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45

Input-to-State Stability

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45.1 Introduction

The notion of input-to-state stability (ISS) was introduced in [21]. Together with several variants, also discussed in this article, it provides theoretical concepts that describe various desirable stability features of a mapping $u(\cdot) \mapsto (\cdot)$ from (time-dependent) inputs to outputs (or internal states). Prominent among these features are that inputs that are bounded, "eventually small," "integrally small," or convergent, should lead to outputs with the respective property. In addition, ISS and related notions quantify in what manner initial states affect transient behavior. The discussion in this article focuses on stability notions relative to globally attractive steady states, but a more general theory is also possible, that allows consideration of more arbitrary attractors, as well as robust and/or adaptive concepts. The reader is referred to the cited literature, as well as the textbooks [5,7,8,12,14,15,20], for extensions of the theory as well as applications, The paper [26] may also be consulted for further references and an exposition of many extensions of the concepts and results discussed in this chapter.

45.1.1 Operator and Lyapunov Stability

Broadly speaking, there are two main competing approaches to system stability: the *state-space approach* usually associated with the name of Lyapunov, and the *operator* approach, of which George Zames was one of the main proponents and developers and which was the subject of major contributions by Sandberg, Willems, Safonov, and others. In the operator approach, one studies the i/o mapping:

$$\left(x^0, u(\cdot)\right) \mapsto y(\cdot),$$
$$\mathbb{R}^n \times \left[\mathcal{L}_q(0, +\infty)\right]^m \to \left[\mathcal{L}_q(0, +\infty)\right]^p,$$

that sends initial states and input functions into output functions. This includes the special case when the output is the internal state of a system. The notation $\mathcal{L}_q$ refers to spaces of functions whose qth power is integrable; typical choices are $q = 2$ or $q = \infty$. This approach permits the use of Hilbert or Banach space techniques, and elegantly generalizes to nonlinear systems many properties of linear systems, especially in the context of robustness analysis. The state-space approach, in contrast, is geared to the study of systems without inputs, but is better suited to the study of nonlinear dynamics, and it allows the use of geometric and topological ideas. The ISS notion combines these dual views of stability.

45.1.2 The Class of Systems

This chapter considers systems with inputs and outputs in the usual sense of control theory [24]:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t))$$

(the arguments "t" is often omitted). There are n state variables, m input channels, and p output channels. States $x(t)$ take values in Euclidean space $\mathbb{R}^n$, and the inputs (also called "controls" or "disturbances" depending on the context) are measurable locally essentially bounded maps $u(\cdot) : [0, \infty) \to \mathbb{R}^m$. Output values $y(t)$ take values in $\mathbb{R}^p$, The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz with $f(0, 0) = 0$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous with $h(0) = 0$. These two properties mean that the state $x = 0$ is an equilibrium when the input is $u = 0$, and the corresponding output is $y = 0$. Many of these assumptions can be weakened considerably, and the cited references should be consulted for more details. The solution, defined on some maximal interval $[0, t_{\max}(x^0, u))$, for each initial state x^0 and input u, is denoted as $x(t, x^0, u)$, and in particular, for systems with no inputs

$$\dot{x}(t) = f(x(t)),$$

just as $x(t, x^0)$. The *zero-system* associated to $\dot{x} = f(x, u)$ is by definition the system with no inputs $\dot{x} = f(x, 0)$. Euclidean norm is written as $|x|$. For a function of time, typically an input or an output, $\|u\|$, or $\|u\|_\infty$ for emphasis, is the (essential) supremum or "sup" norm (possibly $+\infty$, if u is not bounded). The norm of the restriction of a signal to an interval I is denoted by $\|u_I\|_\infty$ (or just$\|u_I\|$).

45.1.3 Notions of Stability

It is convenient to introduce "comparison functions" to quantify stability. A *class $\mathcal{K}_\infty$ function* is a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is continuous, strictly increasing, unbounded, and satisfies $\alpha(0) = 0$ and a *class $\mathcal{KL}$ function* is a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each t and $\beta(r, t)$ decreases to zero as $t \to \infty$, for each fixed r.

For a system with no inputs $\dot{x} = f(x)$, there is a well-known notion of global asymptotic stability (GAS), or "*0-GAS*" when referring to the zero-system $\dot{x} = f(x, 0)$ associated to a given system with inputs $\dot{x} = f(x, u)$) due to Lyapunov, and usually defined in "ε-δ" terms. It is an easy exercise to show that this

standard definition is in fact equivalent to the following statement:

$$(\exists \beta \in \mathcal{KL}) \quad |x(t, x^0)| \le \beta\left(|x^0|, t\right), \quad \forall x^0, \ \forall \ge 0.$$

Observe that, since β decreases on t, in particular:

$$|x(t, x^0)| \ \le \ \beta\left(|x^0|, 0\right), \quad \forall x^0, \ \forall t \ge 0,$$

which provides the Lyapunov-stability or "small overshoot" part of the GAS definition (because $\beta\left(|x^0|, 0\right)$ is small whenever $|x^0|$ is small, by continuity of $\beta(\cdot, 0)$ and $\beta(0, 0) = 0$), while the fact that $\beta \to 0$ as $t \to \infty$ gives

$$|x(t, x^0)| \le \beta\left(|x^0|, t\right) \underset{t \to \infty}{\longrightarrow} 0, \quad \forall x^0,$$

which is the attractivity (convergence to steady state) part of the GAS definition.

In [23, Proposition 7], it is shown that for each $\beta \in \mathcal{KL}$ there exist two class $\mathcal{K}_\infty$ functions α_1, α_2 such that:

$$\beta(r, t) \ \le \ \alpha_2\left(\alpha_1(r)e^{-t}\right), \quad \forall s, t \ge 0,$$

which means that the GAS estimate can be also written in the form:

$$|x(t, x^0)| \ \le \ \alpha_2\left(\alpha_1(|x^0|)e^{-t}\right)$$

and thus suggests a close analogy between GAS and an exponential stability estimate $|x(t, x^0)| \le |x^0| \, e^{-at}$.

In general, 0-GAS does not guarantee good behavior with respect to inputs. To explain why this is relevant, let us briefly recall the case of linear systems. A *linear* system in control theory is the one for which both f and h are linear mappings:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$. It is well-known that a linear system is 0-GAS (or "internally stable") if and only if the matrix A is a Hurwitz matrix, that is to say, all the eigenvalues of A have negative real parts. Such a 0-GAS linear system automatically satisfies all reasonable i/o stability properties: bounded inputs result in bounded state trajectories as well as outputs, inputs converging to zero imply solutions (and outputs) converging to zero, and so forth [24]. But *the 0-GAS property is not equivalent*, in general, to i/o, or even input/state, stability of any sort. The implication that 0-GAS implies i/o stability is in general false for nonlinear systems. For a simple example, consider the following one-dimensional ($n = 1$) system, with scalar ($m = 1$) inputs:

$$\dot{x} = -x + (x^2 + 1)u.$$

This system is clearly 0-GAS, since it reduces to $\dot{x} = -x$ when $u \equiv 0$. On the other hand, solutions diverge even for some inputs that converge to zero. For example, take the control $u(t) = (2t + 2)^{-1/2}$ and $x^0 = \sqrt{2}$, This results in the unbounded trajectory $x(t) = (2t + 2)^{1/2}$. This is in spite of the fact that the unforced system is GAS. Thus, the converging-input converging-state property does not hold. Even worse, the bounded input $u \equiv 1$ results in a finite-time explosion. This example is not artificial, as it arises in feedback-linearization design, mentioned below.

45.1.4 Gains for Linear Systems

For linear systems, the three most typical ways of defining i/o stability in terms of operators

$$\left\{L^2, L^\infty\right\} \to \left\{L^2, L^\infty\right\}$$

are as follows. The statements below should be read, more precisely, as asking that there should exist positive c and λ such that the given estimates hold for all $t \ge 0$ and all solutions of $\dot{x} = Ax + Bu$ with

$x(0) = x^0$ and arbitrary inputs $u(\cdot)$. The estimates are written in terms of states $x(t)$, but similar notions can be defined for more general outputs $y = Cx$.

$$\text{``}L^\infty \to L^\infty\text{''}: \ c\,|x(t, x^0, u)| \ \le \ |x^0|\,e^{-\lambda t} + \sup_{s \in [0,t]} |u(s)|$$

$$\text{``}L^2 \to L^\infty\text{''}: \ c\,|x(t, x^0, u)| \ \le \ |x^0|\,e^{-\lambda t} + \int_0^t |u(s)|^2 \ ds$$

$$\text{``}L^2 \to L^2\text{''}: \ c \int_0^t |x(s, x^0, u)|^2 \ ds \ \le \ |x^0| + \int_0^t |u(s)|^2 \ ds$$

(the missing case $L^\infty \to L^2$ is less interesting, being too restrictive).

For linear systems, these estimates are all equivalent in the following sense: if an estimate of one type exists, then the other two estimates exist too, although the actual numerical values of the constants c, λ appearing in the different estimates are not necessarily the same: they are associated to various types of norms on input spaces and spaces of solutions, such as "H_2" and "H_∞" gains, [4]. It is easy to see that existence of the above estimates is simply equivalent to the requirement that the A matrix be Hurwitz, that is to say, to 0-GAS, the asymptotic stability of the unforced system $\dot{x} = Ax$.

45.1.5 Nonlinear Coordinate Changes

A "geometric" view of nonlinear dynamics suggests that *notions of stability should be invariant under (nonlinear) changes of variables*: under a change of variables, a system which is stable in some technical sense should remain stable, in the same sense, when written in the new coordinates. This principle leads to the ISS notion when starting from the above linear notions, as elaborated next. In this article, a *change of coordinates* is any map

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

such that the following properties hold: $T(0) = 0$ (this fixes the equilibrium at $x = 0$), T is continuous, and it admits an inverse map $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, which is well-defined and continuous as well. In other words, T is a homeomorphism which fixes the origin. One could add the requirement that T should be differentiable, or that it be differentiable at least for $x \neq 0$, but the discussion to follow does not require this additional condition. Now suppose that a system $\dot{x} = f(x)$ is exponentially stable:

$$|x(t, x^0)| \ \le \ c\,|x^0|e^{-\lambda t} \quad \forall\, t \ge 0 \quad (\text{some } c, \lambda > 0)$$

and that a change of variables is performed:

$$x(t) = T(z(t)).$$

Consider, for this transformation T, the following two functions:

$$\underline{\alpha}(r) := \min_{|x| \ge r} |T(x)| \quad \text{and} \quad \overline{\alpha}(r) := \max_{|x| \le r} |T(x)|,$$

which are well-defined because T and its inverse are both continuous, and are both functions of class $\mathcal{K}_\infty$ (easy exercise). Then,

$$\underline{\alpha}(|x|) \ \le \ |T(x)| \ \le \ \overline{\alpha}(|x|), \quad \forall\, x \in \mathbb{R}^n$$

and therefore, substituting $x(t, x^0) = T(z(t, z^0))$ in the exponential stability estimate:

$$\underline{\alpha}(|z(t, z^0)|) \ \le \ c\,\overline{\alpha}(|z^0|)\,e^{-\lambda t},$$

where $z^0 = T^{-1}(x^0)$. Thus, the estimate in z-coordinates takes the following form:

$$|z(t, z^0)| \ \le \ \beta\left(|z^0|, t\right),$$

where $\beta(r, t) = \underline{\alpha}^{-1}\left(c\overline{\alpha}\left(re^{-\lambda t}\right)\right)$ is a function of class $\mathcal{KL}$. (As remarked earlier, any possible function of class $\mathcal{KL}$ can be written in this factored form, actually.)

In summary, the concept of GAS is rederived simply by making coordinate changes on globally exponentially stable systems. The same considerations, applied to systems with inputs, lead to ISS and related notions. In addition to the state transformation $x(t) = T(z(t))$, there is now also a transformation $u(t) = S(v(t))$, where S is a change of variables in the space of input values $\mathbb{R}^m$. Arguing analogously as for systems without inputs, one arrives to the following three concepts:

$$L^\infty \to L^\infty \rightsquigarrow \alpha\left(|x(t)|\right) \; \leq \; \beta(|x^0|, t) + \sup_{s \in [0,t]} \gamma(|u(s)|),$$

$$L^2 \to L^\infty \rightsquigarrow \alpha\left(|x(t)|\right) \; \leq \; \beta(|x^0|, t) + \int_0^t \gamma(|u(s)|)\, \mathrm{d}s$$

$$L^2 \to L^2 \rightsquigarrow \int_0^t \alpha\left(|x(s)|\right)\, \mathrm{d}s \; \leq \; \alpha_0(|x^0|) + \int_0^t \gamma(|u(s)|)\, \mathrm{d}s$$

($x(t)$ is written instead of the more cumbersome $x(t, x^0, u)$). If more general outputs $y = h(x)$ instead of states are the object of interest, these notions can be modified in several ways, as discussed later in the chapter. Unless otherwise stated, the convention when giving an estimate like the ones above is that there should exist comparison functions ($\alpha, \alpha_0 \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$) such that the estimates hold for all inputs and initial states. These three notions will be studied one at a time.

45.2 ISS and Feedback Redesign

The "$L^\infty \to L^\infty$" estimate, under changes of variables, leads to the concept of ISS: there should exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|x(t)| \; \leq \; \beta(|x^0|, t) + \gamma\left(\|u\|_\infty\right) \qquad \text{(ISS)}$$

holds for all solutions (meaning that the estimate is valid for all inputs $u(\cdot)$, all initial conditions x^0, and all $t \geq 0$). Note that there is now no function "α" is the left-hand side because, redefining β and γ, one can assume, without loss of generality, that α is the identity: if $\alpha(r) \leq \beta(s, t) + \gamma(t)$ holds, then also $r \leq \alpha^{-1}(\beta(s, t) + \gamma(t)) \leq \alpha^{-1}(2\beta(s, t)) + \alpha^{-1}(2\gamma(t))$; since $\alpha^{-1}(2\beta(\cdot, \cdot)) \in \mathcal{KL}$ and $\alpha^{-1}(2\gamma(\cdot)) \in \mathcal{K}_\infty$, an estimate of the same type, but now with no "α," is obtained. In addition, note that the supremum $\sup_{s \in [0,t]} \gamma(|u(s)|)$ over the interval $[0, t]$ is the same as $\gamma(\|u_{[0,t]}\|_\infty) = \gamma(\sup_{s \in [0,t]}(|u(s)|))$, because the function γ is increasing, so one may replace this term by $\gamma(\|u\|_\infty)$, where $\|u\|_\infty = \sup_{s \in [0,\infty)} \gamma(|u(s)|)$ is the sup norm of the input, because the solution $x(t)$ depends only on values $u(s), s \leq t$ (so, one could equally well consider the input that has values $\equiv 0$ for all $s > t$).

Note that a potentially weaker definition might be that the ISS estimate should hold merely for all $t \in [0, t_{\max}(x^0, u))$. However, this potentially different definition turns out to be equivalent. Indeed, if the estimate holds *a priori* only on such a maximal interval of definition, then, since the right-hand is bounded on $[0, T]$, for any $T > 0$ (recall that inputs are by definition assumed to be bounded on any bounded interval), it follows that the maximal solution of $x(t, x^0, u)$ is bounded, and therefore that $t_{\max}(x^0, u) = +\infty$ (e.g., Proposition C.3.6 in [24]). In other words, the ISS estimate holds for all $t \geq 0$ automatically, if it is required to hold merely for maximal solutions.

Since, in general, $\max\{a, b\} \leq a + b \leq \max\{2a, 2b\}$, one can restate the ISS condition in a slightly different manner, namely, asking for the existence of some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ (in general, different from the ones in the ISS definition) such that

$$|x(t)| \leq \max\left\{\beta(|x^0|, t), \, \gamma\left(\|u\|_\infty\right)\right\}$$

holds for all solutions. Such redefinitions, using "max" instead of sum, are also possible for each of the other concepts to be introduced later.
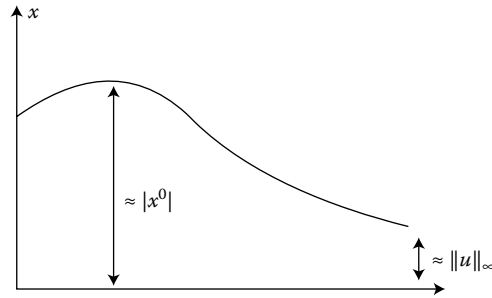
FIGURE 45.1   ISS combines overshoot and asymptotic behavior.

Intuitively, the definition of ISS requires that, for t large, the size of the state must be bounded by some function of the sup norm—that is to say, the amplitude—of inputs (because $\beta(|x^0|, t) \to 0$ as $t \to \infty$). On the other hand, the $\beta(|x^0|, 0)$ term may dominate for small t, and this serves to quantify the magnitude of the transient (overshoot) behavior as a function of the size of the initial state x^0 (Figure 45.1).

The *ISS superposition theorem*, discussed later, shows that ISS is, in a precise mathematical sense, the conjunction of two properties, one of them dealing with asymptotic bounds on $|x^0|$ as a function of the magnitude of the input, and the other one providing a transient term obtained when one ignores inputs.

45.2.1 Linear Case, for Comparison

For internally stable linear systems $\dot{x} = Ax + Bu$, the variation of parameters formula gives immediately the following inequality:

$$|x(t)| \leq \beta(t) |x^0| + \gamma \|u\|_\infty,$$

where

$$\beta(t) = \left\| e^{tA} \right\| \to 0 \quad \text{and} \quad \gamma = \|B\| \int_0^\infty \left\| e^{sA} \right\| ds < \infty.$$

This is a particular case of the ISS estimate, $|x(t)| \leq \beta(|x^0|, t) + \gamma\left(\|u\|_\infty\right)$, with linear comparison functions.

45.2.2 Feedback Redesign

The notion of ISS arose originally as a way to precisely formulate, and then answer the following question. Suppose that, as in many problems in control theory, a system $\dot{x} = f(x, u)$ has been stabilized by means of a feedback law $u = k(x)$ (Figure 45.2), that is to say, k was chosen such that the origin of the closed-loop system $\dot{x} = f(x, k(x))$ is globally asymptotically stable. (See e.g. [25] for a discussion of mathematical
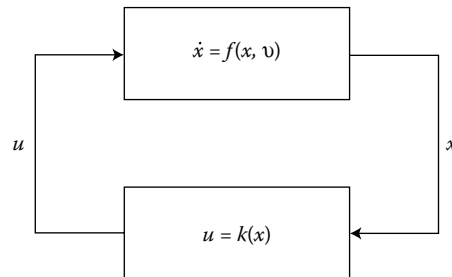

FIGURE 45.2   Feedback stabilization, closed-loop system $\dot{x} = f(x, k(x))$.
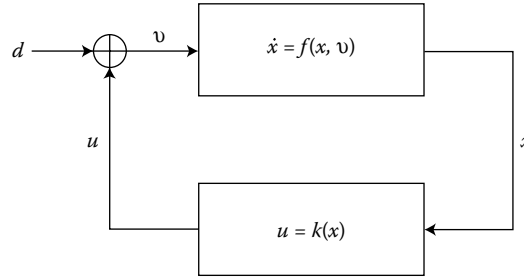
FIGURE 45.3 Actuator disturbances, closed-loop system $\dot{x} = f(x, k(x) + d)$.

aspects of state feedback stabilization.) Typically, the design of k was performed by ignoring the effect of possible *input disturbances* $d(\cdot)$ (also called actuator disturbances). These "disturbances" might represent true noise or perhaps errors in the calculation of the value $k(x)$ by a physical controller, or modeling uncertainty in the controller or the system itself. What is the effect of considering disturbances? In order to analyze the problem, d is incorporated into the model, and one studies the new system $\dot{x} = f(x, k(x) + d)$, where d is seen as an input (Figure 45.3). One may then ask what is the effect of d on the behavior of the system.

Disturbances d may well destabilize the system, and the problem may arise even when using a routine technique for control design, feedback linearization. To appreciate this issue, take the following very simple example. Given is the system

$$\dot{x} = f(x, u) = x + (x^2 + 1)u.$$

In order to stabilize it, substitute $u = \tilde{u}/(x^2 + 1)$ (a preliminary feedback transformation), rendering the system linear with respect to the new input $\tilde{u}$: $\dot{x} = x + \tilde{u}$, and then use $\tilde{u} = -2x$ in order to obtain the closed-loop system $\dot{x} = -x$. In other words, in terms of the original input u, the feedback law is

$$k(x) = \frac{-2x}{x^2 + 1},$$

so that $f(x, k(x)) = -x$. This is a GAS system. The effect of the disturbance input d is analyzed as follows. The system $\dot{x} = f(x, k(x) + d)$ is

$$\dot{x} = -x + (x^2 + 1)\, d.$$

As seen before, this system has solutions that diverge to infinity even for inputs d that converge to zero; moreover, the constant input $d \equiv 1$ results in solutions that explode in finite time. Thus $k(x) = -2x/(x^2 + 1)$ was not a good feedback law, in the sense that its performance degraded drastically once actuator disturbances were taken into account.

The key observation for what follows is that, if one adds a correction term "$-x$" to the above formula for $k(x)$, so that now

$$\tilde{k}(x) = \frac{-2x}{x^2 + 1} - x,$$

then the system $\dot{x} = f(x, \tilde{k}(x) + d)$ with disturbance d as input becomes, instead

$$\dot{x} = -2x - x^3 + (x^2 + 1)\, d$$

and this system is much better behaved: it is still GAS when there are no disturbances (it reduces to $\dot{x} = -2x - x^3$) but, in addition, it is ISS (easy to verify directly, or appealing to some of the characterizations mentioned later). Intuitively, for large x, the term $-x^3$ serves to dominate the term $(x^2 + 1)d$, for all bounded disturbances $d(\cdot)$, and this prevents the state from getting too large.
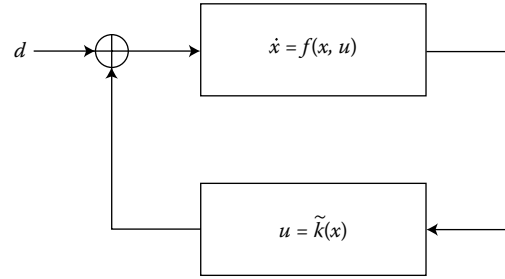
FIGURE 45.4   Different feedback ISS-stabilizes.

45.2.3 A Feedback Redesign Theorem for Actuator Disturbances

This example is an instance of a general result, which says that whenever there is some feedback law that stabilizes a system, there is also a (possibly different) feedback so that the system with external input d (Figure 45.4) is ISS.

Theorem 45.1: [21]

Consider a system affine in controls

$$\dot{x} = f(x, u) = g_0(x) + \sum_{i=1}^{m} u_i g_i(x) \quad \big(g_0(0) = 0\big)$$

and suppose that there is some differentiable feedback law $u = k(x)$ so that

$$\dot{x} = f(x, k(x))$$

has $x = 0$ as a GAS equilibrium. Then, there is a feedback law $u = \tilde{k}(x)$ such that

$$\dot{x} = f(x, \tilde{k}(x) + d)$$

is ISS with input $d(\cdot)$

The proof is very easy, once the appropriate technical machinery has been introduced: one starts by considering a smooth Lyapunov function V for GAS of the origin in the system $\dot{x} = f(x, k(x))$ (such a V always exists, by classical converse theorems); then $\hat{k}(x) := -(L_G V(x))^T = -(\nabla V(x) G(x))^T$, where G is the matrix function whose columns are the $g_i, i = 1, \ldots, m$ and T indicates transpose, provides the necessary correction term to add to k. This term has the same degree of smoothness as the vector fields making up the original system. Somewhat less than differentiability of the original k is enough for this argument: continuity is enough. However, if no continuous feedback stabilizer exists, then no smooth V can be found. (Continuous stabilization of nonlinear systems is basically equivalent to the existence of what are called smooth control-Lyapunov functions, see e.g. [25].) In that case, if only discontinuous stabilizers are available, the result can still be generalized, see [17], but the situation is harder to analyze, since even the notion of "solution" of the closed-loop system $\dot{x} = f(x, k(x))$ has to be carefully defined.

There is also a redefinition procedure for systems that are not affine on inputs, but the result as stated above is false in that generality, and is much less interesting; see [22] for a discussion.

The above feedback redesign theorem is merely the beginning of the story. The reader is referred to the book [15], and the references given later, for many further developments on the subjects of recursive feedback design, the "backstepping" approach, and other far-reaching extensions.

45.3 Equivalences for ISS

This section reviews results that show that ISS is equivalent to several other notions, including asymptotic gain (AG), existence of robustness margins, dissipativity, and an energy-like stability estimate.

45.3.1 Nonlinear Superposition Principle

Clearly, if a system is ISS, then the system with no inputs $\dot{x} = f(x, 0)$ is GAS: the term $\|u\|_\infty$ vanishes, leaving precisely the GAS property. In particular, then, the system $\dot{x} = f(x, u)$ is *0-stable*, meaning that the origin of the system without inputs $\dot{x} = f(x, 0)$ is stable in the sense of Lyapunov: for each $\varepsilon > 0$, there is some $\delta > 0$ such that $|x^0| < \delta$ implies $|x(t, x^0)| < \varepsilon$. (In comparison function language, one can restate 0-stability as: there is some $\gamma \in \mathcal{K}$ such that $|x(t, x^0)| \leq \gamma(|x^0|)$ holds for all small x^0.)

On the other hand, since $\beta(|x^0|, t) \to 0$ as $t \to \infty$, for t large one has that the first term in the ISS estimate $|x(t)| \leq \max\{\beta(|x^0|, t), \gamma(\|u\|_\infty)\}$ vanishes. Thus, an ISS system satisfies the following *"AG" property*: there is some $\gamma \in \mathcal{K}_\infty$ so that:

$$\overline{\lim_{t \to +\infty}} |x(t, x^0, u)| \leq \gamma(\|u\|_\infty), \quad \forall x^0, u(\cdot) \qquad \text{(AG)}$$

(see Figure 45.5). In words, for all large enough t, the trajectory exists, and it gets arbitrarily close to a sphere whose radius is proportional, in a possibly nonlinear way quantified by the function γ, to the amplitude of the input. In the language of robust control, the estimate (AG) would be called an "ultimate boundedness" condition; it is a generalization of attractivity (all trajectories converge to zero, for a system $\dot{x} = f(x)$ with no inputs) to the case of systems with inputs; the "lim sup" is required since the limit of $x(t)$ as $t \to \infty$ may well not exist. From now on (and analogously when defining other properties), we will just say "the system is AG" instead of the more cumbersome "satisfies the AG property."

Observe that, since only large values of t matter in the limsup, one can equally well consider merely tails of the input u when computing its sup norm. In other words, one may replace $\gamma(\|u\|_\infty)$ by $\gamma(\overline{\lim}_{t \to +\infty} |u(t)|)$, or (since γ is increasing), $\overline{\lim}_{t \to +\infty} \gamma(|u(t)|)$.
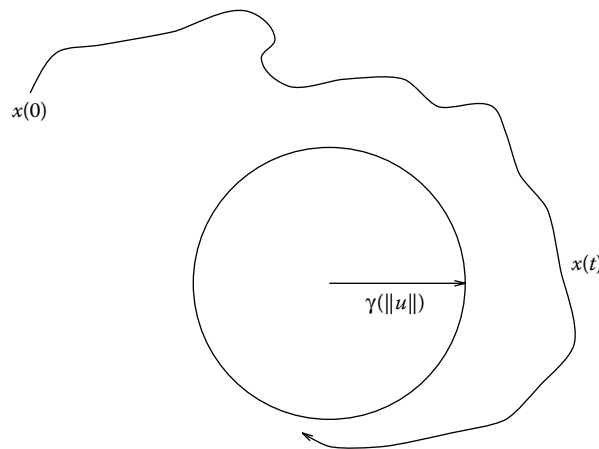

FIGURE 45.5   Asymptotic gain property.

The surprising fact is that these two necessary conditions are also sufficient. This is summarized by the *ISS superposition theorem*:

Theorem 45.2: [29]

A system is ISS if and only if it is 0-stable and AG.

The basic difficulty in the proof of this theorem is in establishing uniform convergence estimates for the states, that is, in constructing the β function in the ISS estimate, independently of the particular input. As in optimal control theory, one would like to appeal to compactness arguments (using weak topologies on inputs), but there is no convexity to allow this. The proof hinges upon a lemma given in [29], which may be interpreted [6] as a relaxation theorem for differential inclusions, relating GAS of an inclusion $\dot{x} \in F(x)$ to GAS of its convexification.

A minor variation of the above superposition theorem is as follows. Let us consider the *limit property (LIM)*:

$$\inf_{t \geq 0} |x(t, x^0, u)| \leq \gamma(\|u\|_\infty), \quad \forall x^0, \, u(\cdot) \qquad \text{(LIM)}$$

(for some $\gamma \in \mathcal{K}_\infty$).

Theorem 45.3: [29]

A system is ISS if and only if it is 0-stable and LIM.

45.3.2 Robust Stability

In this article, a system is said to be *robustly stable* if it admits a *margin of stability* ρ, that is, a smooth function $\rho \in \mathcal{K}_\infty$ so the system

$$\dot{x} = g(x, d) := f(x, d\rho(|x|))$$

is GAS uniformly in this sense: for some $\beta \in \mathcal{KL}$,

$$\left|x(t, x^0, d)\right| \leq \beta(\left|x^0\right|, t)$$

for all possible $d(\cdot) : [0, \infty) \to [-1, 1]^m$. An alternative way to interpret this concept (*cf.* [28]) is as uniform GAS of the origin with respect to all possible time-varying feedback laws Δ bounded by ρ: $|\Delta(t, x)| \leq \rho(|x|)$. In other words, the system

$$\dot{x} = f(x, \Delta(t, x))$$

(Figure 45.6) is stable uniformly over all such perturbations Δ. In contrast to the ISS definition, which deals with all possible "open-loop" inputs, the present notion of robust stability asks about all possible closed-loop interconnections. One may think of Δ as representing uncertainty in the dynamics of the original system, for example.

Theorem 45.4: [28]
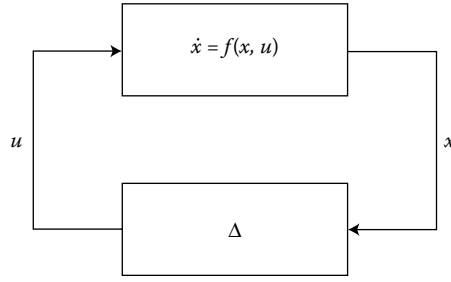

A system is ISS if and only if it is robustly stable.

FIGURE 45.6 Margin of robustness.

Intuitively, the ISS estimate $|x(t)| \leq \max \left\{ \beta(|x^0|, t), \gamma \left(\|u\|_\infty \right) \right\}$ says that the β term dominates as long as $|u(t)| \ll |x(t)|$ for all t, but $|u(t)| \ll |x(t)|$ amounts to $u(t) = d(t).\rho(|x(t)|)$ with an appropriate function ρ. This is an instance of a "small gain" argument, see below. One analog for linear systems is as follows: if A is a Hurwitz matrix, then $A + Q$ is also Hurwitz, for all small enough perturbations Q; note that when Q is a nonsingular matrix, $|Qx|$ is a $\mathcal{K}_\infty$ function of $|x|$.

45.3.3 Dissipation

Another characterization of ISS is as a dissipation notion stated in terms of a Lyapunov-like function. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a *storage function* if it is positive definite, that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, and proper, that is, $V(x) \to \infty$ as $|x| \to \infty$. This last property is equivalent to the requirement that the sets $V^{-1}([0, A])$ should be compact subsets of $\mathbb{R}^n$, for each $A > 0$, and in the engineering literature it is usual to call such functions *radially unbounded*. It is an easy exercise to show that $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if and only if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \quad \forall x \in \mathbb{R}^n$$

(the lower bound amounts to properness and $V(x) > 0$ for $x \neq 0$, while the upper bound guarantees $V(0) = 0$). For convenience, $\dot{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the function

$$\dot{V}(x, u) := \nabla V(x) \cdot f(x, u),$$

which provides, when evaluated at $(x(t), u(t))$, the derivative $dV(x(t))/dt$ along solutions of $\dot{x} = f(x, u)$.

An *ISS-Lyapunov function* for $\dot{x} = f(x, u)$ is by definition a smooth storage function V for which there exist functions $\gamma, \alpha \in \mathcal{K}_\infty$ so that

$$\dot{V}(x, u) \leq -\alpha(|x|) + \gamma(|u|), \quad \forall x, u. \qquad \text{(L-ISS)}$$

Integrating, an equivalent statement is that, along all trajectories of the system, there holds the following dissipation inequality:

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} w(u(s), x(s)) \, ds,$$

where, using the terminology of [31], the "supply" function is $w(u, x) = \gamma(|u|) - \alpha(|x|)$. For systems with no inputs, an ISS-Lyapunov function is precisely the same object as a Lyapunov function in the usual sense.

Theorem 45.5: [28]

A system is ISS if and only if it admits a smooth ISS-Lyapunov function.

Since $-\alpha(|x|) \le -\alpha(\bar{\alpha}^{-1}(V(x)))$, the ISS-Lyapunov condition can be restated as

$$\dot{V}(x,u) \le -\tilde{\alpha}(V(x)) + \gamma(|u|), \quad \forall\, x, u$$

for some $\tilde{\alpha} \in \mathcal{K}_\infty$. In fact, one may strengthen this a bit [19]: for any ISS system, there is a always a smooth ISS-Lyapunov function satisfying the "exponential" estimate $\dot{V}(x,u) \le -V(x) + \gamma(|u|)$.

The sufficiency of the ISS-Lyapunov condition is easy to show, and was already in the original paper [21]. A sketch of the proof is as follows, assuming for simplicity a dissipation estimate in the form $\dot{V}(x,u) \le -\alpha(V(x)) + \gamma(|u|)$. Given any x and u, either $\alpha(V(x)) \le 2\gamma(|u|)$ or $\dot{V} \le -\alpha(V)/2$. From here, one deduces by a comparison theorem that, along all solutions,

$$V(x(t)) \le \max\left\{ \beta(V(x^0), t),\ \alpha^{-1}(2\gamma(\|u\|_\infty)) \right\},$$

where the $\mathcal{KL}$ function $\beta(s, t)$ is the solution $y(t)$ of the initial value problem

$$\dot{y} = -\frac{1}{2}\alpha(y) + \gamma(u), \quad y(0) = s.$$

Finally, an ISS estimate is obtained from $V(x^0) \le \bar{\alpha}(x^0)$.

The proof of the converse part of the theorem is based upon first showing that ISS implies robust stability in the sense already discussed, and then obtaining a converse Lyapunov theorem for robust stability for the system $\dot{x} = f(x, d\rho(|x|)) = g(x, d)$, which is asymptotically stable uniformly on all Lebesgue-measurable functions $d(\cdot) : \mathbb{R}_{\ge 0} \to B(0, 1)$. This last theorem was given in [16], and is basically a theorem on Lyapunov functions for differential inclusions. The classical result of Massera [18] for differential equations (with no inputs) becomes a special case.

45.3.4 Using "Energy" Estimates Instead of Amplitudes

In linear control theory, H_∞ theory studies $L^2 \to L^2$-induced norms, which under coordinate changes leads to the following type of estimate:

$$\int_0^t \alpha\left(|x(s)|\right) ds \le \alpha_0(|x^0|) + \int_0^t \gamma(|u(s)|)\, ds$$

along all solutions, and for some $\alpha, \alpha_0, \gamma \in \mathcal{K}_\infty$. Just for the statement of the next result, a system is said to *satisfy an integral–integral estimate* if for every initial state x^0 and input u, the solution $x(t, x^0, u)$ is defined for all $t > 0$ and an estimate as above holds. (In contrast to ISS, this definition explicitly demands that $t_{\max} = \infty$.)

Theorem 45.6: [23]

A system is ISS if and only if it satisfies an integral–integral estimate.

This theorem is quite easy to prove, in view of previous results. A sketch of the proof is as follows. If the system is ISS, then there is an ISS-Lyapunov function satisfying $\dot{V}(x,u) \le -V(x) + \gamma(|u|)$, so, integrating along any solution:

$$\int_0^t V(x(s))\, ds \le \int_0^t V(x(s))\, ds + V(x(t)) \le V(x(0)) + \int_0^t \gamma(|u(s)|)\, ds$$

and thus an integral–integral estimate holds. Conversely, if such an estimate holds, one can prove that $\dot{x} = f(x, 0)$ is stable and that an AG exists.

45.4 Cascade Interconnections

One of the main features of the ISS property is that it behaves well under composition: a cascade (Figure 45.7) of ISS systems is again ISS, see [21]. This section sketches how the cascade result can also be seen as a consequence of the dissipation characterization of ISS, and how this suggests a more general feedback result. For more details regarding the rich theory of ISS small-gain theorems, and their use in nonlinear feedback design, the references should be consulted. Consider a cascade as follows:

$$\dot{z} = f(z, x),$$
$$\dot{x} = g(x, u),$$

where each of the two subsystems is assumed to be ISS. Each system admits an ISS-Lyapunov function V_i. But, moreover, it is always possible (see [27]) to redefine the V_i's so that the comparison functions for both are matched in the following way:

$$\dot{V}_1(z, x) \leq \theta(|x|) - \alpha(|z|),$$
$$\dot{V}_2(x, u) \leq \tilde{\theta}(|u|) - 2\theta(|x|).$$

Now it is obvious why the full system is ISS: simply use $V := V_1 + V_2$ as an ISS-Lyapunov function for the cascade:

$$\dot{V}((x, z), u) \leq \tilde{\theta}(|u|) - \theta(|x|) - \alpha(|z|).$$

Of course, in the special case in which the x-subsystem has no inputs, this also proved that the cascade of a GAS and an ISS system is GAS.

More generally, one may allow a "small gain" feedback as well (Figure 45.8). That is, one allows inputs $u = k(z)$ as long as they are small enough:

$$|k(z)| \leq \tilde{\theta}^{-1}((1 - \varepsilon)\alpha(|z|)).$$

The claim is that the closed-loop system

$$\dot{z} = f(z, x),$$
$$\dot{x} = g(x, k(x))$$

is GAS. This follows because the same V is a Lyapunov function for the closed-loop system; for $(x, z) \neq 0$:

$$\tilde{\theta}(|u|) \leq (1 - \varepsilon)\alpha(|z|) \rightsquigarrow \dot{V}(x, z) \leq -\theta(|x|) - \varepsilon\alpha(|z|) < 0.$$

A far more interesting version of this result, resulting in a composite system with inputs being itself ISS, is the *ISS small-gain theorem* due to Jiang et al. [10].
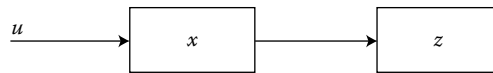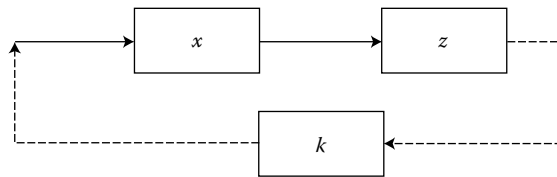

FIGURE 45.7 Cascade.

FIGURE 45.8 Adding a feedback to the cascade.

45.5 Integral ISS

Several different properties, including "integral-to-integral" stability, dissipation, robust stability margins, and AG properties, were all shown to be exactly equivalent to ISS. Thus, it would appear to be difficult to find a general and interesting concept of nonlinear stability that is truly distinct from ISS. One such concept, however, does arise when considering a mixed notion which combines the "energy" of the input with the amplitude of the state. It is obtained from the "$L^2 \to L^\infty$" gain estimate, under coordinate changes, and it provides a genuinely new concept [23].

A system is said to be *integral-input-to-state stable (iISS)* provided that there exist $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the estimate

$$\alpha\left(|x(t)|\right) \leq \beta(|x^0|, t) + \int_0^t \gamma(|u(s)|)\, ds \qquad\qquad \text{(iISS)}$$

holds along all solutions. Just as with ISS, one could state this property merely for all times $t \in t_{\max}(x^0, u)$. Since the right-hand side is bounded on each interval $[0, t]$ (because, inputs are by definition assumed to be bounded on each finite interval), it is automatically true that $t_{\max}(x^0, u) = +\infty$ if such an estimate holds along maximal solutions. So forward-completeness (solution exists for all $t > 0$) can be assumed with no loss of generality.

45.5.1 Other Mixed Notions

A change of variables starting from a system that satisfies a finite operator gain condition from L^p to L^q, with $p \neq q$ both finite, leads naturally to the following type of "weak integral-to-integral" mixed estimate:

$$\int_0^t \underline{\alpha}(|x(s)|)\, ds \leq \kappa(|x^0|) + \alpha\left(\int_0^t \gamma(|u(s)|)\, ds\right)$$

for appropriate $\mathcal{K}_\infty$ functions (note the additional "α"). See [3] for more discussion on how this estimate is reached, as well as the following result:

Theorem 45.7: [3]

A system satisfies a weak integral-to-integral estimate if and only if it is iISS.

Another interesting variant is found when considering mixed *integral/supremum* estimates:

$$\underline{\alpha}(|x(t)| \leq \beta(|x^0|, t) + \int_0^t \gamma_1(|u(s)|)\, ds + \gamma_2(\|u\|_\infty)$$

for suitable $\beta \in \mathcal{KL}$ and $\underline{\alpha}, \gamma_i \in \mathcal{K}_\infty$. One then has:

Theorem 45.8: [3]

A system satisfies a mixed estimate if and only if it is iISS.

45.5.2 Dissipation Characterization of iISS

Recall that a storage function is a continuous $V : \mathbb{R}^n \to \mathbb{R}$ which is positive definite and proper. A smooth storage function V is an *iISS-Lyapunov function* for the system $\dot{x} = f(x, u)$ if there are a $\gamma \in \mathcal{K}_\infty$ and an

$\alpha : [0, +\infty) \to [0, +\infty)$ which is merely *positive definite* (i.e., $\alpha(0) = 0$ and $\alpha(r) > 0$ for $r > 0$) such that the inequality

$$\dot{V}(x, u) \leq -\alpha(|x|) + \gamma(|u|) \qquad \text{(L-iISS)}$$

holds for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. To compare, recall that an ISS-Lyapunov function is required to satisfy an estimate of the same form but where α is required to be of class $\mathcal{K}_\infty$; since every $\mathcal{K}_\infty$ function is positive definite, an ISS-Lyapunov function is also an iISS-Lyapunov function.

Theorem 45.9: [2]

A system is iISS if and only if it admits a smooth iISS-Lyapunov function.

Since an ISS-Lyapunov function is also an iISS one, ISS implies iISS. However, iISS is a strictly weaker property than ISS, because α may be bounded in the iISS-Lyapunov estimate, which means that V may increase, and the state become unbounded, even under bounded inputs, so long as $\gamma(|u(t)|)$ is larger than the range of α. This is also clear from the iISS definition, since a constant input with $|u(t)| = r$ results in a term in the right-hand side that grows like rt.

An interesting general class of examples is given by *bilinear* systems

$$\dot{x} = \left(A + \sum_{i=1}^m u_i A_i \right) x + Bu$$

for which the matrix A is Hurwitz. Such systems are always iISS (see [23]), but they are not in general ISS. For instance, in the case when $B = 0$, boundedness of trajectories for all constant inputs already implies that $A + \sum_{i=1}^m u_i A_i$ must have all eigenvalues with nonpositive real part, for all $u \in \mathbb{R}^m$, which is a condition involving the matrices A_i (e.g., $\dot{x} = -x + ux$ is iISS but it is not ISS).

The notion of iISS is useful in situations where an appropriate notion of detectability can be verified using LaSalle-type arguments. There follow two examples of theorems along these lines.

Theorem 45.10: [2]

A system is iISS if and only if it is 0-GAS and there is a smooth storage function V such that, for some $\sigma \in \mathcal{K}_\infty$:

$$\dot{V}(x, u) \leq \sigma(|u|)$$

for all (x, u).

The sufficiency part of this result follows from the observation that the 0-GAS property by itself already implies the existence of a smooth and positive-definite, but not necessarily proper, function V_0 such that $\dot{V}_0 \leq \gamma_0(|u|) - \alpha_0(|x|)$ for all (x, u), for some $\gamma_0 \in \mathcal{K}_\infty$ and positive-definite α_0 (if V_0 were proper, then it would be an iISS-Lyapunov function). Now one uses $V_0 + V$ as an iISS-Lyapunov function (V provides properness).

Theorem 45.11: [2]

A system is iISS if and only if there exists an output function $y = h(x)$ (continuous and with $h(0) = 0$), which provides zero-detectability ($u \equiv 0$ and $y \equiv 0 \Rightarrow x(t) \to 0$) and dissipativity in the following

sense: there exists a storage function V and $\sigma \in \mathcal{K}_\infty$, α *positive definite, so that:*

$$\dot{V}(x, u) \leq \sigma(|u|) - \alpha(h(x))$$

holds for all (x, u).

The paper [3] contains several additional characterizations of iISS.

45.5.3 Superposition Principles for iISS

There are also AG characterizations for iISS. A system is *bounded energy weakly converging state (BEWCS)* if there exists some $\sigma \in \mathcal{K}_\infty$ so that the following implication holds:

$$\int_0^{+\infty} \sigma(|u(s)|) \, ds < +\infty \implies \liminf_{t \to +\infty} \left| x(t, x^0, u) \right| = 0 \qquad \text{(BEWCS)}$$

(more precisely: if the integral is finite, then $t_{\max}(x^0, u) = +\infty$ and the lim inf is zero). It is *bounded energy frequently bounded state (BEFBS)* if there exists some $\sigma \in \mathcal{K}_\infty$ so that the following implication holds:

$$\int_0^{+\infty} \sigma(|u(s)|) \, ds < +\infty \implies \liminf_{t \to +\infty} \left| x(t, x^0, u) \right| < +\infty \qquad \text{(BEFBS)}$$

(again, meaning that $t_{\max}(x^0, u) = +\infty$ and the lim inf is finite).

Theorem 45.12: [1]

The following three properties are equivalent for any given system $\dot{x} = f(x, u)$:

- *The system is iISS*
- *The system is BEWCS and 0-stable*
- *The system is BEFBS and 0-GAS*

45.6 Output Notions

Until now, the chapter discussed only stability of states with respect to inputs. For systems with outputs $\dot{x} = f(x, u)$, $y = h(x)$, several new notions can be introduced.

45.6.1 Input-to-Output Stability

If one simply replaces states by outputs in the left-hand side of the estimate defining ISS, there results the notion of *input-to-output stability (IOS)*: there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\left| y(t) \right| \leq \beta(|x^0|, t) + \gamma \left(\|u\|_\infty \right) \qquad \text{(IOS)}$$

holds for all solutions, where $y(t) = h(x(t, x^0, u))$. (Meaning that the estimate is valid for all inputs $u(\cdot)$, all initial conditions x^0, and all $t \geq 0$, and imposing as a requirement that the system be forward complete, that is, $t_{\max}(x^0, u) = \infty$ for all initial states x^0 and inputs u.) As earlier, $x(t)$, and hence $y(t) = h(x(t))$, depend only on past inputs ("causality"), so one could have used just as well simply used the supremum of $|u(s)|$ for $s \geq t$ in the estimate.

A system is *bounded-input bounded-state stable (BIBS)* if, for some $\sigma \in \mathcal{K}_\infty$, the following estimate

$$|x(t)| \leq \max \left\{ \sigma(|x^0|), \ \sigma(\|u\|_\infty) \right\}$$

holds along all solutions. (Note that forward completeness is a consequence of this inequality, even if it is only required on maximal intervals, since the state is upper bounded by the right-hand side expression.)

An *IOS-Lyapunov function* is any smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that, for some $\alpha_i \in \mathcal{K}_\infty$:

$$\alpha_1(|h(x)|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

and, for all x, u:

$$V(x) > \alpha_3(|u|) \implies \nabla V(x) f(x, u) < 0.$$

One of the key results for IOS is as follows:

Theorem 45.13: [30]

A BIBS system is IOS if and only if it admits an IOS-Lyapunov function.

45.6.2 Detectability and Observability

Recall (see [24] for precise definitions) that an *observer* for a given system with inputs and outputs $\dot{x} = f(x, u)$, $y = h(x)$ is another system which, using only information provided by past input and output signals, provides an asymptotic (i.e., valid as $t \to \infty$) estimate $\hat{x}(t)$ of the state $x(t)$ of the system of interest (Figure 45.9).

One may think of the observer as a physical system or as an algorithm implemented by a digital computer. The general problem of building observers is closely related to "incremental" ISS-like notions, a subject not yet studied enough. This chapter will limit itself to an associated but easier question. When the ultimate goal is that of stabilization to an equilibrium, say $x = 0$ in Euclidean space, sometimes a weaker type of estimate suffices: it may be enough to obtain a *norm-estimator* which provides merely an *upper bound* on the norm $|x(t)|$ of the state $x(t)$; see [9,11,19].

Suppose that an observer exists for a given system. Since $x^0 = 0$ is an equilibrium for $\dot{x} = f(x, 0)$, and also $h(0) = 0$, the solution $x(t) \equiv 0$ is consistent with $u \equiv 0$ and $y \equiv 0$. Thus, the estimation property $\hat{x}(t) - x(t) \to 0$ implies that $\hat{x}(t) \to 0$. Now consider *any* state x^0 for which $u \equiv 0$ and $y \equiv 0$, that is, so that $h(x(t, x^0, 0)) \equiv 0$. The observer output, which can only depend on u and y, must be the same $\hat{x}$ as when $x^0 = 0$, so $\hat{x}(t) \to 0$; then, using once again the definition of observer $\hat{x}(t) - x(t, x^0, 0) \to 0$, it follows that $x(t, x^0, 0) \to 0$. In summary, a *necessary* condition for the existence of an observer is that the "subsystem" of $\dot{x} = f(x, u)$, $y = h(x)$, consisting of those states for which $u \equiv 0$ produces the output $y \equiv 0$, must have $x = 0$ as a GAS state (Figure 45.10); one says in that case that the system is *zero detectable*. (For *linear* systems, zero detectability is equivalent to detectability or "asymptotic observability" [24]: two trajectories than produce the same output must approach each other. But this equivalence need not hold for nonlinear systems.) In a nonlinear context, zero detectability is not "well-posed" enough: to get a well-behaved notion, one should add explicit requirements to ask that small inputs and outputs imply that internal states are small too (Figure 45.11), and that inputs and outputs converging to zero
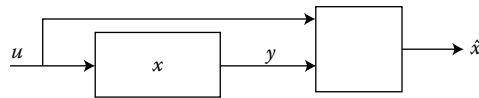

FIGURE 45.9 Observer provides estimate $\hat{x}$ of state x; $\hat{x}(t) - x(t) \to 0$ as $t \to \infty$.
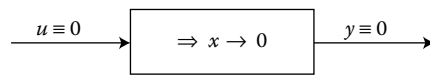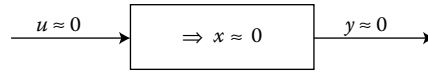

FIGURE 45.10 Zero detectability.

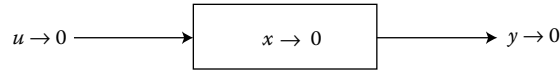**FIGURE 45.11**　Small inputs and outputs imply small states.

FIGURE 45.12　Converging inputs and outputs imply convergent states.

as $t \to \infty$ implies that states do, too (Figure 45.12). These properties are needed so that "small" errors in measurements of inputs and outputs processed by the observer give rise to small errors. Furthermore, one should impose asymptotic bounds on states as a function of input/output bounds, and it is desirable to quantify "overshoot" (transient behavior). This leads us to the following notion.

45.6.3 Dualizing ISS to OSS and IOSS

A system is *input/output to state stable (IOSS)* if, for some $\beta \in \mathcal{KL}$ and $\gamma_u, \gamma_y \in \mathcal{K}_\infty$,

$$x(t) \leq \beta(|x^0|, t) + \gamma_1 \left(\left\| u_{[0,t]} \right\|_\infty \right) + \gamma_2 \left(\left\| y_{[0,t]} \right\|_\infty \right) \qquad \text{(IOSS)}$$

for all initial states and inputs, and all $t \in [0, T_{\xi,u})$. Just as ISS is stronger than 0-GAS, IOSS is stronger than zero detectability. A special case is when one has no inputs, *output to state stability:*

$$|x(t, x^0)| \leq \beta(|x^0|, t) + \gamma \left(\left\| y|_{[0,t]} \right\|_\infty \right)$$

and this is formally "dual" to ISS, simply replacing inputs u by outputs in the ISS definition. This duality is only superficial, however, as there seems to be no useful way to obtain theorems for OSS by dualizing ISS results. (Note that the outputs y depend on the state, not vice versa.)

45.6.4 Lyapunov-Like Characterization of IOSS

To formulate a dissipation characterization, we define an *IOSS-Lyapunov function* as a smooth storage function so that

$$\nabla V(x) f(x, u) \leq -\alpha_1(|x|) + \alpha_2(|u|) + \alpha_3(|y|)$$

for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$. The main result is

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Theorem 45.14: [13]

A system is IOSS if and only if it admits an IOSS-Lyapunov function.

45.6.5 Norm-Estimators

A *state-norm-estimator* (or *state-norm-observer*) for a given system is another system

$$\dot{z} = g(z, u, y), \quad \text{with output } \ k : \mathbb{R}^\ell \times \mathbb{R}^p \to \mathbb{R}_{\geq 0}$$

evolving in some Euclidean space $\mathbb{R}^\ell$, and driven by the inputs and outputs of the original system. It is required that the output k should be IOS with respect to the inputs u and y, and the true state should be asymptotically bounded in norm by some function of the norm of the estimator output, with a transient (overshoot) which depends on both initial states. Formally

- there are $\hat{\gamma}_1, \hat{\gamma}_2 \in \mathcal{K}$ and $\hat{\beta} \in \mathcal{KL}$ so that, for each initial state $z^0 \in \mathbb{R}^\ell$, and inputs $\mathbf{u}$ and $\mathbf{y}$, and every t in the interval of definition of the solution $z(\cdot, z^0, \mathbf{u}, \mathbf{y})$:

$$k\left(z(t, z^0, \mathbf{u}, \mathbf{y}), \mathbf{y}(t)\right) \leq \hat{\beta}(|z^0|, t) + \hat{\gamma}_1\left(\|\mathbf{u}|_{[0,t]}\|\right) + \hat{\gamma}_2\left(\|\mathbf{y}|_{[0,t]}\|\right);$$

- there are $\rho \in \mathcal{K}$, $\beta \in \mathcal{KL}$ so that, for all initial states x^0 and z^0 of the system and observer, and every input $\mathbf{u}$:

$$\left|x(t, x^0, \mathbf{u})\right| \leq \beta(|x^0| + |z^0|, t) + \rho\left(k\left(z(t, z^0, \mathbf{u}, \mathbf{y}_{x^0, \mathbf{u}}), \mathbf{y}_{x^0, \mathbf{u}}(t)\right)\right)$$

for all $t \in [0, t_{\max}(x^0, \mathbf{u}))$, where $\mathbf{y}_{x^0, \mathbf{u}}(t) = y(t, x^0, \mathbf{u})$.

Theorem 45.15: [13]

A system admits a state-norm-estimator if and only if it is IOSS.

45.7 The Fundamental Relationship among ISS, IOS, and IOSS

The definitions of the basic ISS-like concepts are consistent and related in an elegant conceptual manner, as follows:
A system is ISS if and only if it is both IOS and IOSS.
In informal terms:

$$\text{External stability and detectability} \iff \text{Internal stability}$$

as it is the case for linear systems. Intuitively, consider the three possible signals in Figure 45.13

The basic idea of the proof is as follows. Suppose that external stability and detectability hold, and take an input so that $u \to 0$. Then $y \to 0$ (by external stability), and this then implies that $x \to 0$ (by detectability). Conversely, if the system is internally stable, then it is i/o stable and detectable. Suppose that $u \to 0$. By internal stability, $x \to 0$, and this gives $y(t) \to 0$ (i/o stability). Detectability is even easier: if both $u(t) \to 0$ and $y(t) \to 0$, then in particular $u \to 0$, so $x \to 0$ by internal stability. The proof that ISS is equivalent to the conjunction of IOS and IOSS must keep careful track of the estimates, but the idea is similar.

45.8 Response to Constant and Periodic Inputs

Systems $\dot{x} = f(x, u)$ that are ISS have certain noteworthy properties when subject to constant or, more generally periodic, inputs.

Let V be an ISS-Lyapunov function that satisfies the inequality $\dot{V}(x, u) \leq -V(x) + \gamma(|u|)$ for all x, u, for some $\gamma \in \mathcal{K}_\infty$. To start with, suppose that $\bar{u}$ is any fixed bounded input, and let $a := \gamma(\|\bar{u}\|_\infty)$, pick any initial state x^0, and consider the solution $x(t) = x(t, x^0, \bar{u})$ for this input. Letting $v(t) := V(x(t))$, it follows that $\dot{v}(t) + v(t) \leq a$ so, using e^t as an integrating factor, $v(t) \leq a + e^{-t}(v(0) - a)$ for all $t \geq 0$. In particular, if $v(0) \leq a$ it will follow that $v(t) \leq a$ for all $t \geq 0$, that is to say, the sublevel set $K := \{x \mid V(x) \leq a\}$ is a forward-invariant set for this input: if $x^0 \in K$ then $x(t) = x(t, x^0, \bar{u}) \in K$ for all $t \geq 0$. Therefore,
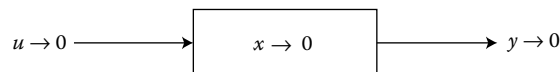

FIGURE 45.13 Convergent input, state, and/or output.

$M_T : x^0 \mapsto x(T, x^0, \bar{u})$ is a continuous mapping from K into K, for each fixed $T > 0$, and thus, provided that K has a fixed-point property (every continuous map $M : K \to K$ has some fixed point), then for each $T > 0$ there exists some state x^0 such that $x(T, x^0, \bar{u}) = x^0$. The set K indeed has the fixed-point property, as does any sublevel set of a Lyapunov function. To see this, note that V is a Lyapunov function for the zero-input system $\dot{x} = f(x, 0)$, and thus, if B is any ball which includes K in its interior, then the map $Q : B \to K$ which sends any $\xi \in B$ into $x(t_\xi, \xi)$, where t_ξ is the first time such that $x(t, \xi) \in K$ is continuous (because the vector field is transversal to the boundary of K since $\nabla V(x).f(x, 0) < 0$), and is the identity on K (that is, Q is a topological retraction). A fixed point of the composition $M \circ Q : B \to B$ is a fixed point of M.

Now suppose that $\bar{u}$ is periodic of period T, $\bar{u}(t + T) = \bar{u}(t)$ for all $t \geq 0$, and pick any x^0 which is a fixed point for M_T. Then the solution $x(t, x^0, \bar{u})$ is periodic of period T as well. In other words, *for each periodic input, there is a solution of the same period*. In particular, if $\bar{u}$ is constant, one may pick for each $h > 0$ a state x_h so that $x(h, x_h, \bar{u}) = x_h$, and therefore, picking a convergent subsequence $x_h \to \bar{x}$ gives that $0 = (1/h)(x(h, x_h, \bar{u}) - x_h) \to f(\bar{x}, \bar{u})$, so $f(\bar{x}, \bar{u}) = 0$. Thus one also has the conclusion that *for each constant input, there is a steady state.*

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