An observability result related to active sensing

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1 Abstract

For a general class of translationally invariant systems with a specific category of nonlinearity in the output, this paper presents necessary and sufficient conditions for global observability. Critically, this class of systems cannot be stabilized to an isolated equilibrium point by dynamic output feedback. These analyses may help explain the active sensing movements made by animals when they perform certain motor behaviors, despite the fact that these active sensing movements appear to run counter to the primary motor goals. The findings presented here establish that active sensing underlies the maintenance of observability for such biological systems, which are inherently nonlinear due to the presence of the high-pass sensor dynamics.

2 Introduction

Active sensing is the process of expending energy, typically through movement, for the purpose of sensing [1–3]. Animals use this strategy to enhance sensory information across sensory modalities e.g., echolocation [4,5], whisking [6,7] and other forms of touch [8,9], electrosense [10–12], and vision [13,14]. It is well established that conditions of decreased sensory acuity leads to increased active movements [5,11,12,14–21] but its actual role in relation to task-level control remains underexplored. The ubiquity of active sensing in nature motivates us to explore the mathematical conditions that might necessitate active sensing. Our theory is that active sensing is at least in part borne out of the needs of nonlinear state estimation. We hypothesize that animals—through active sensing—generate time-varying motor commands that continuously stimulate their sensory receptors so that the system states can be estimated with satisfactory error bounds from the sensor measurements. In essence, these movements aim to maintain the observability of the system.

A dominant paradigm in control systems engineering involves designing state feedback and state estimation independently, an approach can be applied successfully to a wide range of system designs. Indeed, for linear plants corrupted by Gaussian noise, there is a separation principle: it is not only satisfactory to separate state estimation from the task-level control design, but, in fact, it is *optimal* to perform this decomposition. In particular, the linear-quadratic-Gaussian (LQG) controller decomposes into a linear-quadratic regulator (LQR) applied to the optimal state estimate which comes from a Kalman filter. Critically, the Kalman filter does not depend on the LQR cost function, and the LQR gains do not depend on the sensor noise and process noise. Conceptually speaking, “active sensing” is the opposite approach to applying a separation principle: control inputs are specifically designed to excite sensory receptors, presumably in service to the state estimator. This may be, at least in part, because biological sensory systems often stop responding to persistent (i.e. "DC") stimuli, via sensory "adaptation" [22–26] or "perceptual fading" [27, 28].

In this paper, we formalize a class of nonlinear systems that have a simple high-pass sensory output that mimics sensory adaptation or perceptual fading. Under some simplified modeling assumptions, reviewed below, this implies that linear observability is lost, which means the usual LQG-style framework does not apply. However, under some interesting modeling conditions, nonlinear observability persists. Critically, nonlinear observability does not necessarily afford a separation principle: the control signal may need to contain ancillary energy that is expressly for the purpose of state estimation, and may be in conflict with task goals. Indeed, the energy expended for active sensing movements do not necessarily directly serve a motor control goal, and are instead believed to improve sensory feedback and prevent perceptual fading [12, 27, 28].

The organization of the paper is as follows. Section 3 motivates the model structure from prior work and Section 4 generalizes the model and presents the main theorem. Section 5 has the proof of the main theorem. The Appendix provides some background concepts for the ease of understanding the tools used in Section 3 and 5.

3 Biological motivation and simplified system

Station keeping behavior in weakly electric fish, *Eigenmannia virescens*, provides an ideal system for investigating the interplay between active sensing and task-level control [11, 12, 29, 30]. These fish routinely maintain their position relative to a moving refuge and uses both vision and electrosense to collect the necessary sensory information from its environment [31–34]. While tracking the refuge position (i.e., task-level control), the fish additionally produce rapid "whisking-like" forward and backward swimming movements (i.e., active sensing). When vision is limited (for example, in darkness), the fish increase their active sensing movements [12, 30]. This increased motion compensates the lack of visual cues [11].

Suppose x is the position of an animal and $z = \dot{x}$ is its velocity as it moves in one degree of freedom. We assume that a sensory receptor measures only the local rate of change of a stimulus, $s(x)$ as the animal moves relative to the sensory scene, i.e. $y = \dfrac{d}{dt}s(x)$. Defining $\gamma(x) := \dfrac{d}{dx}s(x)$, we arrive at a 2-dimensional, single-input, single-output normalized mass-damper system of the following form [35, 36]:

$$
\begin{aligned}
\dot{x} &= z, && x \in \mathbb{R} \\
\dot{z} &= -z + u, && z, u \in \mathbb{R} \\
y &= \frac{d}{dt}s(x) = \gamma(x)\, z, && y \in \mathbb{R}
\end{aligned}
\tag{1}
$$

where the mass and the damping constant both are assumed to be unity. Linearization of the above system (1) around any equilibrium, $(x^*, 0)$, is given by (A, B, C) as follows:

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \gamma^* \end{bmatrix},
$$

where $\gamma^* = \gamma(x^*)$. Clearly (A,C) is not observable irrespective of γ^* [37]. Indeed, the output introduces a zero at the origin that cancels a pole at the origin, rendering x unobservable. Assuming no input u, we can write the system (1) as,

$$
\dot{\xi} = f(\xi), \quad y = h(\xi), \tag{2}
$$

where $\xi = (x,\ z)^\top$, $f = (z,\ -z)^\top$ and $h(\xi) = \gamma(x)z$. We can construct the observation space, $\mathcal{O}$ (set of all infinitesimal observables) by taking $y = \gamma(x)z$ with all repeated time derivatives

$$
y^{(k)} = L_f^{(k)}(\gamma(x)z)
$$

as in [38, 39]. The superscript "(k)" indicates kth order derivative. Note that $L_f^{(k)}((\gamma(x)z))$ lies in the span of the functions $\gamma^{(j)}(x)z^{j+1}$, $j = 0, 1, \ldots, k$. The rank condition on the observability codistribution [38, 39] implies a sufficient condition for local observability as follows [36]:

$$z^2(2(\gamma'(x))^2 - \gamma(x)\gamma''(x)) \neq 0. \tag{3}$$

Clearly for an non-hyperbolic γ ($\neq 1/(\alpha x + \beta)$, with constants of integration α, β), the non-zero velocity requirement ($z \neq 0$) implies the need for active sensing to maintain the local observability of the system [36, 40].

What does this simplified example say about the need for active sensing? As the proposition below illustrates, for such a system, dynamic output feedback cannot asymptotically stabilize the origin $(0, 0)$, indicating the need for extra inputs to perform active state estimation:

Proposition 3.1 Consider the system (1). Let

$$\begin{aligned} \dot{q} &= g(q, y) \\ u &= k(y, q) \end{aligned} \tag{4}$$

be a dynamic output feedback (Fig 1). Suppose $(x^*, z^*, q^*) = (0, 0, q^*)$ is an equilibrium of the coupled system. Then all points $(\xi^*, 0, q^*)$, $\xi^* \in \mathbb{R}$, are equilibiria. □

Proof. Since $(0, 0, q^*)$ is an equilibrium, we see from the second equation in (1) that $k(0, q^*) = 0$. That means that $k(\gamma(\xi^*) \cdot 0, q^*) = 0$, i.e. $(\xi^*, 0, q^*)$ is also an equilibrium, for all $\xi^* \in \mathbb{R}$. ∎

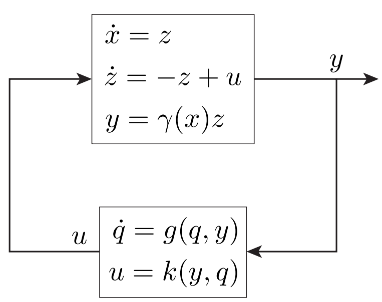

Figure 1: The system (1) cannot be stabilized to an equilibrium point by the dynamic feedback in (4).

Remark 3.2 The impossibility of stabilizing a system with high-pass sensing to an equilibrium point, using only dynamic output feedback, generalizes to the class of systems described below. □

4 The class of systems and main result

Now we consider a general $2n$-dimensional, single-input, n-output system of the following form:

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= F(z) + bu \\ y &= H(x)\, z \end{aligned}$$

where the state space variable $\xi \in \mathbb{R}^{2n}$ is partitioned as $\xi = (x^\top, z^\top)^\top$ into variables $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$, and $H(x) = \operatorname{diag}(\gamma_1(x_1), \ldots, \gamma_n(x_n))$ is a diagonal $n \times n$ matrix. The entries of the column vector function $F(z)$ as well the functions $\gamma_i(x_i)$ are real-analytic functions of their arguments, and b is a column vector of size n all whose entries are nonzero.

Let us introduce the following notations:

$$
\begin{aligned}
f(\xi) &:= \begin{pmatrix} z \\ F(z) \end{pmatrix} \\
g &:= \begin{pmatrix} 0 \\ b \end{pmatrix} \\
h(\xi) &:= (h_1(\xi), \ldots, h_n(\xi))^\top \;=\; (\gamma_1(x_1)\, z_1 \ldots, \gamma_n(x_n)\, z_n)^\top ,
\end{aligned}
$$

so that, in standard form for nonlinear systems (see e.g. [38]), our system becomes

$$\dot{\xi} = f(\xi) + gu,\; y = h(\xi)\,. \tag{5}$$

Given an input $u : [0, \infty) \to \mathbb{R}$ and an initial state ξ_0, we denote by $\varphi_{\xi_0,u}(t)$ the solution of (5), that is, $(d/dt)\varphi_{\xi_0,u}(t) = f(\varphi_{\xi_0,u}(t)) + gu(t)$ and $\varphi(0) = \xi_0$. Note that $\varphi_{\xi_0,u}(t)$ is defined on some nontrivial time interval $[0, T)$ containing $t = 0$.

Remark 4.1 Our results will hold in total generality, for inputs $u(\cdot)$ assumed to be Lebesgue measurable locally bounded functions, in which case a solution is an absolutely continuous function and the equality $\dot{\xi}(t) = f(\xi(t)) + gu(t)$ holds almost-everywhere. If the reader prefers, one can restrict to inputs u which are piecewise continuous functions of time (with well-defined one-sided limits at points of discontinuity) and solutions are piecewise differentiable functions. □

We recall that two states ξ_0 and $\hat{\xi}_0$ are said to be *distinguishable* (by input/output measurements) if there is some input $u(\cdot)$ and some time t such that $h(\varphi_{\xi_0,u}(t)) \neq h(\varphi_{\hat{\xi}_0,u}(t))$, and that the system (5) is said to be *observable* provided that every pair of distinct states is distinguishable. (An appendix reviews characterizations of observability in terms of Lie derivatives, and these results are used in proofs.)

We will say that a function $\theta : \mathbb{R} \to \mathbb{R}$ is *periodic* if there is some nonzero $T \in \mathbb{R}$ (a "period") such that $\theta(x) = \theta(x + T)$ for all $x \in \mathbb{R}$, An *aperiodic* function is one that is not periodic.

We will say that the system (5) is *aperiodic* if none of the functions γ_i are periodic.

Our main result is as follows:

Theorem 1 *The system* (5) *is observable if and only if it is aperiodic.*

The proof of Theorem follows immediately from Lemmas 5.3 and 5.4.

5 Proof of the main result

The key property that we need is as follows.

Lemma 5.1 For each integer $k \geq 0$, and each $i = 1, \ldots, n$, the following formulas hold:

$$(L_f L_g)^k h_i(\xi) \;=\; \gamma_i^{(k)}(x_i)\, b_i^k\, z_i \tag{6}$$

$$L_g (L_f L_g)^k h_i(\xi) \;=\; \gamma_i^{(k)}(x_i)\, b_i^{k+1} \tag{7}$$

(where the superscript "(k)" indicates kth order derivative).

Proof. Fix any $i \in \{1, \ldots, n\}$. For $k = 0$, formula (6) states $h_i(\xi) = \gamma_i(x_i) z_i$. which is true by definition of h_i. Using induction, we will prove that, if formula (6) holds for a given k then (7) holds for the same k, and (6) holds for $k + 1$. Suppose that (6) is true. Since

$$L_g (L_f L_g)^k h_i \;=\; \nabla[(L_f L_g)^k h_i] \cdot g$$

it follows that

$$L_g(L_fL_g)^k h_i \;=\; \nabla[\gamma_i^{(k)}(x_i)\, b_i^k\, z_i]\cdot g \;=\; \begin{pmatrix} \gamma_i^{(k+1)}(x_i)\, b_i^k\, z_i\, e_i \\ \gamma_i^{(k)}(x_i)\, b_i^k\, e_i \end{pmatrix}^{\top} \begin{pmatrix} 0 \\ b \end{pmatrix} \;=\; \gamma_i^{(k)}(x_i)\, b_i^{k+1}$$

where e_i is the n-vector with a "1" in position i and zeroes elsewhere . Since

$$(L_fL_g)^{k+1}h_i \;=\; L_f(L_g(L_fL_g)^k h_i) \;=\; \nabla[(L_g(L_fL_g)^k h_i]\cdot f\,,$$

we have that

$$(L_fL_g)^{k+1}h_i \;=\; \nabla[\gamma_i^{(k)}(x_i)\, b_i^{k+1}]\cdot f \;=\; \begin{pmatrix} \gamma_i^{(k+1)}(x_i)\, b_i^{k+1} e_i \\ 0 \end{pmatrix}^{\top} \begin{pmatrix} z \\ F(z) \end{pmatrix} \;=\; \gamma_i^{(k+1)}(x_i)\, b_i^{k+1} z_i\,.$$

This completes the induction step. ∎

Lemma 5.2 Suppose that $\gamma : \mathbb{R} \to \mathbb{R}$ is an aperiodic analytic function. Then, for each two distinct $r, s \in \mathbb{R}$ there is some nonnegative integer k such that $\gamma^{(k)}(r) \neq \gamma^{(k)}(s)$.

Proof. Suppose, by way of contradiction, that there would be some pair $r \neq s$ for which $\gamma^{(k)}(r) = \gamma^{(k)}(s)$ for all $k \geq 0$. Let $T := s - r$, so that $s = r + T$. Consider the function $\beta(x) := \gamma(x+T) - \gamma(x)$, so that

$$\beta^{(k)}(r) \;=\; \gamma^{(k)}(r+T) - \gamma^{(k)}(r) \;=\; \gamma^{(k)}(s) - \gamma^{(k)}(r) \;=\; 0$$

for each $k \geq 0$. Since the function β is analytic, it follows that $\beta(x) \equiv 0$, which means that γ would be periodic of period T. ∎

Lemma 5.3 If the system (5) is aperiodic, then it is observable.

Proof. Consider two different states ξ_0 and $\hat{\xi}_0$. We need to show that these two states are distinguishable. We write these states in partitioned form as follows:

$$\xi_0 \;=\; \begin{pmatrix} x \\ z \end{pmatrix} \;=\; \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ z_1 \\ \vdots \\ z_n \end{pmatrix}, \quad \hat{\xi}_0 \;=\; \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} \;=\; \begin{pmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \\ \hat{z}_1 \\ \vdots \\ \hat{z}_n \end{pmatrix}$$

and consider two possible cases: (a) $x = \hat{x}$, $z \neq \hat{z}$ and (b) $x \neq \hat{x}$.

Consider first case (a), so that $x_i = \hat{x}_i$ for all i. The set

$$\mathcal{I} := \{i \in \{1, \ldots, n\} \,|\, z_i \neq \hat{z}_i\}$$

is nonempty. Pick any $i \in \mathcal{I}$. Since γ_i is not periodic, it follows that $\gamma_i^{(k)}(x_i) \neq 0$ for some nonnegative integer k (which may depend on i). (If all derivatives were zero at a point, analyticity would imply $\gamma_i \equiv 0$, but the zero function is periodic.) For any such i and k, $\gamma_i^{(k)}(x_i) = \gamma_i^{(k)}(\hat{x}_i)$ is nonzero and $z_i \neq \hat{z}_i$, so $\gamma_i^{(k)}(x_i)\, b_i^k z_i \neq \gamma_i^{(k)}(\hat{x}_i)\, b_i^k \hat{z}_i$. From Equation (6) we have then that

$$(L_fL_g)^k h_i(\xi_0) \;=\; \gamma_i^{(k)}(x_i)\, b_i^k z_i \;\neq\; \gamma_i^{(k)}(\hat{x}_i)\, b_i^k \hat{z}_i \;=\; (L_fL_g)^k h_i(\hat{\xi}_0)$$

This means that the infinitesimal observable $(L_fL_g)^k h_i$ separates the states ξ_0 and $\hat{\xi}_0$, so by Lemma A.1 these states are distinguishable, as we claimed.

Next consider case (b). Pick any $i \in \{1, \ldots, n\}$ for which $x_i \neq \hat{x}_i$. By Lemma 5.2, there is some nonnegative integer k such that $\gamma^{(k)}(x_i) \neq \gamma^{(k)}(\hat{x}_i)$. Then

$$L_g(L_f L_g)^k h_i(\xi_0) \;=\; \gamma_i^{(k)}(x_i)\, b_i^{k+1} \;\neq\; \gamma_i^{(k)}(\hat{x}_i)\, b_i^{k+1} \;=\; L_g(L_f L_g)^k h_i(\hat{\xi}_0)$$

implies that the infinitesimal observable $L_g(L_f L_g)^k h_i$ separates the states ξ_0 and $\hat{\xi}_0$, so by Lemma A.1 these states are distinguishable, as we claimed. ∎

Lemma 5.4 If the system (5) is not aperiodic, then it is not observable.

Proof. Since (5) is not aperiodic, there is some $i_0 \in \{1, \ldots, n\}$ and some $T_{i_0} \neq 0$ such that $\gamma_{i_0}(x + T_{i_0}) = \gamma_{i_0}(x)$ for all $x \in \mathbb{R}$. Let T be the n-vector that has this number T_{i_0} as its i_0th coordinate and zero in all other coordinates. Thus, $T \neq 0$, and $\gamma_i(x + T_i) = \gamma_i(x)$ for all i and all $x \in \mathbb{R}$. Consider the following two distinct states ξ_0 and $\hat{\xi}_0$:

$$\xi_0 \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \hat{\xi}_0 \;=\; \begin{pmatrix} T \\ 0 \end{pmatrix},$$

Consider any input $u(\cdot)$ and the respective solutions $\xi(t) = \varphi_{\xi_0,u}(t)$ and $\hat{\xi}(t) = \varphi_{\hat{\xi}_0,u}(t)$. In terms of the x and z components, we have that $z(t) = \hat{z}(t)$ for all $t \geq 0$, because the z component of the system does not depend on the x component and the two initial conditions coincide on their z components. Thus, from

$$x(t) \;=\; \int_0^\infty z(t)\, dt$$

and

$$\hat{x}(t) \;=\; T + \int_0^\infty \hat{z}(t)\, dt \;=\; T + \int_0^\infty z(t)\, dt$$

we conclude that

$$\hat{x}(t) \;=\; T + x(t)$$

for all $t \geq 0$. Since $\gamma_i(x_i + T_i) = \gamma_i(x_i)$ for all i and all $x_i \in \mathbb{R}$, substituting $x_i = x_i(t)$ we have that the output coordinates satisfy

$$\hat{y}_i(t) \;=\; \gamma_i(\hat{x}_i(t))\, \hat{z}_i(t) \;=\; \gamma_i(x_i(t))\, z_i(t) \;=\; \gamma_i(x_i(t))\, z_i(t) \;=\; y(t)$$

for all $t \geq 0$. This means that ξ_0 and $\hat{\xi}_0$ are indistinguishable. ∎

Remark 5.5 We stated Lemmas 5.3 and 5.4 separately because the latter one does not require real-analyticity of any of the functions γ_i, and also holds for a more general class of systems, namely $\dot{x} = K(z)$, $\dot{z} = F(z, u)$. No regularity properties whatsoever are required, except for existence and uniqueness of solutions of the differential equations. □

6 Conclusion

Design of output feedback controllers commonly relies on the separation principle, which allows designers to independently design observers (based on sensor inputs) and controllers (designed assuming full state measurements). In biological systems, this requirement for separability may be violated. Specifically, high-pass sensing (which we use to model adapting or perceptual fading sensory systems) causes loss of observability for a class of systems. This manuscript presents necessary and sufficient conditions for global observability for a class of nonlinear systems with high-pass sensors. Though the system structure was motivated by the locomotion dynamics of weakly electric fish, it can be adopted to model behaviors of other animals with translationally invariant plant dynamics and appropriately modeled output measurements.

The goal of this work is to elucidate conditions that guarantee the existence of inputs so that any two states can be distinguished, but a characterization of "good" inputs that make the system "sufficiently" observable remains an open question. Observability and its dual, controllability are generic properties of a system, although in practice are not always realizable due to practical reasons, such as numerical conditioning [41]. Thus, it remains unclear how best to design output feedback systems that achieve a task-level control objective given the inseparability of control and state estimation; a significant challenge in control engingeering is to design a common framework to address both active sensing and task-level control in a single design framework.

A Relevant concepts related to nonlinear observability

We review here a test for observability based on the "infinitesimal observables" associated to a system.

Given any differentiable function $\alpha : \mathbb{R}^n \to \mathbb{R}$ and any vector field X, one defines the *Lie derivative of α along X* as the new function with values

$$L_g\alpha \,:\, \mathbb{R}^n \to \mathbb{R} \,:\, x \mapsto \nabla\alpha(x) \cdot X(x)$$

where $\nabla\alpha$ is the gradient of α and "$\cdot$" indicates the dot or inner product. (This is the same as what in elementary calculus is called the "directional derivative" of the function α in the direction of X.) This operation is multilinear: $L_{X+Y}\alpha = L_X\alpha + L_Y\alpha$ and $L_X(\alpha+\beta) = L_X\alpha + L_x\beta$.

Consider a (generally multi-input multi-output) system in which inputs appear linearly:

$$\dot{x} \;=\; g_0(x) \,+\, \sum_{i=1}^{m} g_i(x)\, u_i \tag{8}$$

and with p outputs $y_j = h_j(x)$, $j = 1, \ldots, p$ (in our application, $m = 1$, $p = n$, $g_0 = f$, $g_1 = g$). We only assume at first that all vector fields (that is, vector functions) g_i as well as the functions h_j are infinitely differentiable; later we impose real-analyticity (convergent power series around each state).

For any vector of nonnegative integers (though of as indices of vector fields)

$$\mu = (\mu_k, \ldots, \mu_1) \in \{0, \ldots, m\}^k$$

with $k \geq 1$, we define the *infinitesimal observable* function

$$L_\mu h : \mathbb{R}^n \to \mathbb{R} \tag{9}$$

by the iteration $L_{g_{\mu_k}}(L_{g_{\mu_{k-1}}}(\ldots(L_{g_{\mu_1}}h)\ldots))$, which we also write as $L_{g_{\mu_k}}L_{g_{\mu_{k-1}}}\ldots L_{g_{\mu_1}}h$. We use power notation in the obvious form; for example, $L^0_{g_1}h$ is just h, and $L^2_{g_1}h$ is the same as $L_{(1,1)}h$, that is, $L_{g_1}(L_{g_1}h)$. Finally, we let $\mathcal{O}$ be the set of all infinitesimal observables.

We say that two states ξ and $\hat{\xi}$ are *separated by* $\mathcal{O}$ if there exists some $\alpha \in \mathcal{O}$ such that $\alpha(\xi) \neq \alpha(\hat{\xi})$. We have the following well-known fact (see e.g. [38] and references there), which we prove here for ease of reference.

Lemma A.1 If two states are separated by $\mathcal{O}$, then they are distinguishable.

Proof. We prove the contrapositive: if two states ξ and $\hat{\xi}$ are not distinguishable, then they cannot be separated by $\mathcal{O}$, that is, $\alpha(\xi) = \alpha(\hat{\xi})$ for all $\alpha \in \mathcal{O}$. Pick ξ and $\hat{\xi}$ that are not distinguishable, and consider a piecewise constant input on k intervals: $u(\cdot)$ has a constant value u^1 on an interval $[0, t_1)$, a constant value u^2 on $[t_1, t_1+t_2)$, $\ldots$, and u^k on $[t_1+\ldots+t_{k-1}, t_1+\ldots+t_k)$. For small enough t_i's there is a solution of the differential equation from both initial conditions ξ and $\hat{\xi}$. Since these two states are indistinguishable, the resulting output at time $t = t_1 + \ldots + t_k$ is the same, when starting from either initial state. In general, let us denote the jth coordinate of this output value by

$$h_j(t_1, t_2, \ldots, t_k, u^1, u^2, \ldots, u^k, \widetilde{\xi}) \tag{10}$$

when the initial state is $\widetilde{\xi}$. It follows that the derivatives with respect to the t_i's of this output are also equal, for ξ and $\hat{\xi}$, for every such piecewise constant input. One may prove by induction that

$$\left.\frac{\partial^k}{\partial t_1 \ldots \partial t_k}\right|_{t_1=t_2=\ldots=0} h_j(t_1, t_2, \ldots, t_k, u^1, u^2, \ldots, u^k, \widetilde{\xi}) \;=\; L_{X_1} L_{X_2} \ldots L_{X_k} h_j(\widetilde{\xi})$$

where $X_l(x) = g_0(x) + \sum_{i=1}^m u_i^l g_i(x)$. In summary,

$$L_{X_1} L_{X_2} \ldots L_{X_k} h_j(\xi) \;=\; L_{X_1} L_{X_2} \ldots L_{X_k} h_j(\hat{\xi})$$

for any k and any vector $(u^1, \ldots, u^k) \in \mathbb{R}^k$. Using multilinearity, $L_{X_1} L_{X_2} \ldots L_{X_k} h_j(x)$ can be expanded as a polynomial on the $u^1, \ldots, u^k$ whose coefficients are exactly the elementary observables. For example, if $k = 2$ and $m = 1$,

$$\begin{aligned}
L_{X_1} L_{X_2} h_j &= L_{g_0+u^1 g_1}(L_{g_0+u^2 g_1} h_j) \\
&= L_{g_0}(L_{g_0+u^2 g_1} h_j) + u^1 L_{g_1(x)}(L_{g_0+u^2 g_1} h_j) \\
&= L_{g_0}(L_{g_0} h_j + u^2 L_{g_1} h_j) + u^1 L_{g_1(x)}(L_{g_0} h_j + u^2 L_{g_1} h_j) \\
&= L_{g_0}^2 h_j + u^1 L_{g_1} L_{g_0} h_j + u^2 L_{g_0} L_{g_1} h_j + u^1 u^2 L_{g_1}^2 h_j \,.
\end{aligned}$$

Since two polynomial functions are equal if and only if their coefficients of equal powers are equal, it follows that $L_\mu h_j(\xi) = L_\mu h_j(\hat{\xi})$ for all k and all indices μ, and hence these states cannot be separated by $\mathcal{O}$. ∎

We'll say that *observables separate states* if any two states can be separated by $\mathcal{O}$. A consequence of the above is:

Corollary A.2 If observables separate states, then the system is observable. □

To rephrase the corollary, a sufficient condition for observability is that the mapping

$$\xi \;\mapsto\; \{\alpha(\xi), \alpha \in \mathcal{O}\}\,,$$

which sends each state into the infinite sequence of possible observables evaluated at that state, be one-to-one.

This condition is also necessary when all the functions are real-analytic:

Theorem 2 *Suppose that the vector fields g_i as well as the functions h_j are real analytic. Then, the following two properties are equivalent:*

1. *The system is observable.*
2. *Observables separate states.*

Proof. One implication is given by Corollary A.2. To prove the converse, we need to show that if observables do not separate states then the system is not observable. Indeed, suppose that there is some pair of states ξ and $\hat{\xi}$ such that $\alpha(\xi) = \alpha(\hat{\xi})$ for all $\alpha \in \mathcal{O}$. We want to show that these two states are not distinguishable. We first show that these two states are not distinguishable by means of piecewise constant inputs. This follows from the construction in Lemma A.1. For any given piecewise constant input, we have that

$$\begin{aligned}\left.\frac{\partial^k}{\partial t_1 \ldots \partial t_k}\right|_{t_1=t_2=\ldots=0} h_j(t_1,t_2,\ldots,t_k,u^1,u^2,\ldots,u^k,\xi) &= L_{X_1}L_{X_2}\ldots L_{X_k}h_j(\xi) \\ \left.\frac{\partial^k}{\partial t_1 \ldots \partial t_k}\right|_{t_1=t_2=\ldots=0} h_j(t_1,t_2,\ldots,t_k,u^1,u^2,\ldots,u^k,\hat{\xi}) &= L_{X_1}L_{X_2}\ldots L_{X_k}h_j(\hat{\xi})\end{aligned}$$

and the two right-hand sides coincide because observables do not separate these two states. Now, the maps

$$h_j(t_1,t_2,\ldots,t_k,u^1,u^2,\ldots,u^k,\widetilde{\xi})$$

are analytic functions of the times t_i (see e.g. [38]), and two analytic functions that have the same derivatives at one point must be the same, so

$$h_j(t_1,t_2,\ldots,t_k,u^1,u^2,\ldots,u^k,\xi) \ = \ h_j(t_1,t_2,\ldots,t_k,u^1,u^2,\ldots,u^k,\hat{\xi})$$

for any such piecewise constant input. To finalize the proof, observe that piecewise constant inputs are dense in the set of measurable inputs (see e.g. [38], Remark C.1.2), and that the state (and hence output) is continuous with respect to the weak topology on inputs (see e.g. [38], Theorem 1). Thus the states ξ and $\hat{\xi}$ are not distinguishable. ∎

Remark A.3 There is a completely different proof of the same fact, using Corollary 5.1 in [42]. This Corollary says that two states are indistinguishable if and only if for every polynomial-in-time input, the derivatives at time zero of the output $y(t)$ are the same. Since polynomial inputs are dense in the set of all inputs, the result follows by a density argument. □

B Relevant concepts related to local nonlinear observability

We review here a test for local observability based on the observation space, $\mathcal{O}$. For a general autonomous multi-output system (with $u = 0$ in eq. 8)

$$\begin{aligned}\dot{x} &= g_0(x), \quad x \in X \\ y_j &= h_j(x), \quad j \in p\end{aligned} \tag{11}$$

one can construct $\mathcal{O}$ using p outputs, $y_j = h_j(x)$ and the repeated time derivatives, $y_j^{(k)} = L_{g_0}^{(k)}h_j(x)$. Consequently, one can define the observability codistribution, $d\mathcal{O}$ as

$$d\mathcal{O}(\xi) = \text{span}\left\{\frac{\partial\alpha}{\partial x} \mid \alpha \in \mathcal{O}\right\}, \quad \xi \in X,$$

Theorem 3 *The system (11) is locally observable at ξ_0 if*

$$\dim d\mathcal{O}(\xi_0) = \dim(X).$$

The proof of the theorem 3 is given in [38] (Remark 6.4.2, pp. 281-282) or [39] (pp. 95-96).

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