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REMARKS ON PIECEWISE-LINEAR ALGEBRA

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This note studies some of the basic properties of the category whose objects are finite unions of (open and closed) polyhedra and whose morphisms are (not necessarily continuous) piecewise-linear maps.

Introduction. A function $f\colon V \to W$ between real vector spaces is *piecewise-linear* (PL) if there exists a partition of V into "open polyhedra" X_i (i.e., relative interiors of polyhedra) such that f is affine on each X_i. (As distinct to the case of PL-topology, no continuity is required of f.) Images and preimages under PL-maps give rise to finite unions of open polyhedra, or PL-*sets*; conversely PL functions can be characterized by the fact that their graphs are PL-sets. This paper studies some basic algebraic properties of the category $\boldsymbol{PL}$, proving in particular that it is an exact category, and in fact a pretopos. A classification is given for the isomorphism classes of objects of $\boldsymbol{PL}$, in terms of a two-generator semiring.

The first section recalls without proof some facts from polyhedral geometry needed in the paper. Except for the setting of a unified notation and for minor generalizations, the material there is well known. The second section defines PL maps and sets, and studies the category. The main results (leading to the classification theorem) are given in the last section.

1. Review of polyhedral geometry. The following conventions and definitions hold throughout. All vector spaces are finite-dimensional spaces over the reals $\boldsymbol{R}$; a *flat* means an affine submanifold of some such space V, and the *closed half-spaces* associated to a linear $f\colon V \to \boldsymbol{R}$ and an r in $\boldsymbol{R}$ (or associated to the hyperplane $\{x \mid f(x) = r\}$,) are the sets $\{x \mid f(x) \leqq r\}$ and $\{x \mid f(x) \geqq r\}$. The corresponding *open half-spaces* are obtained by using strict inequalities in the above. A *half-line* (closed or open) is the intersection of a line L in V with a (closed or open) half-space not containing L.

A (convex) closed *polyhedron* in V is by definition an intersection of finitely many closed half-spaces. The *dimension* of a nonempty polyhedron P is the dimension of aff(P), the smallest flat containing P; the *relative interior* ri(P) is the interior of P relative to the usual topology on aff(P). An *open polyhedron* P is by definition the relative interior of some closed polyhedron $c\ell(P)$ ($c\ell$ denoting usual topological closure); this is equivalent to P being an intersection of finitely many flats and open half-spaces. A (closed or open) polytope is a polyhedron which is bounded (=contains no half-lines). The *relative boundary* rb (P) is the set-theoretic difference $cl(P) \backslash \mathrm{ri}\,(P)$.

We review now a few other notions and results which are needed later; for proofs the reader is referred to Rockaffelar [7], especially §§8, 9, 17, 18, and 19, and to Grünbaum [5]. (The terminology "polyhedron" is usually reserved for closed ones and results are stated for these; results for "open polyhedra" are implicit in results about relative interiors. In our case, open polyhedra will be more relevant.)

A *proper* ("exposed") *face* of a closed polyhedron P (or, more generally, of a convex set,) is the intersection of P with a nontrivial supporting hyperplane H (i.e., an H such that H intersects P and P is contained entirely in one of the half-spaces associated to H). A *face* of P is either P itself, the empty set, or a proper face. A zero-dimensional, one-dimensional, or maximal proper face is called respectively a *vertex* (or *extreme point*), *edge*, or *facet*. An *extreme direction* (or "extreme point at infinity") is the direction of a half-line contained in some edge (by *direction* one means a translation-equivalence class of half-lines dir (y) = family of sets $\{x + \lambda y,\ \lambda \geqq 0\}$, x in V, for a nonzero vector y).

If P is an irredundant intersection of half-spaces each associated to a hyperplane H_i, the facets of P are precisely the intersections of P with the various H_i. The facets of facets of P are intersections of facets of P, and every face of P is an intersection of facets. The set $\boldsymbol{F}(P)$ of faces of P, ordered by inclusion, is a complete lattice, the meet being the intersection. The following result (see e.g., Rockaffelar [7, Theorem 18.2]) will be useful.

THEOREM 1.1. *The family of relative interiors* ri (F), *F in $\boldsymbol{F}(P)$, gives a partition of P.*

Note that the ri (F) for F proper give a partition of rb (P).

A face of an *open* polyhedron P is by definition the same thing as a face of $cl(P)$.

A (closed or open) polyhedron P *recedes in direction* $d = \mathrm{dir}\,(y)$ if $x + \lambda y$ is in P whenever x is in P and λ is nonnegative. This is *equivalent* to requiring that there exist at least one x in P with $\{x + \lambda y, \lambda \geqq 0\}$ contained in P. The *recession cone* 0^+P is the set consisting of zero and of all *vectors* y such that dir (y) is a direction of recession of P, or equivalently, the set of y with $P + y \subseteqq P$. (Here $C + y$, or more generally $A + B$, indicates as usual the set of all sums.) For any P, it holds that $0^+P = 0^+(\mathrm{ri}\,(P))$.

The (closed) *convex hull* conv (S) of a finite subset S of V is the smallest convex subset of V containing S; it is the closed polyhedron consisting of all convex combinations $\sum \lambda_i s_i$, $\sum \lambda_i = 1$, $\lambda_i \geqq 0$, of elements of S. The *open convex hull* opconv (S) is obtained by restricting to $\lambda_i > 0$ in the above; this is the same as ri (conv (S)). If D is a set of directions of V, ray (D) is the union of zero and the set of all vectors y whose directions belong to D. For a finite set of points S and directions D, conv (S, D) is the smallest convex set containing S and receding in all directions of D, i.e., conv $(S+\text{ray}(D))$. The latter is also equal to conv (S) + cone (D), where cone (D) = conv (ray (D)); thus conv (S, D) is the set of all sums $\sum \lambda_i x_i + \sum \mu_j y_j$, where the x_i are in S, the y_j are arbitrary vectors with directions in D, and the λ_i, μ_j are nonnegative with $\sum \lambda_i = 1$. The corresponding open convex hull opconv (S, D) = ri (conv (S, D)) is obtained by restricting the λ_i, μ_j to be strictly positive. One of the basic results on polyhedra, due to Minkowski and Weyl, can now be stated (see Rockaffelar [7, Theorem 19.1]):

THEOREM 1.2. *The following are equivalent*:
(a) *P is a (closed) polyhedron*;
(b) *P is a closed convex set with finitely many faces*;
(c) *P is finitely generated*, i.e., $P = \text{conv}(S, D)$ *for some finite* S, D.
Further, if P is a line-free polyhedron, then S above [resp., D] *can be chosen as the set of extreme points* [resp., *extreme directions*,] *of* P.

Polyhedral sets are preserved by images and preimages under linear transformations; this is clear from (c) above. But the larger class of *projective transformations* is also compatible with the polyhedral structure. A projective transformation is obtained by embedding the given affine space into a projective space and then restricting to a different affine open set. In local coordinates, say for $V = \boldsymbol{R}^n$, this process induces a partial map of the form

$$x \longmapsto (\langle c, x\rangle + d)^{-1}(Ax + b) ,$$

where A is linear and $\|c\|^2 + d^2 \neq 0$. Grünbaum's book discusses these transformations in detail. Projective transformations preserve colinearity and the polyhedral sets in their domain. Questions about polyhedra can then be reduced to the case of polytopes. For example, consider the nonsingular projective transformation:

$$(1.3) \qquad \alpha\colon x \longmapsto (1 + x_1 + \cdots + x_n)^{-1}(x_1, \cdots, x_n) .$$

This maps the open positive orthant $\boldsymbol{R}^n_+$ of $\boldsymbol{R}^n$ into the following open polytope Δ_n of $\boldsymbol{R}^n$,

$$x_1 + \cdots + x_n < 1 , \quad x_i \geqq 0 , \tag{1.4}$$

in such a way that directions in $\boldsymbol{R}^n_+$ become identified to just *points* in $x_1 + \cdots + x_n = 1$, $x_i \geqq 0$. Open polyhedra P in $\boldsymbol{R}^n_+$ correspond to open polytopes Q contained in Δ_n, extreme points [resp., extreme directions] of P corresponding to extreme points of Q [resp., extreme points of Q lying in $x_1 + \cdots + x_n = 1$].

A (*generalized*) *m-simplex* is the (closed or open) convex hull of $m + 1$ affinely independent points and directions, i.e., of a set $S \cup D$ of cardinality $m + 1$ for which aff (conv (S, D)) has dimension m. Since projective equivalences preserve affine independence in their domains, simplexes correspond to simplexes under (1.3), and the usual (polytope) baricentric subdivision theorem implies, (after if necessary subdividing a given polyhedron into its intersection with orthants,):

LEMMA 1.5. *Every (open or closed) polyhedron is a disjoint union of open simplexes.*

This lemma will be useful in the last section.

2. PL-sets and maps.

DEFINITION 2.1. The PL-*subsets* of a vector space V are those belonging to the Boolean algebra PL (V) generated by all the open (or all the closed) half-spaces of V. A PL-*set* is a PL-subset of some V. A PL-*relation* $R: X \to Y$ is one which is a PL-set as a subset of $X \times Y$; a PL-*map* $f: X \to Y$ is a map which is a PL-relation.

A PL-set is thus the same as a disjoint union of open polyhedra. A number of facts are therefore immediate consequences of those known for polyhedra. For example, images and preimages under linear maps preserve PL-sets, and the product $X \times Y$ of PL-subsets of V, W is a PL-subset of $V \times W$.

LEMMA 2.2. *Let* $R: X_1 \to X_2$ *and* $S: X_2 \to X_3$ *be* PL-*relations, with the* X_i *in* PL (V_i). *Then*

$$R \,\&\, S := \{(x, y, z) \text{ in } X \times Y \times Z \mid x R y \text{ and } y S z\}$$

is a PL-*set.*

Proof. Just note that $R \,\&\, S = (R \times V_3) \cap (V_1 \times S)$. □

Projection of the above on V_1 and V_3 yields:

COROLLARY 2.3. *A composition of* PL-*relations is a* PL-*relation.*

Thus PL-maps are also closed under composition, and we have a well-defined category ***PL***. Note that since PL-maps are defined through their graphs, f^{-1} is a PL-relation for any PL-map f, and therefore *isomorphisms* in ***PL*** are the same as PL-bijections. Before studying general properties of ***PL*** we need several technical facts. A PL-map is a "piecewise linear" map in the following sense.

LEMMA 2.4. *If $f: X \to Y$, X is in* PL(V), *Y is in* PL(W), *and $X = \cup X_i$ is a finite covering such that each restriction $f \mid X_i$ is a* PL-*map, then f is a* PL-*map. Conversely, assume that $f: X \to Y$ is a* PL-*map and let $X = \cup X_i'$ and $Y = \cup Y_j'$ be (finite) partitions into open polyhedra. Then there exist (finite) open polyhedral partitions $X = \cup X_i$ and $Y = \cup Y_j$ which refine the original ones and such that each restriction $f \mid X_i$ is the restriction of a affine map $V \to W$, and maps X_i into some Y_j. Further, if f is one-to-one, then there exist simultaneous open refinements as above $X = \cup X_i$, $f(X) = \cup Y_i$, with $f(X_i) = Y_i$ and each $f \mid X_i$ the restriction of an invertible linear map from* aff(X_i) *onto* aff(Y_i).

Proof. The first assertion is clear from the fact that (the graph of) f is the union of (the graphs of) the $f \mid X_i$.

To prove the remaining assertions, let $X = \cup X_i'$, $Y = \cup Y_j'$, and $G = \cup G_k$ be partitions of X, Y, and $G =$ graph of f, into open polyhedra. Let G_{ijk} be the intersection of G_k, $pr_1^{-1}(X_i')$, and $pr_2^{-1}(Y_j')$, where pr_i denotes projection of $V \times W$ into the ith factor. Each G_{ijk} is an open polyhedron and projects into X_i', Y_j'. Since f is a function, the family of all $pr_1(G_{ijk})$ is a partition of X into open polyhedra X_{ijk}, which refines the original one; note that X_{ijk} maps into Y_j'. When f is one-to-one, the $Y_{ijk} := pr_2(G_{ijk})$ are disjoint, and $f(X_{ijk}) = Y_{ijk}$. The last statement follows from the fact that the projections restricted to aff(G_{ijk}) establish isomorphisms with aff(X_{ijk}), aff(Y_{ijk}), since they are one-to-one on the open subset G_{ijk} of aff(G_{ijk}). □

A PL-map is in fact, up to PL-automorphisms of its domain, a *linear* map, since $X \cong$ graph(f). This fact is itself not too useful, (since the automorphism carries all the nonlinearity,) but is implicit in arguments like the above. (The terminology "PL" should not be confused with the very different notion that appears in combinatorial topology, where all (PL) maps are *continuous* and all polyhedra are closed polytopes.)

A general way of constructing PL-sets is the following. Let L be the first-order language over the alphabet having: constants r for each real number r, variables $x_1, \cdots, x_n, \cdots$, unary function symbols "$r.(\ )$" for each r real, binary function symbol $+$, and relational symbols $>$, $=$.

LEMMA 2.5. *Every sentence S in L defines a* PL-*set,* i.e., *if $x_1, \cdots, x_n$ are the free variables in S then*

$$\text{(2.6)} \qquad \operatorname{Dom}(S) := \{(x_1, \cdots, x_n) \text{ in } \boldsymbol{R}^n \mid S(x_1, \cdots, x_n)\}$$

is a PL-*set. Conversely, any* PL-*subset of an $\boldsymbol{R}^n$ can be defined in this fashion.*

Proof. The converse part is clear, since every open half-space, or hyperplane, can be defined by an equation $a_1x_1 + \cdots + a_nx_n < r$, or $=r$. To prove that Dom (S) is always a PL-set, it is enough to prove that all atomic formulas define PL-sets and that closing under $\neg$, $\wedge$, $\exists$, preserves the PL-structure. But atomic formulas are all linear equations or inequalities, so they define hyperplanes or half-spaces, and they are therefore PL-sets. Closure under negation and conjunction holds by the Boolean closure (complements, intersections). Finally, $\{x \mid (\exists y)S(x, y)\}$ is the projection of $\{(x, y) \mid S(x, y)\}$; it is therefore PL if the latter is. $\square$

Another way of expressing the conclusion of the above lemma is by saying that any set defined using existential or universal quantifiers can be also defined using only propositional connectives. In using the above one usually extends the language (informally) to include sentences containing PL-functions (since these are defined by their graphs, which are PL-sets), and arbitrary PL-sets (since under isomorphism a P in PL (V) is a subset of some $\boldsymbol{R}^n$). Similarly, one can bound quantifiers, as in "for all x in $S, \cdots$", when S is known to be a PL-set.

For example, consider the situation in applications in which one has a family of PL-maps

$$\text{(2.7)} \qquad f_y = f(\cdot, y) \colon X \longrightarrow Z\,, \qquad y \text{ in } Y\,,$$

where $f\colon X \times Y \to Z$ is a fixed PL-map. Here it is natural to ask whether the sets of the y in Y where f is one-to-one, or onto, are PL-sets. In view of 2.5, the answer is (trivially) yes: for instance, f is one-to-one for the y in:

$$\{y \text{ in } Y \mid f(x, y) = f(x', y) \text{ implies } x = x'\}\,,$$

which is expressable as a first-order sentence. Similarly, given a z_0 in Z, the set of common "zeros"

$$\text{(2.8)} \qquad \{x \text{ in } X \mid f_y(x) = z_0 \text{ for all } y\}$$

is a PL-set. Related to this is the set of all x for which there is *some* y solving the equations:

$$(2.9) \qquad X_0 := \{x \text{ in } X \mid \text{there exists } y \text{ with } f(x, y) = z_0\} ,$$

which is again PL. One of the most useful properties of the category ***PL*** is the validity of the following global implicit function theorem:

THEOREM 2.10. *Let* $f: X \times Y \to Z$ *be a* PL-*map*, z_0 *in* Z, *and* X_0 *as in* (2.9). *Then there exists a* PL-*map* $s: X_0 \to Y$ *such that* $f(x, s(x)) = z_0$ *for all* x *in* X_0.

Letting R be the relation "$x R y$ iff $f(x, y) = z_0$", this becomes a consequence of the first part of the following stronger result:

THEOREM 2.11. *Let* $R: X \to Y$ *be a* PL-*relation with domain* X_0. *There exists then a* PL-*map* $s: X_0 \to Y$ *such that*:
(a) $s(x)$ *is in* $R(x)$ *for all* x, *and*
(b) $R(x) = R(x')$ *implies* $s(x) = s(x')$.

Proof. With a slightly different terminology, the problem is that of showing that for each PL-subset R of $V \times W$ there is a section s of pr_1 such that $s(x)$ depends only on $pr_1^{-1}(x)$. Without loss of generality we assume $V = \boldsymbol{R}^n$, $W = \boldsymbol{R}^k$. The result will follow by induction on k once it is established for $k = 1$. Indeed, assume it is true for k and let R be a PL-subset of $\boldsymbol{R}^{n+k+1}$. Let X_1 be the projection of R on the first $n + 1$ coordinates. The result being true for $k = 1$ means that there is an $s_1: X_0 \to \boldsymbol{R}$ with $s_1(x)$ in $R_1(x)$ for all x and $s_1(x) = s_1(x')$ if $R_1(x) = R_1(x')$, where R_1 is just X_1 seen as a relation $\boldsymbol{R}^n \to \boldsymbol{R}$; note that X_0 is the domain both of R and of R_1. By the inductive hypothesis there is also an $s_2: X_1 \to \boldsymbol{R}^k$ satisfying (a), (b). The desired s can be then obtained defining $s(x) := s_2(x, s_1(x))$.

Thus the problem reduces to the case $k = 1$. Further, we may assume that R is an *open polyhedron*. If R is a more general PL-set we can write R as a union of the disjoint open polyhedra $R_1, \cdots, R_t$. If for each of these there exists an $s_i: pr_1(R_i) \to \boldsymbol{R}$, an $s: pr_1(R) \to \boldsymbol{R}$ can be constructed using 2.4 by defining $s := s_i$ on X_i, where X_1 is $pr_1(R_1)$ and X_i is $pr(R_i) \backslash X_{i-1}$ for $i > 1$. This s will satisfy both (a), (b).

We divide the case $R =$ open polyhedron in $\boldsymbol{R}^n \times \boldsymbol{R}$ into four disjoint cases: (i) the vector

$$e_{n+1} := (0, \cdots, 0, 1)$$

is in 0^+R and $-e_{n+1}$ is not, (ii) e_{n+1} is not in 0^+R but $-e_{n+1}$ is, (iii) both $\pm e_{n+1}$ are in 0^+R, and (iv) neither of $\pm e_{n+1}$ is in 0^+R. Let

$$r_1(x) := \inf \{r \mid (x, r) \text{ is in } R\} ,$$
$$r_2(x) := \sup \{r \mid (x, r) \text{ is in } R\}$$

(these may be $\pm\infty$). If (i) holds, we define $s(x) := r_1(x) + 1$; if (ii) holds, $s(x) := r_2(x) - 1$; if (iii) then $s(x) := 0$; and if (iv) then $s(x) := 1/2(r_1(x) + r_2(x))$. Property (b) is trivially true for such a definition of s, since the $r_i(x)$ depend only on the corresponding $R(x)$. With respect to (a), case (i) corresponds to the situation in which $(\{x\}\times \boldsymbol{R}) \cap c\ell(R)$ is always a half-line not bounded "above", thus $(x, r_1(x)) + e_{n+1}$ is in R, since $(x, r_1(x))$ is in $c\ell(R)$ and $0^+R = 0^+(c\ell R)$. The same argument applies to case (ii). Case (iii) corresponds to $(\{x\}\times \boldsymbol{R}) \cap c\ell(R)$ being a finite interval with endpoints $(x, r_1(x))$ and $(x, r_2(x))$; thus $(x, s(x))$ is in $\text{ri}\,(c\ell(R)) = R$. In case (iv) the entire line $\{x\} \times \boldsymbol{R}$ is in R and thus $(x, 0)$ is in R.

It only remains to prove that s is piecewise linear. For this it is enough to show that r_1 and r_2 are PL-maps when finite-valued. We work with r_1; the argument for r_2 being the same. Let $F_1^0, \cdots, F_t^0$ be the relative interiors of the proper faces of R. These cover $rb(R)$, so that $r_1(x)$ is always in a suitable F_j^0 when finite. We drop from the list all those F_j^0 for which $\{x\} \times \boldsymbol{R}$ either doesn't intersect F_j^0 or intersects F_j^0 at more than one point. Note that, for the latter, $r_1(x)$ will be in a face of F_j^0, which is in turn a lower-dimensional F_i^0. Since the projections $pr_1 | F_j^0$ are now all one-to-one, they admit PL-sections (whose graphs are the F_j^0 themselves). If $r_1(x)$ belongs to a certain F_j^0 then also $r_1(x')$ is in the same F_j^0 for any other x' in $pr_1(F_j^0)$: otherwise, there is some $\varepsilon > 0$ such that

$$z' := (1 + \varepsilon)r_1(x) - \varepsilon z$$

is in F_j^0, where z is the unique point in F_j^0 projecting into x', since F_j^0 is relatively open and convex. But then

$$(1 + \varepsilon)^{-1}(z' + \varepsilon r_1(x'))$$

is in $c\ell(R)$, projects into x, and is strictly less than $r_1(x)$, contradicting the definition of the latter. Thus r_1 is a PL-map on each of the (disjoint) $pr_1(F_j^0)$, and is therefore PL itself.

The above remarks help in characterizing the category $\boldsymbol{PL}$. The terminology "subset" will be used in the set-theoretic sense, while "subobject" will be used in the categorical sense. Note that the category of finite sets is equivalent to a full subcategory of $\boldsymbol{PL}$; the subcategory of finite sets of integers (as PL subsets of $\boldsymbol{R}$) gives the representation. By arguments as in **Sets**, (and using the above results,) it is easy to establish that: (i) $\mathbf{0} = \varnothing$ and $\mathbf{1} = \{0\}$ are initial and final objects respectively; (ii) $\mathbf{0}$ is a generator for the category, and $2 = \{0, 1\}$ is a cogenerator; and (iii) equalizers and finite products and coproducts exist (first order constructions). Thus PL is finitely complete with finite coproducts, with the respective (co)limits preserved by the forgetful functor to **Sets**. It is equivalent for a PL-map to be a monomorphism, one-to-one, a coretraction, or an equalizer. Similarly, epimorphism, onto, retraction, and coequalizer are equivalent properties. A theory of congruences exists, as with algebraic theories. A *congruence* on a PL-set X is a PL-relation $R: X \to X$ which is an equivalence relation. Then one can prove that a relation R is a congruence iff $R = \operatorname{Ker} f = \{(x, y) \mid f(x) = f(y)\}$ for some PL-map f with domain R. The induced-homomorphism property holds for congruences. Thus PL is a pretopos (Johnstone [6]).

It follows that $\boldsymbol{PL}$ admits a full, limit-preserving, and finite coproduct-preserving embedding into the geometric topos shv $(\boldsymbol{PL}, J)$ of sheaves for the precanonical (Grothendieck) topology J on $\boldsymbol{PL}$. This embedding is for example useful in system theory, when one studies the category Mach $(\boldsymbol{PL})$ of machines (see Arbib and Manes [1]) over $\boldsymbol{PL}$. The canonical realization functor does not admit a adjoint over $\boldsymbol{PL}$, but adjoints do exist over the larger category of sheaves, since the latter is in particular cartesian closed and has epi/mono factorizations (see Goguen [4]). For the "uniqueness of canonical realizations" results, one can work on the sheaf category and then descend to the (full) subcategory $\boldsymbol{PL}$. In fact, observable and bounded-time reachable realizations over $\boldsymbol{PL}$ are also canonical over the larger category because the embedding preserves finite coproducts and limits. Other applications to system theory of the $\boldsymbol{PL}$ concepts introduced in this paper are given in Sontag [8].

3. The objects of $\boldsymbol{PL}$.

The main purpose of this section is to obtain a classification of the isomorphism classes of PL-sets. This classification is of course not as simple as that for the subcategories of finite sets and of finite-dimensional vector spaces, but is nonetheless easy to understand after introducing the proper algebraic structure. The theory will be developed through a series of technical remarks.

A. Ranks and labels.

For each fixed n, $\boldsymbol{D}_n$ is the set of all finite disjoint families of open polyhedra contained in $\boldsymbol{R}^n$. The following defines an equivalence relation in $\boldsymbol{D}_n$:

$$DE_1D' \quad \text{iff} \quad \cup\{P, P \text{ in } D\} = \cup\{Q, Q \text{ in } D'\}. \tag{3.1}$$

The relation E_0 is defined as follows: DE_0D' is and only if $D =$

$\{P_1, \cdots, P_r\}$ and there is some hyperplane H intersecting P_r such that $D' = \{P_1, \cdots, P_{r-1}, P_r^0, P_r^+, P_r^-\}$, where P_r^0, P_r^+, and P_r^- denote the intersection of P_r with H and with the two half-spaces associated to H.

LEMMA 3.2. *The smallest equivalence relation on* $\boldsymbol{D}_n$ *containing* E_0 *is* E_1. *In fact,* E_1 *is the union, over all* i, j, *of the relations* $E_0^i \circ E_0^{-j}$.

Proof. Since E_1 contains E_0, we need only to prove that, if $D = \{P_1, \cdots, P_r\}$ and $D' = \{Q_1, \cdots, Q_s\}$ have the same union, then there exists a third family D'' of open polyhedra $\{L_1, \cdots, L_t\}$ such that DE_0^iD'' and DE_0^jD'' for some i, j. In other words, D'' must be such that there is a chain $D = D_0, D_1, \cdots, D_i = D''$ with $D_kE_0D_{k+1}$ for all k, and similarly for D'. Let $\{H_\lambda\}$ be a finite family of hyperplanes constructed as follows. For each P in D or in D', pick hyperplanes $K_1, \cdots, K_n$ such that P is the intersection of some of the K_j and of open half-spaces associated to the rest of the K_j; the H_λ are then obtained by considering all the K_j obtained in this way, for P in D and D'.

Let D'' be the family of all those minimally-nonempty intersections T of the H_λ and open half-spaces associated to the H_λ for which T is included in the union of the sets in D (or D'). These intersections are disjoint, by minimality, so D'' is in $\boldsymbol{D}_n$. Note that DE_1D''. To obtain a chain from D to D'', let $D: = \{P_1, \cdots, P_r\}$, and write each P_j as an intersection of the H_λ and corresponding half-spaces. If any of the P_i, say P_r, is not a minimal intersection, there exists an H_λ inducing a proper subdivision P_r^+, P_r^-, P_r^0. Then DE_0D_1, where D_1 is $\{P_1, \cdots, P_{r-1}, P_r^0, P^+, P^-\}$, and the argument can be repeated with D_1. Eventually one reaches a D_i with all P_j minimal (so a member of D''). But every T in D'' appears in D_i, since $D''E_1D_i$ and every T in D'' is included in the union of the P_j in D. So $D_i = D''$. The same argument gives a chain from D' to D''. □

Pairs of integers are naturally ordered by: $(i, j) \leqq (i', j')$ whenever $i \leqq i'$ and $j \leqq j'$. With this ordering:

DEFINITION 3.3. The *p-rank of a* (nonempty) *open polyhedron* P is

$$p\text{-rank}\,(P): = (\dim P, \dim 0^+P)\ .$$

The *rank* $r(X)$ *of a nonempty* PL-*set* X is defined as the maximal possible p-rank of a polyhedral subobject of X when there is a unique such maximal rank.

DEFINITION 3.4. The (open or closed) polyhedron P in $\boldsymbol{R}^n$ is *acute* if it is included in the orthant $\boldsymbol{R}^n_+$.

The above subclass of polyhedra is introduced for purely technical purposes, mainly because the projective equivalence α introduced in (1.3) will permit reducing many questions about acute polyhedra to questions about polytopes (even though α is not a PL-map).

LEMMA 3.5. *Let P be an acute open polyhedron in $\boldsymbol{R}^n$ of p-rank (n, m), with $n \neq 0$, and H be a hyperplane intersecting P. Let P^0, P^+, and P^- be the intersections of P with H and its associated open half-spaces, and let (n^0, n^0), (n^+, n^+), and (n^-, m^-) denote the respective p-ranks. Then exactly one of the following possibilities holds*:

(a) $(n^+, m^+) = (n^-, m^-) = (n, m)$ *and* $(n^0, m^0) = (n-1, m)$;

(b) $m \geqq 2$, $(n^+, m^+) = (n^-, m^-) = (n, m)$, *and* $(n^0, m^0) = (n-1, m-1)$;

(c) *for some* $k < m$, $(n^+, m^+) = (n, m)$, $(n^-, m^-) = (n, k)$, *and* $(n^0, m^0) = (n-1, k)$;

(d) *idem to* (c), *with* $+$, $-$ *reversed.*

Proof. We first note that P^+, P^- (and P^0) are all nonempty. This is because P is open in $\boldsymbol{R}^n$ and H is the boundary of its associated half-spaces. In fact, P^+ and P^- are again open, so also of dimension n, while P^0 is relatively open in H and hence of dimension $n-1$. To understand the dimensions "at infinity" m^0, m^+, m^-, we consider the projective equivalence $\alpha\colon \boldsymbol{R}^n \to \Delta_n = \Delta$. Let Q, Q^+, Q^-, Q^0, K be the images of P, P^+, P^-, P^0, and $H \cap \boldsymbol{R}^n_+$ under this transformation, denoting also by K the span of the above K. Let K_∞ denote the hyperplane

$$x_1 + \cdots + x_n = 1 .$$

Since directions of recession of acute polyhedra correspond to points in K_∞ with all $x_i \geqq 0$, one has for each of $A := Q, Q^0, Q^+, Q^-$ and $A' := P, P^0, P^+, P^-$ (and defining the dimension of the empty set as -1):

$$\dim 0^+A' = 1 + \dim(c\ell(A) \cap K_\infty) . \tag{3.6}$$

There are then four cases to consider:

(i) $c\ell(Q) \cap K_\infty = \emptyset$. Here $c\ell(A) \cap K_\infty$ is empty for all of the above A, and hence case (a) holds with $m = 0$.

(ii) $c\ell(Q) \cap K_\infty \neq \emptyset$ but $K \cap c\ell(Q) \cap K_\infty = \emptyset$. In this case, $c\ell(Q^0) \cap K_\infty = c\ell(K \cap Q \cap K_\infty) = K \cap c\ell(Q) \cap K_\infty = \emptyset$, so one of $c\ell(Q^+) \cap K_\infty$ or $c\ell(Q^-) \cap K_\infty$ must also be empty (otherwise a line in

K_∞ joining points in both of the latter would intersect $c\ell(Q^0) \cap K_\infty$). If $c\ell(Q^-) \cap K_\infty$ is empty, then $c\ell(Q^+) \cap K_\infty = c\ell(Q) \cap K_\infty$, and (c) holds with $k = 0$. Otherwise, (d) holds with $k = 0$.

(iii) $K \cap c\ell(Q) \cap K_\infty \neq \emptyset$ and $c\ell(Q) \cap K_\infty$ is contained in $K \cap K_\infty$. In this case all $c\ell(A) \cap K_\infty$ are equal, so (a) holds.

(iv) As above but with $c\ell(Q) \cap K_\infty$ not contained in $K \cap K_\infty$. Necessarily $c\ell(Q) \cap K_\infty$ has more than one point so (since a polytope) dimension at least one, and hence $m \geqq 2$. Here $K \cap K_\infty$ is a hyperplane of K_∞ intersecting the polytope $c\ell(Q) \cap K_\infty$ at $c\ell(Q^0) \cap K_\infty$. Let k be the dimension of the latter. If $K \cap K_\infty$ intersects $Q \cap K_\infty$ then $k = m$ and (b) holds. If $K \cap K^\infty$ intersects only the boundary then $k < m$ and either $c\ell(Q^0) \cap K_\infty = c\ell(Q^-) \cap K_\infty$ and (c) holds, or $c\ell(Q^0) \cap K_\infty = c\ell(Q^+) \cap K_\infty$ and (d) holds instead. □

DEFINITION 3.7. The *(n, m)-simplex* $s(n, m)$ is the (generalized) open n-simplex $\text{opconv}(S, D)$, where S is the set of points $0, e_1, \cdots, e_{n-m}$ and D is the set of directions of $e_{n-m+1}, \cdots, e_n$ (e_i denotes here the ith canonical vector in $\boldsymbol{R}^n$).

Thus $s(n, m)$ is an acute polyhedron of p-rank (n, m), namely,

$$\{(x_1, \cdots, x_n) \text{ in } \boldsymbol{R}^n \mid \text{all } x_i > 0 \text{ and } x_1 + \cdots + x_{n-m} < 1\}, \tag{3.8}$$

(just a point if $n = 0$) or equivalently, the product $P_{n-m} \times \boldsymbol{R}_+^m$, where P_k is the interior of the standard (bounded) k-simplex and $\boldsymbol{R}_+$ is $(0, \infty)$. Every simplex of p-rank (n, m) is isomorphic to $s(n, m)$ under an invertible linear map between their spans.

LEMMA 3.9. *For open polyhedra the p-rank coincides with the rank.*

Proof. We need to show that if there is a one-to-one PL-map $f\colon Q \to P$ between open polyhedra then $p\text{-rank}(Q) \leqq p\text{-rank}(P)$. For this it will be sufficient to show that the p-rank is invariant under isomorphism, since it is clear from the definition that the p-rank of a subset of P cannot exceed that of P. Assume then that f is bijective.

Consider for each of P, Q, the partitions into open subpolyhedra obtained by intersecting with the various orthants of $\text{aff}(P)$, $\text{aff}(Q)$ and their respective faces (without loss of generality assume $\text{aff}(P) = \boldsymbol{R}^s$, $\text{aff}(Q) = \boldsymbol{R}^t$). By (2.4) there exist refinements into open polyhedral partitions $\{P_i\}$, $\{Q_i\}$ such that f is a linear isomorphism in each Q_i. (By (1.5) we may in fact assume that each element of the partition is a generalized simplex, with corresponding P_i, Q_i linearly isomorphic.) Since p-rank is invariant under linear isomorphism, it will be enough to show that, whenever one has such partitions, p-rank (P) = max. of the p-ranks of the P_i (and the same for Q). If P is acute, this is a consequence of (3.2) and (3.5). In the general case, it will be enough to show that the intersection P' of P with at least one of the open orthants of $\boldsymbol{R}^s$ has the same rank as P. But 0^+P is open in aff (0^+P) and thus there is some orthant, say $\boldsymbol{R}^s_+$ under a linear change of coordinates, such that $c\ell(\boldsymbol{R}^s_+)$ intersects aff (0^+P) and 0^+P. Thus P recedes in dim (0^+P) linearly independent directions of vectors in $\boldsymbol{R}^s_+$ and the corresponding P' has equal p-rank. One may construct the p_i so that some refine p', and the result follows from the case of acute polyhedra. □

A polyhedron is by (1.5) a disjoint union of (generalized) simplexes. Let

$$F := N[\{s_{ij},\ i \geqq j \geqq 0\}] \tag{3.10}$$

be the free abelian monoid on the symbols s_{ij} (here N denotes the nonnegative integers, and elements of F are polynomials in the symbols s_{ij} with coefficients in N and "termwise" addition).

DEFINITION 3.11. An element $a = \sum a_{ij}s_{ij}$ of of F is a *label* for a PL-set X if there exists a partition of X into open simplexes consisting of precisely a_{ij} simplexes of rank (i, j).

Of course labels are far from unique. Note that each PL-set has labels and that for each element a of F there is a PL-set X having a as a label; in fact the latter X can be assumed to be in an orthant $\boldsymbol{R}^n_+$ (one only needs to suitably embed the orthants of an $\boldsymbol{R}^{n-1}$ containing X in disjoint hyperplanes of $\boldsymbol{R}^n$).

LEMMA 3.12. *Two* PL-*sets are isomorphic if and only if they have a label in common.*

Proof. Clear from 2.4 and the previous remarks. □

LEMMA 3.13. *Let P be an open polyhedron. If $\{P\}E^i_0\{Q_1, Q_2, \cdots\}$ and each Q_i is isomorphic to a (generalized) simplex, then P is also isomorphic to a simplex.*

Proof. We must prove that if P^+, P^-, P^0 are all isomorphic to generalized simplices then P also is. Since $s(n, m) = P_{n-m} \times \boldsymbol{R}^m_+$, this is by 3.5 equivalent to showing that the coproducts of each of:

(a) P_k, P_k, P_{k-1},

(b) $\boldsymbol{R}_+^2$, $\boldsymbol{R}_+^2$, $\boldsymbol{R}_+$,

(c) $\boldsymbol{R}_+$, P_n, P_{n-1}, $n \geqq 1$,

are all simplices (respectively, P_k, $\boldsymbol{R}_+^2$, $\boldsymbol{R}_+^n$). The simplex $s(n, m)$ is isomorphic to the cube $I^{n-m} \times \boldsymbol{R}_+^m$, where I is the open interval $(0, 1)$. This is proved in the first paragraph of (3.14) below, and will be used here. Since I is isomorphic to the coproduct of two I's and a point, case (a) follows by multiplication by I^{k-1}. Case (b) follows by slicing $P = \boldsymbol{R}_+^2$ by the hyperplane $K = \{x_1 - x_2 = 0\}$: each of P^+, P^- is a simplex (isomorphic to) $\boldsymbol{R}_+^2$, and P_0 is a half-line. We now study case (c). Consider $P = \boldsymbol{R}_+^n$ and let K_∞ be the hyperplane $\{x_1 + \cdots + x_n = 1\}$. Let P^- be the bounded component. Then P^- is isomorphic to P_n, and P_0 is P_{n-1}. It will be enough then to prove that P^+ is isomorphic to $\boldsymbol{R}_+^n$. For this, define $f: P^+ \to \boldsymbol{R}_+^n$ as the identity if $x_1 + \cdots + x_{n-1} \geqq 1$, and $(x_1, \cdots, x_{n-1}, x_1 + \cdots + x_n - 1)$ if $x_1 + \cdots + x_{n-1} < 1$; this is a PL-isomorphism. □

PROPOSITION 3.14. *Every acute open polyhedron is isomorphic to a (generalized) open simplex.*

Proof. It is proved in combinatorial topology that a closed polytope is isomorphic to a closed simplex, with the relative interiors mapping to each other, via a *continuous* PL-map; see for instance Zeeman [9, Lemma 8]. Recalling the form of $s(n, m)$, this implies that any product $X \times \boldsymbol{R}_+^m$, with X a polytope, is isomorphic to a simplex, so the cubes $I^i \times \boldsymbol{R}_+^j$ used in 3.13 are indeed isomorphic to simplices.

For a general acute polyhedron P the result will be proved by a refinement of the argument for polytopes, using the projective equivalence α. "Slicing" first by a suitable hyperplane $x_1 + \cdots + x_n = a$, it is enough by 3.13 to prove the result for a P such that $\alpha(P)$ is in any desired neighborhood of K_∞ (notation as in 3.5). A further linear transformation on P can be used to insure that $\alpha(P)$ is in fact contained in a neighborhood of any given point x in K_∞ with all $x_i > 0$. For $\alpha(P)$ small enough, there exists then a simplex Q such that (i) $\alpha(Q)$ contains and has the same dimension as $\alpha(P)$, and (ii) $Q_\infty = c\ell(\alpha(Q)) \cap K_\infty$ has the same dimension as $P_\infty = c\ell(\alpha(P)) \cap K_\infty$.

Pick now a point z in the relative interior of the face P_∞ of $\alpha(P)$. Define a "pseudo-radial projection" into z, as in Zeeman [9, Lemma 8], from a cell subdivision of the boundary of $\alpha(Q)$ into one for the boundary of $\alpha(P)$. Since z is in K_∞ and P_∞, Q_∞ have equal dimensions, this means joining to z that $\alpha(P)$ and $\alpha(Q)$ admit simultaneous partitions into isomorphic open simplexes $\alpha(P_i)$, $\alpha(Q_i)$ with $(P_i)_\infty$ and $(Q_i)_\infty$ of equal dimensions. So P_i, Q_i have the same rank and are isomorphic to each other, for each i. Thus P is isomorphic to Q, as wanted. □

B. **Classification.** We consider now the set PL_0 of isomorphism classes of PL-sets, as an abelian monoid, with coproduct as the binary operation and whose identity is the empty set. By 1.5, PL_0 is generated by the generalized simplices. Thus the monoid homomorphism

$$\lambda\colon F \longrightarrow PL_0 \tag{3.15}$$

induced by

$$\lambda(s_{ij})\colon = \text{isoclass of } \; s(i, j)\ , \qquad i \geqq j \geqq 0\ , \tag{3.16}$$

is surjective.

PROPOSITION 3.17. *As a monoid congruence, the kernel of* λ *is generated by the following set of equations*:

(a) $s_{ij} = 2s_{ij} + s_{i-1,j}$, $i \geqq 1$, $j \geqq 0$,

(b) $s_{ij} = 2s_{ij} + s_{i-1,j-1}$, $i \geqq j \geqq 2$, *and*

(c) $s_{ij} = s_{ij} + s_{ik} + s_{i-1,k}$, $j \geqq k$, $i - 1 \geqq k \geqq 0$.

Proof. Each of the above equations is in the kernel of λ, since they can be realized by slicing the simplex $s(i, j)$ by appropriate hyperplanes, as in the proof of 3.13. Conversely, assume that

$$\lambda(a) = \lambda(b)\ . \tag{3.18}$$

We want to prove that there exists a sequence of elements of F, all mapping into the same isoclass, and such that each of these elements is obtained from the preceeding or succeeding one by one of the above types of substitutions.

Let X and Y be PL-sets in $\boldsymbol{R}^n_+$ with labels a and b respectively. Since $\lambda(a) = \lambda(b)$, X and Y have by 3.12 a label in common, say c. It is enough to prove that a (and hence by the same argument b) is equivalent to c using the above transformations. Let D be a partition of X that gives the label c. Then $\{X\}E^i_0D_1E^i_0D_2 \cdots E^i_0D$, with each $i = +1$ or -1. Each D_i has a label c_i with $\lambda(c_i) = \lambda(c_{i+1})$. Since the polyhedra D_i are all acute, 3.5 can be applied at each step, with each of the cases (a), (b), (c) resulting from the corresponding cases in 3.5. □

One has then an algebraic representation for the isoclasses of PL, as elements of the quotient monoid $F/(\ker \lambda)$. Given a PL-set, a label for it can be obtained immediately from any partition of F

into open simplices, or more generally, (by 3.14), from any partition into open polyhedra contained in orthants.

Matters can be further simplified by the introduction of the two-generator abelian semiring with identity:

$$N[x, y] = \{\sum a_{ij}x_i y_j,\ a_{ij} \geqq 0\} . \tag{3.19}$$

Additively, this is just again F, when one identifies

$$s_{ij} = x^{i-j}y^j . \tag{3.20}$$

The map λ, thought of now as defined on $N[x, y]$, is a *semiring* homomorphism when PL_0 is viewed as a semiring using (cartesian, or categorical) product as multiplication. The multiplicative identity 1 is the one-point set, $\lambda(x)$ is the open unit interval $I = (0, 1)$, and $\lambda(y)$ is $\boldsymbol{R}_+$. The semiring structure permits giving a finite presentation for PL_0:

THEOREM 3.21. *Let* PL_0 *be the set of isomorphism classes of objects of* PL, *thought of as a semiring with coproduct as addition and product as multiplication. Then* PL_0 *is isomorphic to the semiring*

$$S: = N[x, y]/E , \tag{3.22}$$

where E is the semiring congruence generated by the three equations

$$\begin{aligned} &(\mathrm{a})\quad x = 2x + 1 ,\\ &(\mathrm{b})\quad y^2 = 2y^2 + y ,\\ &(\mathrm{c})\quad y = x + y + 1 . \end{aligned} \tag{3.23}$$

Proof. In terms of x and y, a set of equations generating $E = \ker \lambda$ is known to be, from (4.17),

$$\begin{aligned} &(\mathrm{a})\quad x^n y^m = 2x^n y^m + x^{n-1}y^m , \qquad n \geqq 1 , \qquad m \geqq 0 ,\\ &(\mathrm{b})\quad x^n y^m = 2x^n y^m + x^n y^{m-1} , \qquad n \geqq 0 , \qquad m \geqq 2 ,\\ &(\mathrm{c})\quad x^d y^{k+t} = x^d y^{k+t} + x^{d+t}y^k + x^{d+t-1}y^k ,\\ &\qquad d \geqq 0 , \qquad t \geqq 0 , \qquad k \geqq 0 , \qquad d + t \geqq 1 . \end{aligned} \tag{3.24}$$

As generators of a *semiring* congruence, these equations are in turn equivalent to the simpler set consisting of (3.23a), (3.23b) and

$$y^t = y^t + x^t + x^{t-1} , \qquad t \geqq 1 . \tag{3.25}$$

Since (3.23c) is the particular case $t = 1$ of this, the proof will be complete once that we establish that (3.25) follows from (3.23). By induction assume that (3.25) is true for $t - 1$ ($t \geqq 2$); then:

$$\begin{aligned} y^t &= y^{t-2}y^2 = y^{t-2}(2y^2 + y) \\ &= 2y^2 + y^{t-1} \\ &= 2y^{t-1}(x + y + 1) + y^{t-1} \\ &= (2y^t + 3y^{t-1} + xy^{t-1}) + xy^{t-1} \\ &= (2y^t + 3y^{t-1} + xy^{t-1}) + x(y^{t-1} + x^{t-1} + x^{t-2}) \\ &= (2y^t + 2xy^{t-1} + 3y^{t-1}) + (x^t + x^{t-1}) , \end{aligned}$$

and we need to show that the first of these is again y^t. But this term is

$$\begin{aligned} y^{t-2}[(2y^2 + y) + 2y + 2xy] &= y^{t-2}(y^2 + 2y + 2xy) \\ &= y^{t-1}[y + 1 + (1 + 2x)] \\ &= y^{t-1}(y + 1 + x) \\ &= y^t . \end{aligned}$$ □

Let

(3.26) $$\hat{\chi}\colon \text{PL-sets} \longrightarrow \boldsymbol{S}$$

be the *characteristic* of ***PL***, the map assigning to each PL-set its equivalence class of labels. This is a useful object to consider when answering questions about the category. Note that $\hat{\chi}$ can be used to translate into purely algebraic problems even questions not exclusively about isomorphism. For instance, asking whether there is an epimorphism $f\colon X \to Y$ between PL-sets amounts by the results of §3 to asking if Y is a subobject of X, and this is in turn equivalent to deciding if $\hat{\chi}(Y)$ is a summand of $\hat{\chi}(X)$ in $\boldsymbol{S}$. Similarly, Y is a factor of X (f "splits") if and only if $\hat{\chi}(Y)$ divides $\hat{\chi}(X)$. As an example, we solve here the epi/mono question for the case of well-defined ranks (the solution in the general case follows easily from this).

PROPOSITION 3.27. *Let X, Y be* PL-*sets with well-defined ranks. Then there exists a monomorphism $f\colon X \to Y$ if and only if either one of the following conditions hold*:

(i) $\dim X > 1$ *and* $r(X) \leqq r(Y)$.

(ii) $\dim X = \dim Y = 1$ *and the coefficient of y in a label of X never exceeds that of y in a label of Y.*

(iii) $\dim Y > \dim X = 1$ *and* $r(X) \leqq r(Y)$.

(iv) X *is finite and* $\text{card}(X) \leqq \text{card}(Y)$.

Proof. The condition $r(X) \leqq r(Y)$ is necessary, by definition of rank. Assume now that $\dim X = n + m > 1$ and let $r(X) = (n + m, m) \leqq (u + v, v) = r(Y)$. Let a, b be labels of X, Y. Each monomial $x^i y^j$

is a summand of $x^n y^m$, and hence of $x^u y^v$, so a is a summand of $rx^u y^v$ for some integer r. Iterating (3.24a), the latter is in fact a summand of a single monomial $x^u y^v$. But b contains one such monomial, since $r(Y) = (u + v, v)$. Thus a is a summand of b, and (i) is proved. When the dimension of X is 1, the possibilities are $r(X) = (1, 0)$ or $(1, 1)$. In the first case, $a = \alpha x + \beta$ is a summand of x and hence of b. In the second case, $a = \alpha x + \beta y + \gamma$, which is a summand of a suitable ky. Since $r(Y) \geqq (2, 1)$, b has *a* term xy. By (3.24a), this admits a summand kxy, which by (3.23a) has a summand ky. Thus (ii) follows. To prove (iii), note that $a = \alpha x + \beta y + \gamma$, $b = \alpha' x + \beta' y + \gamma'$, $\alpha\alpha' \neq 0$, $\beta' \geqq \beta$. By (3.23a), $\alpha x + \gamma$ is a summand of x, hence of $\alpha' x$. And βy is a summand of $\beta' y$, so a is a summand of b, as wanted. □

It is interesting to remark that the classical theorem of Euler on counting faces of polyhedra is a consequence of the form of S. Let

$$\chi\colon \text{PL-sets} \longrightarrow N \tag{3.28}$$

be obtained by composing $\hat{\chi}$ with the evaluation $x: = -1$, $y: = 0$ (well-defined by the form of E). In terms of S, a PL-set X is isomorphic to an acute polyhedron iff it is isomorphic to a cube, i.e., of the form $\lambda(x^i y^j)$. In particular, an open polytope of dimension n has label x^n. Thus $\hat{\chi}(X) = (-1)^n$. A closed polytope P is (see proof of 3.14) isomorphic to a closed cube, i.e., it has a label $(x + 2)^n$, so $\hat{\chi}(P) = 1$ always. Writing a closed polytope P as a disjoint union of the relative interiors of its faces, one has, if P has d_i faces of dimension i, that $\sum_{i=0}^{\infty} (-1)^i d_i = 1$ (Euler's theorem). The map χ can be extended to ψ: PL-sets $\rightarrow \boldsymbol{Z}^2$, by evaluation at $x: = (-1, -1)$, $y: = (0, -1)$. This map is universal for groups, and establishes $\boldsymbol{Z}^2$ as the Grothendieck group of PL. Note that by studying the free abelian monoid (rather than group) generated by the open simplexes, one obtains a complete characterization of isoclasses, from which the Grothendieck group can be in turn derived.

We turn now to the word problem for S, i.e., deciding for given a, b in $N[x, y]$ if they are equivalent under E. Since the generators of E all preserve degree, it is clear that in deciding if $a\,E\,b$ one may restrict attention to the free abelian monoid

$$F_n: = N[\{s_{ij} \mid 0 \leqq j \leqq i \leqq n\}]\,, \tag{3.29}$$

where n is the dimension of $\lambda(a)$ and $\lambda(b)$, modulo the congruence generated by those equations in (3.15) which involve only $i \leqq n$. Note that F_n is just the product monoid N^{2n}.

There are a considerable number of results each of which implies the decidability of word problems in a monoid $M = N^{2n}$. For example, one may use the results on integer equations given by Ginsburg and Spanier [3]. Another approach is to note that every monoid congruence on the (finitely generated commutative) monoid M is a rational subset of $M \times M$, as proved in Filenberg and Schützenberger [2] (in fact, they also prove that there are always rational cross-sections for such congruences). A rational subset of $M \times M$ is recognizable by a generalized finite automaton (Eilenberg [2, Theorem VII. 10.1]). Thus checking if $a E b$ is equivalent to checking if (a, b) is accepted by a given automaton, a purely algorithmic process.

To rigorously state the implications of the above one would need to give a precise meaning to the phase "given a PL-set". This could mean for example "given by a sentence in the language L" introduced in (2.5), restricting unary operators and constants to computable real numbers. Obtaining a polyhedral partition, i.e., a label, becomes a problem in linear algebra. We shall assume in any case that a PL-set is "given" by specifying a label for it. Decidability of word problems in each F_n gives then:

THEOREM 3.30. *Isomorphism of* PL-*sets is decidable.*

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