly universal inputs (and their genericity); however, there is no *analytic* uniformly universal input. To illustrate this fact, consider the following example.

Example 1. Take any fixed analytic function $\alpha : [0, \infty) \to \mathbb{R}$. For this function, consider the state space system:

$$\dot{x}_1 = 1\,, \quad \dot{x}_2 = 0\,, \quad \dot{x}_3 = (\alpha(x_1) - u)x_2\,, \quad y = x_3\,. \tag{6}$$

When writing the system as

$$\dot{x} = g_0(x) + g_1(x)u\,, \quad y = h(x)\,,$$

one has, in the standard coordinates of $\mathbb{R}^3$, $g_0(x) = (1,\, 0,\, \alpha(x_1)x_2)^\tau$, $g_1(x) = (0,\, 0,\, -x_2)^\tau$ and $h(x) = x_3$, where the superscript "τ" denotes the transpose.

For each $x \in \mathbb{R}^3$, let c_x be the generating series induced by the system with the initial state x, that is, c_x is given by

$$\langle c_x,\ X_{i_1} X_{i_2} \cdots X_{i_r} \rangle \;=\; L_{g_{i_r}} \cdots L_{g_{i_2}} L_{g_{i_1}} h(x)\,,$$

for all multi-indices $i_1 i_2 \ldots i_r$, and all $r \geq 0$. Then c_x is a convergent series, and for any initial state p, and each u, the corresponding output of (6) is given by the "Fliess fundamental formula" ([6]):

$$y(t) = F_{c_p}[u](t)\,.$$

Observe that for system (6), the two particular initial states $p = (0,0,0)$ and $q = (0,1,0)$ can always be distinguished by some input, i.e., $c_p \neq c_q$. (Indeed, whenever $p \neq q$ are two states such that $p_1 = q_1$, the input $u(t) = \alpha(p_1 + t) - 1$ distinguishes these initial states.) But the pair (p, q) cannot be distinguished by u, i.e., $F_{c_p}[u] = F_{c_q}[u]$, if $u(t) = \alpha(t)$. Hence, c_p and c_q cannot be distinguished by $\alpha(\,\cdot\,)$. This shows that for any analytic function $\alpha(\,\cdot\,)$, one can always find a pair (c_p, c_q) which α cannot distinguish but $c_p \neq c_q$. This shows that there is no uniformly universal input which is analytic. $\triangle$

3.1 Universal Input Jets

For each $k \geq 1$, consider the polynomial $\mathfrak{d}_k(\mu)$ in $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1})$ given by

$$\mathfrak{d}_k(\mu) \;=\; \left.\frac{d^k}{dt^k}\right|_{t=0} C[u](t)\,, \tag{7}$$

where u is any input such that $u^{(i)}(0) = \mu_i$. Then one has the following formula for $k \geq 1$:

$$\frac{d^k}{dt^k} F_c[u](0) \;=\; \langle c,\ \mathfrak{d}_k(u(0), u'(0), \ldots, u^{(k-1)}(0))\rangle\,. \tag{8}$$

Let $\mathfrak{d}_0 = 1$. Then if $\mu = (\mu_0, \mu_1, \ldots)$ is such that $\langle c,\ \mathfrak{d}_k(\mu^k)\rangle \neq 0$ for some $k \geq 0$, then $c \not\sim_u 0$, for any $T > 0$ and any $u \in C^\infty[0, T]$ such that $u^{(i)}(0) = \mu_i$ for $0 \leq i \leq k-1$, where $\mu^k \in \mathbb{R}^{mk}$ is given by $\mu^k_i = \mu_i$ for $0 \leq i \leq k-1$.

Let $\mathrm{I\!R}^{\mathrm{m},\infty} = \prod_{i=1}^\infty \mathbb{R}^m$ be endowed with the product topology whose basis of open sets consists of all sets of the form $\prod_{i=1}^\infty U_i$, where each U_i is an open subset of $\mathbb{R}^m$, and only finitely many of them are proper subsets of $\mathbb{R}^m$. Note that $\mathrm{I\!R}^{\mathrm{m},\infty}$ is a Baire space, and hence, any generic subset of $\mathrm{I\!R}^{\mathrm{m},\infty}$ is a dense set. For each $\mu \in \mathrm{I\!R}^{\mathrm{m},\infty}$ and a series c, we let $\langle c,\ \mathfrak{d}(\mu)\rangle$ denote the sequence

$$\langle c,\ \mathfrak{d}_0\rangle, \langle c,\ \mathfrak{d}_1(\mu_0)\rangle,\ \langle c,\ \mathfrak{d}_2(\mu_0, \mu_1)\rangle,\ \langle c,\ \mathfrak{d}_3(\mu_0,\ \mu_1, \mu_2)\rangle,\ \ldots.$$

Let $\mathcal{J}$ be the subset of $\mathrm{I\!R}^{\mathrm{m},\infty}$ defined by

$$\mathcal{J} = \{\mu :\ \langle d,\ \mathfrak{d}(\mu)\rangle \neq 0,\ \forall d \in \mathfrak{C},\ d \neq 0\}\,, \tag{9}$$

where $\mathfrak{C}$ stands for the set of all convergent series. Take $\mu \in \mathcal{J}$. It is easy to see from (8) that for any $u \in C^\infty$ with $u^{(i)}(0) = \mu_i$ for all i, u is a uniformly universal input. We call the elements in $\mathcal{J}$ universal input jets.

Theorem 2. *The set $\mathcal{J}$ of universal input jets is a generic subset of* $\mathrm{I\!R}^{\mathrm{m},\infty}$.

4 Proofs of Theorems 1 and 2

To prove Theorems 1 and 2, we need to study some topological properties of the set $\mathfrak{C}$ of convergent series. This set can be identified with $\mathbb{R}^{\mathbb{N}}$, the set of all maps from $\mathbb{N}$ to $\mathbb{R}$, once the elements of Θ^* are linearly ordered; we again adopt the product topology on this set. With this topology, that a sequence $\{c_j\}$ converges to c means

$$\lim_{j\to\infty} \langle c_j,\ w\rangle \;=\; \langle c,\ w\rangle$$

for each $w \in \Theta^*$. Observe that a subset S of $\mathbb{R}[[\Theta]]$ is compact (in the product topology) if and only if S is closed, and for each w, there exists $M_w > 0$ such that for all $d \in S$,

$$|\langle d,\ w\rangle| \;\le\; M_w .$$

4.1 Equi-Convergent Families

A family S of convergent series is said to be *equi-convergent* if there exist $\rho,\ M > 0$ such that

$$|\langle d,\ w\rangle| \;\le\; M\rho^l l!\,, \qquad \forall |w| = l \tag{10}$$

holds for every $d \in S$. Clearly, every closed equi-convergent family is compact, and if S is equi-convergent, there exists some pair (T, r) that is admissible for every element of S. For such (T, r), we say that (T, r) is admissible for S.

For any convergent series c and $\mu \in \mathbb{R}^{mk}$, we let $\langle c,\ \mathfrak{d}_k(\mu)\rangle_k$ denote the k-vector

$$\Big(\langle c,\ \mathfrak{d}_0\rangle,\ \langle c,\ \mathfrak{d}_1(\mu_0)\rangle\,, \ldots, \langle c,\ \mathfrak{d}_k(\mu_0, \ldots,\ \mu_{k-1})\rangle\Big)\ .$$

For a set S of convergent series, we let

$$\mathcal{J}_S^k \;=\; \{\mu \in \mathbb{R}^{mk}:\ \langle d,\ \mathfrak{d}_k(\mu)\rangle_k \ne\ 0,\ \forall d \in S\}$$

(which maybe an empty set, e.g., in the case when S contains the zero series.) Let $\mu \in \mathbb{R}^{mk}$. We say that ν is a finite extension of μ if $\nu \in \mathbb{R}^{ml}$ for some $l \ge k$ such that $\nu_i = \mu_i$ for $0 \le i \le k-1$. For an equi-convergent family, we have the following conclusion.

Lemma 1. *Assume that S is compact and equi-convergent, and that $\mathcal{J}_S^l \ne \emptyset$ for some l. Then for any $k \ge 1$ and any $\mu \in \mathbb{R}^{mk}$, there exist K and a finite extension ν of μ such that $\nu \in \mathcal{J}_S^K$.*

To prove Lemma 1, we first discuss some continuity properties of the operators defined by the convergent series. Lemma 2.2 of [19] shows that if (T, r) is admissible for c, then the map $\mathcal{V}_T(r) \to C[0, T], u \mapsto F_c[u]$ is continuous using the L_1 norm on $\mathcal{V}_T(r)$ in the special case when $r = 1$. The same proof can be used to prove the following result for equi-convergent families.

Lemma 2. *Assume that S is equi-convergent, and (T, r) is admissible for S. Then the map*

$$\mathcal{V}_T(r) \to C[0,T], \quad u \mapsto F_c[u]$$

is continuous with respect to the L_1 norm on $\mathcal{V}_T(r)$ and the C^0 norm on $C[0,T]$ uniformly for $c \in S$. □

This result can be strengthened further to the following, where the topology on $\mathcal{V}_T(r)$ is the L_1-topology, and the topology on $C[0,T]$ is the C^0 topology.

Lemma 3. *Let S be an equi-convergent family. Then, for any $r > 0$, there exists some $T_1 > 0$ such that for any $T < T_1$ the map*

$$\psi : S \times \mathcal{V}_T(r) \to C[0,T], \quad (c, u) \mapsto F_c[u]$$

is continuous.

Proof. Let S be compact and equi-convergent. Then there exists ρ such that

$$|\langle d,\ w\rangle| \le M\rho^k k! \qquad \forall |w| = k\,,\ \forall d \in S\,. \tag{11}$$

Let $T_1 = \frac{1}{r\rho(m+1)}$. Fix $T \in [0, T_1)$. Then F_d is defined on $\mathcal{V}_T(r)$ for each $d \in S$. For any $c, d \in S$, $u, v \in \mathcal{V}_T(r)$,

$$\|F_c[u] - F_d[v]\|_\infty \le \|F_c[u] - F_c[v]\|_\infty + \|F_c[v] - F_d[v]\|_\infty\,.$$

Hence, by Lemma 2, it is enough to show that the map

$$S \to C[0,T], \quad c \mapsto F_c[v] \tag{12}$$

is equi-continuous for $v \in \mathcal{V}_T(r)$, that is, for any $c \in S$, for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of c such that

$$\|F_c[v] - F_d[v]\|_\infty < \varepsilon$$

for all $d \in \mathcal{N}$ and all $v \in \mathcal{V}_T(r)$.

First note that for each $d \in S$ and $v \in \mathcal{V}_T(r)$, one has

$$|V_w[v](t)| \le \frac{r^k t^k}{k!} \qquad \forall\ |w| = k, \tag{13}$$

and therefore,

$$\left| \sum_{|w| \ge k} \langle d,\ w\rangle V_w[v](t) \right| \le \sum_{j=k}^{\infty} M\rho^j j!\,(m+1)^j\,\frac{r^j T^j}{j!} \le M \sum_{j \ge k} \frac{T^j}{T_1^j}$$

(where we have used the fact that there are at most $(m+1)^j$ elements in Θ^j). Since $0 < T < T_1$, it follows that for any $\varepsilon > 0$, there exists some $k > 0$ such that

$$|F_d[v](t) - F_{d_k}[v](t)| < \varepsilon \quad \forall t \in [0, T], \tag{14}$$

for all $v \in \mathcal{V}_T(r)$, all $d \in S$, where for each d, d_k is the polynomial given by

$$d_k = \sum_{|w| \le k} \langle c, w \rangle w .$$

Let $c \in S$ and $\varepsilon > 0$ be given. Choose k such that (14) holds for all $d \in S$ and $v \in \mathcal{V}_T(r)$ with ε replaced by $\varepsilon/4$. Then,

$$\begin{aligned}|F_c[v](t) - F_d[v](t)| &\le |F_{c_k}[v](t) - F_{d_k}[v](t)| + \varepsilon/2 \\ &= |F_{c_k - d_k}[v](t)| + \varepsilon/2 \le \sum_{|w| \le k} |\langle c - d, w \rangle V_w[v](t)| + \varepsilon/2.\end{aligned}$$

Let

$$R = \max_{0 \le j \le k} \left\{ \frac{r^j T^j}{j!} \right\}.$$

It follows from (13) that $\|V_w[v]\|_\infty \le R$ for all $v \in \mathcal{V}_T(r)$ and for all w with $|w| \le k$. Hence, there exists some $\delta > 0$ such that for any d satisfying $|\langle d, w \rangle - \langle c, w \rangle| < \delta$ for all $|w| \le k$,

$$\sum_{|w| \le k} |\langle c - d, w \rangle V_w[v](t)| < \varepsilon/2.$$

This means that there exists some neighborhood $\mathcal{N}$ of c such that for any $d \in \mathcal{N}$,

$$|F_c[v](t) - F_d[v](t)| < \varepsilon.$$

This shows that the map given in (12) is equi-continuous. □

Proof of Lemma 1. Let $\tilde{\mu} = (\tilde{\mu}_0, \ldots, \tilde{\mu}_{l-1}) \in \mathcal{J}_S^l$, and let $v \in C^\infty[0,1]$ be given by

$$v(t) = \sum_{i=0}^{l-1} \tilde{\mu}_i \frac{t^i}{i!}, \qquad 0 \le t \le 1.$$

Let $r = 2\|v\|_\infty$. Without loss of generality, we assume that $r \ge 1$. Choose $0 < T < 1$ such that (T, r) is admissible for every $d \in S$.

By the assumption on $\tilde{\mu}$, it follows that $v \in \Omega_S^T$, where

$$\Omega_S^T := \{u \in C^\infty[0,T] : \ d \not\sim_u 0, \ \forall d \in S\}.$$

Hence, for any $c \in S$, there exists some $t_c \in [0,T]$ such that

$$|F_c[v](t_c)| = \tau_c > 0.$$

By the continuity property (c.f. Lemma 2), there exists a neighborhood $\mathcal{N}_c$ of c such that for any $d \in \mathcal{N}_c \cap S$,

$$|F_d[v](t_c)| \geq \tau_c/2\,.$$

Since S is compact, there exist $c_1,\, c_2, \ldots, c_n$ such that $S \subseteq \bigcup_{i=1}^n \mathcal{N}_{c_i}$. It then follows that for any $d \in S$, there exists some $1 \leq j \leq n$ such that

$$|F_d[v](t_j)| \geq \tau_{c_j}/2\,, \tag{15}$$

where $t_j = t_{c_j}$.

Let $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1}) \in \mathbb{R}^{mk}$ be given. Let $\{\omega_j\}$ be a sequence of analytic functions defined on $[0, T]$ such that

- $\omega_j^{(i)}(0) = \mu_i$ for $0 \leq i \leq k-1$, $j \geq 1$;
- $\omega_j \to v$ in the L_1 norm (as functions defined on $[0, T]$); and
- for some $M \geq 1$, $\|\omega_j\|_\infty \leq M$ for all $j \geq 1$.

(See Lemma A.3 in [21] for the existence of such sequences.) Reducing the value of T if necessary, one may assume that (T, M) is admissible for all $d \in S$.

Again, as it follows from the continuity property established in Lemma 2, one sees that for some n_0 large enough,

$$|F_d[\omega_{n_0}](t) - F_d[v](t)| \leq \tau/4 \quad \forall\, t \in [0,\, T],\ \forall\, d \in S, \tag{16}$$

where $\tau = \min\{\tau_{c_1}, \tau_{c_2},\, \ldots,\, \tau_{c_n}\}$. It follows from (15) and (16) that for each $d \in S$, there exists some $j > 0$ such that

$$|F_d[\omega_{n_0}](t_j)| \geq \tau/4 > 0\,,$$

from which it follows that $\omega_{n_0} \in \Omega_S^T$. As ω_{n_0} is analytic, it follows that $F_d[\omega_{n_0}]$ is also analytic (see Lemma 2.3 of [19]). This then implies that for any $d \in S$, there exists some $j_d \geq 1$ such that $y_d^{(j_d-1)}(0) \neq 0$, where $y_d(t) = F_d[\omega_{n_0}](t)$, and hence,

$$\langle d,\ \mathfrak{d}_{j_d}(\omega(0),\, \ldots,\, \omega^{(j_d-1)}(0))\rangle_{j_d} \neq 0\,,$$

where for simplicity, we have replaced ω_{n_0} by ω. Note then that this is equivalent to

$$\langle d_{j_d},\ \mathfrak{d}_{j_d}(\omega(0),\, \ldots,\, \omega^{(j_d-1)}(0))\rangle_{j_d} \neq 0\,.$$

Thus, for any $d \in S$, there exists a neighborhood $\mathcal{W}_d$ of d such that for any $\tilde{d} \in \mathcal{W}_d$,

$$\langle \tilde{d}_{j_d},\ \mathfrak{d}_{j_d}(\omega(0),\, \ldots,\, \omega^{(j_d-1)}(0))\rangle_{j_d} \neq 0\,,$$

and consequently,

$$\langle \tilde{d},\ \mathfrak{d}_{j_d}(\omega(0),\, \ldots,\, \omega^{(j_d-1)}(0))\rangle_{j_d} \neq 0\,.$$

Again, by compactness of S, there exists some $K \geq 1$ such that

$$\langle d,\ \mathfrak{d}_K(\omega(0),\, \ldots,\, \omega^{(K-1)}(0))\rangle_K \neq 0\,,$$

for any $d \in S$. Without loss of generality, one may assume that $K \geq k$. Let $\nu \in \mathbb{R}^{mK}$ be given by $\nu_i = \omega^{(i)}(0)$. Then $\nu \in \mathcal{J}_S^K$, and by the choice of $\{\omega_j\}$, ν is a finite extension of μ. □

4.2 Universal Jets for Equi-Convergent Families

For each element $w_0 \in \Theta^*$, and each integer $k > 0$, let $\mathfrak{C}_{w_0,k}$ be the set of all series satisfying:

$$|\langle c,\ w_0\rangle| \geq \frac{1}{k}\,, \tag{17}$$

and

$$|\langle c,\ w\rangle| \leq k^{n+1}n!\,, \qquad \forall |w| = n\,. \tag{18}$$

Clearly, each $\mathfrak{C}_{w,k}$ is compact, equi-convergent, and $d \neq 0$ for any $d \in \mathfrak{C}_{w,k}$. Moreover, it is easy to see that

$$\mathfrak{C} \setminus \{0\} = \bigcup_{w\in\Theta^*,k\geq 1} \mathfrak{C}_{w,k}\,. \tag{19}$$

We now let, for each $w,\ k$, and $T > 0$,

$$\Omega^T_{w,k} = \{u \in C^\infty[0,T] :\ c \not\sim_u 0,\ \forall c \in \mathfrak{C}_{w,k}\}\,.$$

Then it follows from (19) that

$$\Omega^T = \bigcap_{w,k} \Omega^T_{w,k}\,.$$

For a set S of convergent series, we define

$$\mathcal{J}_S = \{\mu \in \mathbb{R}^{\mathrm{m},\infty} :\ \langle d,\ \mathfrak{d}(\mu)\rangle \neq 0,\ \forall d \in S\}\,,$$

and we denote $\mathcal{J}_{\mathfrak{C}_{w,k}}$ by $\mathcal{J}_{w,k}$. Again, by (19), we have

$$\mathcal{J} = \bigcap_{w,k} \mathcal{J}_{w,k}\,.$$

Thus, to prove Theorem 2, it is enough to show that $\mathcal{J}_{w,k}$ is open dense in $\mathbb{R}^{\mathrm{m},\infty}$.

Lemma 4. *Let S be an equi-convergent and compact family so that $0 \notin S$. Then $\mathcal{J}_S$ is open and dense in $\mathbb{R}^{\mathrm{m},\infty}$.*

To prove Lemma 4, we first prove the following result which is stronger than Lemma 1 in that it is no longer a prior requirement that $\mathcal{J}^l_S \neq \emptyset$ for some l.

Lemma 5. *Let S be an equi-convergent and compact family so that $0 \notin S$. Then for any $j \geq 1$ and $\mu^j = (\mu_0, \ldots, \mu_{j-1}) \in \mathbb{R}^{mj}$, there exists a finite extension ν^k of μ^j such that $\nu^k \in \mathcal{J}^k_S$.*

Proof. Let $\mu_j \in \mathbb{R}^{mj}$ be given. Consider any fixed $c \in S$, $c \neq 0$. According to [17, Theorem 1] (see also Lemma A.4 in [21]), there are always some $l \geq j$ and finite extension $\nu_c \in \mathbb{R}^{ml}$ of μ^j such that

$$\langle c,\ \mathfrak{d}_l(\nu_c)\rangle_l \neq 0\,.$$

From here it follows that there exists some neighborhood $\mathcal{N}_c$ of c such that

$$\langle d,\ \mathfrak{d}_l(\nu_c)\rangle_l \neq 0\,,$$

for all $d \in \mathcal{N}_c \cap S$. Since S is Hausdorff and compact, one may assume that $\mathcal{N}_c$ is compact. Applying this argument for each c in S, and using compactness of S, one concludes that there are finitely many $c_1, c_2, \ldots c_n$ such that S is covered by $\cup_{i=1}^n \mathcal{N}_{c_i}$. Write $\mathcal{N}_{c_i} \cap S$ as $\mathcal{N}_i$. Then on each $\mathcal{N}_i$, there exists some finite extension $\nu_{c_i} \in \mathbb{R}^{ml_i}$ of μ^j such that

$$\langle d,\ \mathfrak{d}_{l_i}(\nu_{c_i})\rangle_{l_i} \neq 0\,,$$

for all $d \in \mathcal{N}_i$. In particular, note that, for each i, $\mathcal{N}_i$ is compact and $\mathcal{J}_{\mathcal{N}_i}^{l_i} \neq \emptyset$, so Lemma 1 can be applied to each such $\mathcal{N}_i$. We do this next, inductively.

Start by defining $s_1 = l_1$ and σ_1 as just ν_{c_1}. Then $\sigma_1 \in \mathbb{R}^{ms_1}$ is a finite extension of μ^j and $\sigma_1 \in \mathcal{J}_{\mathcal{N}_1}^{s_1}$. Consider $\mathcal{N}_2$. By Lemma 1, there exists some $s_2 \geq s_1$ and some finite extension σ_2 of σ_1 such that $\sigma_2 \in \mathcal{J}_{\mathcal{N}_2}^{s_2}$. Since σ_2 is an extension of σ_1, it follows that σ_2 is also in $\mathcal{J}_{\mathcal{N}_1}^{s_2}$, and it is also a finite extension of μ^j. Repeating finitely many times, one concludes that there exists some finite extension $\sigma_n \in \mathbb{R}^{ms_n}$ of μ^j such that $\sigma_n \in \mathcal{J}_{\mathcal{N}_i}^{s_n}$ for all $1 \leq i \leq n$. Hence, $\sigma_n \in \mathcal{J}_S^{s_n}$. □

Proof of Lemma 4. Let S be an equi-convergent family so that $0 \notin S$. We first prove the density property of $\mathcal{J}_S$. Pick up any $\mu = (\mu_0, \mu_1, \ldots) \in \mathbb{R}^{\mathrm{m},\infty}$. Let W be a neighborhood of μ (in the product topology). Without loss of generality, one may assume that

$$W = W_0 \times W_1 \times \cdots \times W_{j-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots\,,$$

where W_i is an open subset of $\mathbb{R}^m$ for $0 \leq i \leq j-1$. By Lemma 5, there exists some finite extension ν^N of $\mu^j := (\mu_0, \ldots, \mu_{j-1})$ such that $\nu^N \in \mathcal{J}_S^N$. Note that every extension ν of ν^N is in $\mathcal{J}_S$ as well as in W since it is also an extension of μ^j. Hence, $W \bigcap \mathcal{J}_S \neq \emptyset$.

We now prove the openness property of $\mathcal{J}_S$. Pick $\mu = (\mu_0, \mu_1, \ldots) \in \mathcal{J}_S$. Then for each $c \in S$, there exists some $k \geq 0$ such that

$$\langle c,\ \mathfrak{d}_k(\mu)\rangle_k \neq 0\,. \tag{20}$$

By compactness of S, one can assume that k does not depend on c. Note that (20) involves only finitely many terms, so there are neighborhoods $\mathcal{N}_c$ of $c \in S$ and U_{c,μ^k} of μ^k in $\mathbb{R}^{mk}$ (where $\mu^k = (\mu_0, \ldots, \mu_{k-1})$) such that

$$\langle d,\ \mathfrak{d}_k(\nu)\rangle_k \neq 0$$

for all $d \in \mathcal{N}_c$ and all $\nu \in U_{c,\mu^k}$. Again, using compactness, one can show that there are finitely many $U_{c_1,\mu^k}, \ldots, U_{c_n,\mu^k}$, each of which is open, so that $S \subseteq \bigcup_{i=1}^n \mathcal{N}_{c_i}$, and $U_{c_i,\mu^k} \subseteq \mathcal{J}^k_{\mathcal{N}_{c_i}}$. Let

$$U_{\mu^k} = \bigcap_{i=1}^n U_{c_i,\mu^k}.$$

Then U_{μ^k} is a neighborhood of μ^k in $\mathbb{R}^{mk}$. Since $U_{\mu^k} \subseteq \mathcal{J}^k_{\mathcal{N}_{c_i}}$ for all $1 \le i \le n$, it follows that $U_{\mu^k} \subseteq \mathcal{J}^k_S$. Finally, let $U = U_{\mu^k} \times \mathrm{I\!R}^{\mathrm{m},\infty}$. Then U is an open set containing μ. Furthermore, for any $\nu \in U$, the restriction ν^k of ν is in U_{μ^k}, and therefore, $\nu \in \mathcal{J}_S$. This shows that $U \subseteq \mathcal{J}_S$ and μ is an interior point of $\mathcal{J}_S$. □

4.3 Universal Inputs for Equi-Convergent Families

As discussed in Section 4.2, to prove Theorem 1, it is enough to show the following.

Lemma 6. *Let S be an equi-convergent and compact family so that $0 \notin S$. Then, for any $T > 0$, the set Ω_S^T is open and dense in $C^\infty[0,T]$.*

First of all, we make the following observation.

Remark 1. Suppose that $\Omega_S^{T_0}$ is open and dense in $C^\infty[0,T_0]$ for some T_0, then Ω_S^T is open and dense in $C^\infty[0,T]$ for every $T > T_0$. This can be shown in details as follows.

For each subset U of $C^\infty[0,T]$, let $U_{T_0} = \{v_{T_0} : v \in U\}$, where for $v \in C^\infty[0,T]$, v_{T_0} denotes the restriction of v to the interval $[0,T_0]$. Suppose U is open in $C^\infty[0,T]$, then U_{T_0} is open in $C^\infty[0,T_0]$, and every $u \in U_{T_0}$ can be smoothly extended to a function $\tilde{u} \in U$. Moreover, if $u \in \Omega_S^{T_0}$, then $\tilde{u} \in \Omega_S^T$. Hence, if $\Omega_S^{T_0} \bigcap U_{T_0} \neq \emptyset$, then $\Omega_S^T \bigcap U \neq \emptyset$. This shows the density property of Ω_S^T.

To show the openness property of Ω_S^T, let $u \in \Omega_S^T$. By definition, for any $c \in S$, there exists some $t_c \in [0,T_0]$ such that $F_c[u](t_c) \neq 0$, so $u_{T_0} \in \Omega_S^{T_0}$. By openness of $\Omega_S^{T_0}$, there is a neighborhood U of u_{T_0} in $C^\infty[0,T_0]$ such that $u_{T_0} \in U \subseteq \Omega_S^{T_0}$. Let

$$\tilde{U} = \{v \in C^\infty[0,T] :\ v_{T_0} \in U\}.$$

Then $\tilde{U}$ is a neighborhood of u in $C^\infty[0,T]$, and $\tilde{U} \subseteq \Omega_S^T$. This shows that every u in Ω_S^T is an interior element of Ω_S^T. ◁

Proof of Lemma 6. Assume that S is equi-convergent and compact. Let $T > 0$ be given. We first prove the density property of Ω_S^T. By Remark 1, one may assume that $T < 1/2$. Let $u \in C^\infty[0,T]$, and pick a neighborhood $\mathcal{W}$ of u. Again, without loss of generality, we may assume that

$$\mathcal{W} = \left\{ v \in C^\infty[0,T] : \left\| v^{(i)} - u^{(i)} \right\|_\infty < \delta,\ 0 \le i \le j-1 \right\}$$

for some $j \ge 1$ and some $\delta > 0$. Let $\mu = (\mu_0, \mu_1, \ldots)$ be given by $\mu_i = u^{(i)}(0)$. By Lemma 5, there exists some $K > j$ and a finite extension ν^K of μ^j such that $\nu^K \in \mathcal{J}_S^K$. By Lemma A.4 in [21], one sees that there exists some analytic function w_j such that $w_j^{(i)}(0) = \nu_{j+i} - \mu_{j+i}$ for $i = 0, \ldots, K-j-1$, and $\|w_j\|_{L_1} < \delta$. One then defines w_l inductively for $l = j-1, \ldots, 1, 0$ by

$$w_l(t) = \int_0^t w_{l+1}(s)\, ds.$$

It then can be seen that $w_{l+1}(t) = w_l'(t)$, $w_l(0) = 0$, and $\|w_l\|_\infty < \delta$ for $0 \le l \le j-1$. Consequently, $w_0 \in C^\infty[0,T]$ is a function such that $w_0^{(i)}(0) = 0$ for $0 \le i \le j-1$, and $w_0^{(i)}(0) = \nu_i - \mu_i$ for $j \le i \le K-1$, and $\left\| w_0^{(i)} \right\|_\infty < \delta$ for all $0 \le i \le j-1$.

Let $w(t) = u(t) + w_0(t)$. Then $w \in \mathcal{W}$. Also note that $w^{(i)}(0) = \nu_i$ for $0 \le i \le K-1$. Since $\nu^K \in \mathcal{J}_S^K$, it follows that $w \in \Omega_S^T$. This proves the density property of Ω_S^T.

Next we show the openness property of Ω_S^T. Let $u \in \Omega_S^T$. Again, by Remark 1, we may assume that (T, r) is admissible for every $c \in S$, where $r = \max\{\|u\|_\infty,\ 1\}$, and that $T < T_1$, where T_1 is defined as in Lemma 3. Since S is compact, there exists some $\delta > 0$ such that $\|F_c[u]\|_\infty \ge \delta$ for all $c \in S$. Observe that Lemma 3 still holds when $\mathcal{V}_T(r)$ is endowed with the Whitney topology. Hence, for each $c \in S$, there exist a neighborhood $\mathcal{N}_c$ of c and a neighborhood $U_c \subseteq \mathcal{V}_T(r)$ of u such that

$$\|F_c[v]\|_\infty > \delta/2$$

for all $c \in \mathcal{N}_c$, $v \in U_c$. By compactness of S, there are finitely many $c_1, c_2, \ldots, c_L$ such that $S \subseteq \bigcup_{i=1}^L \mathcal{N}_{c_i}$. Let $U = \bigcap_{i=1}^L U_{c_i}$. Then U is a neighborhood of u, and for each $v \in U$, $\|F_c[v]\|_\infty > \delta/2$ for all $c \in S$. It follows that $U \subseteq \Omega_S^T$. □

5 State Space Systems

Consider an *analytic system*

$$\Sigma : \quad \begin{cases} x'(t) = g_0(x(t)) + \sum_{i=1}^m g_i(x(t)) u_i(t)\,, \\ y(t) \ = h(x(t))\,, \end{cases} \tag{21}$$

where for each t, $x(t) \in \mathcal{M}$, which is an analytic (second countable) manifold of dimension n, $h : \mathcal{M} \longrightarrow \mathbb{R}$ is an analytic function, and $g_0, g_1, \ldots, g_m$ are analytic vector fields defined on $\mathcal{M}$. Inputs are measurable essentially bounded maps $u : [0, T] \longrightarrow \mathbb{R}^m$ defined on $[0, T]$ for suitable choices of $T > 0$. In general, $\varphi(t, x, u)$ denotes the state trajectory of (21) corresponding to an input u and initial state x, defined at least for small t.

Fix any two states p, $q \in \mathcal{M}$ and take an input u. We say p and q are *distinguished by* u, denoted by $p \not\sim_u q$, if $h(\varphi(\,\cdot\,, p, u)) \neq h(\varphi(\,\cdot\,, q, u))$ (considered as functions defined on the common domain of $\varphi(\,\cdot\,, p, u)$ and $\varphi(\,\cdot\,, q, u)$); otherwise we say p and q cannot be distinguished by u, denoted by $p \sim_u q$. If p and q cannot be distinguished by *any* input u, then we say p and q are *indistinguishable*, denoted by $p \sim q$. If for any two states, $p \sim q$ implies $p = q$, then we say that system (21) is *observable*. (See [6] and [13].) See also [9] for other related notions as well as detailed concept of *generic local observability.*

For a given continuous time system Σ, let $\mathcal{F}$ be the subspace of functions $\mathcal{M} \longrightarrow \mathbb{R}$ spanned by the Lie derivatives of h in the directions of $g_0, g_1, \ldots, g_m$, i.e.,

$$\mathcal{F} := \operatorname{span}_{\mathbb{R}} \left\{ L_{g_{i_1}} L_{g_{i_2}} \cdots L_{g_{i_l}} h :\ l \geq 0,\ 0 \leq i_j \leq m \right\}. \tag{22}$$

This is the *observation space* associated to (21); see e.g. [13, Remark 5.4.2].

Now for any $\mu = (\mu_0, \mu_1, \ldots)$ in $\mathrm{I\!R}^{\mathrm{m},\infty}$, we define

$$\psi_i(x, \mu) = \left.\frac{d^i}{dt^i}\right|_{t=0} h(\varphi(t, x, u)) \tag{23}$$

for $i \geq 0$, where u is any C^∞ input with initial values $u^{(j)}(0) = \mu_j$. The functions $\psi_i(x, \mu)$ can be expressed, – applying repeatedly the chain rule, – as polynomials in the $\mu_j = (\mu_{1j}, \ldots, \mu_{mj})$ whose coefficients are analytic functions.

For each fixed $\mu \in \mathrm{I\!R}^{\mathrm{m},\infty}$, let $\mathcal{F}_\mu$ be the subspace of functions from $\mathcal{M}$ to $\mathbb{R}$ defined by

$$\mathcal{F}_\mu = \operatorname{span}_{\mathbb{R}} \left\{ \psi_0(\,\cdot\,, \mu),\ \psi_1(\,\cdot\,, \mu),\ \psi_2(\,\cdot\,, \mu),\ \ldots \right\}, \tag{24}$$

and let $\mathcal{F}_\mu(x)$ be the space obtained by evaluating the elements of $\mathcal{F}_\mu$ at x for each $x \in \mathcal{M}$.

For system (21), we consider the series c_p, for each $p \in \mathcal{M}$, defined by

$$\langle c_p,\ X_{i_1} X_{i_2} \cdots X_{i_l} \rangle\ =\ L_{g_{i_l}} \cdots L_{g_{i_2}} L_{g_{i_1}} h(p). \tag{25}$$

According to [15, Lemma 4.2], this is always a convergent series. Note then that $p \not\sim q$ if and only if $c_p \neq c_q$ (see [6, 17]). Also, for each $i \geq 0$, it holds that

$$\psi_i(p, \mu) = \langle c_p,\ \mathfrak{d}_i(\mu_0, \ldots, \mu_{i-1}) \rangle,$$

where $\mathfrak{d}_i$ is still the same as defined in (7). For each $\mu \in \mathrm{I\!R}^{\mathrm{m},\infty}$, we denote

$$\Psi_\mu(p) \;=\; (\psi_0(p,\mu),\, \psi_1(p,\mu),\, \psi_2(p,\mu), \ldots)\,, \quad p \in \mathcal{M}\,.$$

Consider the set

$$\mathcal{J}_\Sigma := \{\mu \in \mathrm{I\!R}^{\mathrm{m},\infty} : \quad \Psi_\mu(p) \neq \Psi_\mu(q),\ \forall p \not\sim q\},$$

and the set

$$\mathfrak{J} := \bigcap_\Sigma \mathcal{J}_\Sigma,$$

where the intersection is taken over the collection of all analytic systems with m inputs as in (21). Clearly, $\mathfrak{J} \supseteq \mathcal{J}$, and hence, the following is an immediate consequence of Theorem 2.

Corollary 1. *The set* $\mathfrak{J}$ *is a generic subset of* $\mathrm{I\!R}^{\mathrm{m},\infty}$.

Using Corollary 1, one recovers the existence of universal inputs for analytic systems previously established in [14], but in a stronger form, uniformly on all state space systems of all dimensions with input functions taking values in $\mathbb{R}^m$.

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