Passivity gains and the "secant condition" for stability

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Abstract

A generalization of the classical secant condition for the stability of cascades of scalar linear systems is provided for passive systems. The key is the introduction of a quantity that combines gain and phase information for each system in the cascade. For linear one-dimensional systems, the known result is recovered exactly.
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1. Introduction

An often-used tool in the analysis of biological feedback loops is the *secant condition* for linear stability; see the classical papers by Tyson and Othmer [12] and Thron [11], as well as the recent paper [10]. Consider a matrix of the following form:

$$\begin{pmatrix} -\alpha_1 & 0 & \dots & 0 & -\beta_1 \\ \beta_2 & -\alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \beta_n & -\alpha_n \end{pmatrix}, \tag{1}$$

where all $\alpha_i > 0$ and all $\beta_i > 0$. Then, the secant condition states that the matrix is Hurwitz provided that

$$\frac{\beta_1 \dots \beta_n}{\alpha_1 \dots \alpha_n} < \left(\sec \frac{\pi}{n} \right)^n = \frac{1}{(\cos \pi/n)^n}. \tag{2}$$

Since the characteristic polynomial of the matrix in Eq. (1) is

$$p(\lambda) = (\lambda + a_1)(\lambda + a_2) \dots (\lambda + a_n) + \beta_1 \beta_2 \dots \beta_n,$$

an equivalent way to state the secant condition is that this characteristic polynomial has all its roots with real part

negative, provided that condition (2) holds. The secant condition says that a stable system with distinct real eigenvalues and no zeros tolerates negative feedback with a gain much larger than that provided by the small-gain theorem: the corresponding small-gain estimate would just have a "1" in the right-hand side. (The secant expression is always bigger than one. It is singular at $n = 2$—which it should be, since then the matrix is always Hurwitz—and it equals 8 for $n = 3$, 4 for $n = 4$, and ≈ 2.88 for $n = 5$, and tends monotonically to 1 as $n \to \infty$. The bound is achieved exactly when all the α_i's are the same.) The condition is useful because certain models of biological systems are stable for gains larger than those predicted by a simple application of the small-gain theorem. The secant takes advantage simultaneously of phase and gain information on the open-loop system.

Observe that the matrix in Eq. (1) represents the closed-loop dynamics of the cascade of one-dimensional linear systems

$$\dot{x}_1 = -\alpha_1 x_1 + \beta_1 u,$$

$$\dot{x}_i = -\alpha_i x_i + \beta_i x_{i-1} \quad (i = 2, \dots, n) \tag{3}$$

under the negative unity feedback $u = x_n$. We provide here a generalization of the secant condition to cascades of output strictly passive (OSP) systems. We do so in such a manner that, in the special case in which each system is linear and one-dimensional, the classical result is recovered. (For simplicity, we restrict ourselves to systems with scalar inputs

and outputs, but it is obvious how to generalize to more arbitrary systems, as long as inputs and outputs have the same dimension.)

The generalization is based on systematic use of a "gain" associated to OSP systems. It would seem that the use of this quantity might be useful for many other problems as well.

This note is organized as follows. Section 2 introduces the basic concepts and states the main result, which is proved in Section 4 (the proof is actually very easy, given the definitions). Section 3 briefly mentions some extensions of the basic formalism, and Section 5 collects several facts concerning secant gains for the special case of linear systems.

2. Notations, definitions, and statement of main result

As usual, the extended space $L_e^2(0, \infty)$ denotes the set of signals (thought of as time functions) $w : [0, \infty) \to \mathbb{R}$ which have the property that each restriction $w_T = w|_{[0,T]}$ is in $L^2(0, T)$, for every $T > 0$. Given an element $w \in L_e^2(0, \infty)$ and any fixed $T > 0$, one writes $\|w\|_T$ for the L^2 the norm of this restriction w_T, and given two functions $v, w \in L_e^2(0, \infty)$ and any fixed $T > 0$, the inner product of v_T and w_T is denoted by $\langle v, w \rangle_T$. In any Hilbert space, one defines the angle $\theta(v, w) \in [0, \pi]$ between two elements v, w by the formula

$$\cos \theta(v, w) = \frac{\langle v, w \rangle}{\|v\| \|w\|}$$

if v and w are nonzero, and zero otherwise. Given $v, w \in L_e^2(0, \infty)$ and any fixed $T > 0$, we will write $\theta_T(v, w)$ instead of $\theta(v_T, w_T)$, to denote the angle between the restrictions of the signals to $[0, T]$.

We consider continuous-time finite-dimensional systems $\dot{x} = f(x, u)$, $y = h(x)$ in the usual sense of control theory (e.g. [9]), with scalar valued inputs and outputs, and state space $\mathbb{R}^n$, and assume always that the system is L^2-well-posed, in the sense that for each $u \in L_e^2(0, \infty)$ and initial state $x(0) = 0$ there is a unique solution $x(\cdot)$ defined for all $t \geqslant 0$ and the corresponding output $y(t) = h(x(t))$ is also in $L_e^2(0, \infty)$. We call (u, y) is an *input/output* (i/o) *pair* of the system.

We recall the standard notion of an *output strictly passive* ("OSP" for short) system, as given in textbooks such as [5,13,14]. A system is OSP if there is some $\gamma > 0$ such that, for every i/o pair (u, y),

$$\|y\|_T^2 \leqslant \gamma \langle u, y \rangle_T \tag{4}$$

for all $T > 0$. (Allowing an additive constant in the inequality is useful when dealing with arbitrary initial states. As we will study zero-state responses, we do not include a constant.)

If a system is OSP, we call the smallest γ as in (4) the *secant gain* of the system, and denote it as γ_s. (There is a smallest such γ, since the set of γ's that satisfy (4) is a closed set.)

An equivalent definition of γ_s is as the smallest γ with the property that

$$\|y\|_T^2 \leqslant \gamma \|u\|_T \|y\|_T \theta_T(u, y),$$

or equivalently:

$$\|y\|_T \leqslant \gamma \|u\|_T \cos \theta_T(u, y) \tag{5}$$

for all $T > 0$ and all i/o pairs. Since (4) implies that $\langle u, y \rangle_T \geqslant 0$ for all i/o pairs and all T, for OSP systems we always think of the angle as lying in the interval $[0, \pi/2]$, and the cosine is nonnegative.

The Cauchy–Schwartz inequality applied to (4) gives $\|y\|_T \leqslant \gamma \|u\|_T \leqslant \gamma \|u\|$ for all $T > 0$, so in particular $y \in L^2$ if $u \in L^2$, and an OSP system necessarily has finite L^2-induced (or "H_∞") gain $\gamma_\infty \leqslant \gamma_s$ (we remark later that this inequality is in general a strict one). Just as the L^2 gain is the supremum of the expressions $\|y\|_T / \|u\|_T$ over all T and all i/o pairs with nonzero u, the secant gain is obtained by maximizing $\sec \theta_T(u, y) \|y\|_T / \|u\|_T$, hence our terminology.

If $u \in L^2$, so that also $y \in L^2$, taking limits in (4) gives

$$\|y\|^2 \leqslant \gamma \langle u, y \rangle. \tag{6}$$

Conversely, if $u \in L^2 \Rightarrow y \in L^2$ and (6) is true for all $u \in L^2$, then (4) holds. This is a routine exercise in causality, as follows. Pick any i/o pair (u, y) and any $T > 0$. Let $v \in L^2$ be input which equals u on $[0, T]$ and is zero for $t > T$, and z the output corresponding to v. Since $v \in L^2$, also $z \in L^2$. By causality, z restricted to $[0, T]$ is the same as y restricted to $[0, T]$, so $\langle v, z \rangle = \langle u, y \rangle_T$, and $\|y\|_T = \|z\|_T$. Therefore, $\|y\|_T^2 = \|z\|_T^2 \leqslant \|z\|^2 \leqslant \gamma \langle v, z \rangle = \gamma \langle u, y \rangle_T$, and indeed (4) is verified.

We wish to analyze the stability of the closed-loop system $\dot{x} = f(x, u - h(x))$ obtained under negative unity feedback. Specifically, we study a cascade of n subsystems, as shown in the diagram in Fig. 1 and subject to unity negative feedback. Such cascades appear frequently in control theory as well as in biological applications, and, when components are one-dimensional, tend to have especially good dynamical properties such as the validity of the Poincaré–Bendixson Theorem [6]. We will assume that the ith system has a secant gain γ_i, and we write y_i for the output of the ith subsystem. We also assume well-posedness of the closed-loop.

The main result is as follows:

Theorem. *Suppose that*

$$\gamma_1 \gamma_2 \ldots \gamma_n < \left(\sec \frac{\pi}{n} \right)^n.$$

Then the cascade is L^2-stable: there is a number c so that

$$\|y_n\|_T \leqslant c \|u\|_T$$

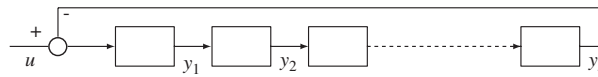*for all input/output pairs in the cascade and all $T > 0$.*

Fig. 1. Closed-loop system.

Of course, this property implies as well that every $\|y_i\|_T$ is bounded by some linear function of $\|u\|_T$, and that the signals y_i belong to L^2 if $u \in L^2$.

The linear cascade in Eq. (3) is a particular case of the type of system shown in Fig. 1, and each system $\dot{x}_i = -\alpha_i x_i + \beta_i u_i$ has secant gain $\gamma_i = \beta_i / \alpha_i$ (see Section 5 for remarks concerning linear systems). Thus, the Theorem generalizes the classical secant condition.

For the special cases $n = 1$ and 2 (secant is infinite), we interpret the inequality in the theorem as saying that the condition holds for any possible values of the γ_i's. For $n = 2$, therefore, the theorem is simply a restatement of the Passivity Theorem as given e.g. in [13], Theorem 2.2.15, Part a (using only the input u). The Passivity Theorem also includes usually a statement ("Part b" in the citation) regarding the case in which the first system is OSP and the second one is only passive, meaning that only $\langle u, y \rangle_T \geqslant 0$ is known for all i/o pairs. We comment later on this fact.

The assumption that the initial state of the cascade is $x(0) = 0$ is easy to dispose of, assuming appropriate reachability of the cascade, as routinely done in going from input/output stability to state space stability, and Barbălat's Lemma combined with either reachability or detectability arguments can be used to show convergence of internal states to zero. As an illustration, we state just one such corollary:

Corollary. *Suppose that the condition in the Theorem is verified, that the composite system shown in Fig. 1 is zero-reachable and that each subsystem is input to state L^2-stable. Then the system with no inputs $(u = 0)$ has the property that all solutions converge to $x = 0$.*

3. Extensions

We have formulated the results in terms of state-space systems only in order to be concrete. One could equally well consider arbitrary operators $L^2 \to L^2$, or even just relations R on $L^2 \times L^2$, where an "i/o pair" is by definition any element of R, and define secant gain γ_s as the smallest number so that (4) holds for all T and all i/o pairs. Nor is it needed for the inputs and outputs to be scalar-valued; one may consider values on arbitrary Hilbert spaces, with inner product and norms taken pointwise in that space. More generally, functions of time are not required: one could consider an arbitrary Hilbert space H and simply ask that u and y belong to H. (To be precise, one needs a Hilbert space together with a resolution of the identity, in order to be able to have a concept of "restriction" of u and y to subintervals; this is

the formalism of *resolution spaces* developed in [8].) Even more generally, if one has a system in which inputs u and outputs y are known to lie in a specific subset $S \subseteq H$, then γ_s can be defined in terms only of i/o pairs that lie in S; the validity of the main theorem is not affected, since it is just an algebraic statement about norms and inner products.

Let us discuss a simple example of an operator defined only on subsets, which is of interest in biomolecular applications ("Michaelis–Menten kinetics"). Suppose that S is the set of all L^2 maps $w : [0, \infty) \to [-a, \infty)$ with any fixed $a > 0$, and that we consider the function $\ell : [-a, \infty) \to \mathbb{R}$ given by

$$\ell(r) = \frac{Vr}{K + a + r} \qquad (7)$$

(with $K, V > 0$ some constants) and the operator $u \mapsto y$ defined on S, where $y(t) = F(u)(t) = \ell(u(t))$. This is an example of a "sector" nonlinearity. The analysis of sector nonlinearities is routine in passivity theory. The operator F is OSP and has $\gamma_s = V/K$, because we have, for all $r \in [-a, \infty)$:

$$[\ell(r)]^2 = \left(\frac{Vr}{K + a + r} \right)^2$$
$$= r \frac{Vr}{K + a + r} \frac{V}{K + a + r} = \frac{V}{K} r \ell(r)$$

(since $K + a + r \geqslant K$), and thus

$$\|y\|_T^2 = \int_0^T \ell(u(t))^2 \, dt$$
$$\leqslant \frac{V}{K} \int_0^T u(t) \ell(u(t)) \, dt = \frac{V}{K} \langle u, y \rangle_T$$

so $\gamma_s \leqslant V/K$, and the equality is verified when $u(t) \equiv -a$.

We gave a definition of L^2 stability that is particularly useful for equilibria associated to zero signals. However, the framework described here can be easily extended to more general situations. These extensions are of interest, particularly, when dealing with problems in biology and chemistry, where quantities represent concentrations of substances, and hence are always nonnegative. We now describe briefly how this extension can be accomplished.

Suppose that one wishes to study a system

$$\dot{x} = f(x, u),$$
$$y = h(x)$$

under the feedback law $u = -y$, and that there is a steady state x^* for this closed-loop system:

$$f(x^*, -h(x^*)) = 0$$

whose stability is of interest to analyze. We assume that the states $x(t)$ evolve in some subset S of $\mathbb{R}^n$, for example, the positive orthant $\mathbb{R}_+^n = \{(x_1, \ldots, x_n), x_i \geqslant 0 \ \forall i\}$, and inputs u of the open-loop system take values on some set U. (In order for the closed-loop system to make sense, one should then have that $-h(S) \subseteq U$, of course.) We perform a change of variables $z = x - x^*$ and define the new system

$$\dot{z} = g(z, v) = f(z + x^*, v - h(x^*)),$$
$$w = \ell(z) = h(z + x^*) - h(x^*),$$

with states $z(t)$ in the state-space $\{x - x^*, x \in S\}$, inputs $v(t)$ in the input-value space $\{u + h(x^*), u \in U\}$, and outputs $w(t)$. Note that $g(0, 0) = 0$. Applying the feedback $v = -\ell(z)$ results in

$$\dot{z} = g(z, -\ell(z)) = f(z + x^*, -h(z + x^*)).$$

Therefore, for each solution $x(t)$ of $\dot{x} = f(x, -h(x))$, the vector function $z(t) = x(t) - x^*$ satisfies $\dot{z} = g(z, -\ell(z))$, and conversely, each solution of the latter system arises from the former. Proving that solutions of $\dot{x} = f(x, -h(x))$ converge to x^* is then equivalent to proving that the solutions of the new system converge to the equilibrium $z = 0$. Thus, we have reduced the analysis to the case treated in this paper.

For example, suppose that we wish to study a positive system, that is, a system whose state space is $\mathbb{R}_+^n$ and inputs are also nonnegative. Furthermore, suppose that, as is often the case in biological feedback loops, one wishes to study an inhibitory feedback of the form

$$u = \frac{M}{K + x_n},$$

where M and K are some positive constants and x_n is the nth coordinate of the state, that is to say, we have $h(x) = -M/(K + x_n)$. In terms of the variables z, we have the output

$$w = \ell(z) = h(z + x^*) - h(x^*)$$
$$= \frac{M}{K + x_n^*} - \frac{M}{K + (z_n + x_n^*)} = \frac{V z_n}{K + x_n^* + z_n}$$

which is the function in (7) with $a = x^*$ and $V = M/(K + x_n^*)$. Since $x_n(t)$ is nonnegative, the state variable $z_n(t)$ takes values in $[-x^*, \infty)$. Thus, we may view the closed-loop system as built from cascading the original system (which may itself be a cascade of several subsystems) with the static system "$y = \ell(u)$", which has $\gamma_s = V/K$, and the previous analysis applies.

This is all particularly simple for a linear system $\dot{x} = f(x, u) = Ax + Bu$. Positivity amounts to asking that all the off-diagonal entries of A as well as all entries of B are nonnegative (see e.g. [2,4]). Since the system is linear and $Ax^* - Bh(x^*) = 0$, we have that $g(z, v) = A(z + x^*) + B(v - h(x^*)) = Az + Bv$, so the same open-loop system results, except that now we are interested in the stability of $z = 0$.

4. Proof of main result

Given an external input u, the solutions of the closed-loop system with initial state zero are so that the signals y_i have the following properties:

$$\|y_1\|_T^2 \leqslant \gamma_1 \langle u + y_0, y_1 \rangle_T,$$
$$\|y_2\|_T^2 \leqslant \gamma_2 \langle y_1, y_2 \rangle_T,$$
$$\vdots$$
$$\|y_n\|_T^2 \leqslant \gamma_n \langle y_{n-1}, y_n \rangle_T$$

for every $T > 0$, where we are writing $y_0 = -y_n$. We expand

$$\langle u + y_0, y_1 \rangle_T = \langle u, y_1 \rangle_T + \langle y_0, y_1 \rangle_T,$$

and use the Cauchy–Schwartz inequality for the first term, upper-bounding it by $\|u\|_T \|y_1\|_T$. Replacing now each $\langle y_{i-1}, y_i \rangle_T$ by $\|y_{i-1}\|_T \|y_i\|_T \cos \theta_T(y_{i-1}, y_i)$ and dividing by $\|y_i\|_T$ (assumed nonzero; otherwise, there will be nothing to prove), we have these estimates:

$$\|y_1\|_T \leqslant \gamma_1 \|y_0\|_T \cos \theta_T(y_0, y_1) + \gamma_1 \|u\|_T,$$
$$\|y_2\|_T \leqslant \gamma_2 \|y_1\|_T \cos \theta_T(y_1, y_2),$$
$$\vdots$$
$$\|y_n\|_T \leqslant \gamma_n \|y_{n-1}\|_T \cos \theta_T(y_{n-1}, y_n)$$

from which we conclude, by recursively substituting the estimates starting from the last one backward towards the first, that:

$$\|y_n\|_T \leqslant \kappa \|y_n\|_T + \alpha \|u\|_T,$$

where

$$\alpha = \gamma_1 \gamma_2 \ldots \gamma_n \cos \theta_T(y_1, y_2) \ldots \cos \theta_T(y_{n-1}, y_n)$$

and

$$\kappa = \alpha \cos \theta_T(y_0, y_1).$$

It is enough to show that $\kappa < 1$, since then we can write $(1 - \kappa) \|y_n\|_T \leqslant \alpha \|u\|_T$, and therefore the result holds with $c = \alpha/(1 - \kappa)$. Let us fix T and write $\theta_i := \theta_T(y_{i-1}, y_i)$ for $i = 1, \ldots, n$. We must show, then, that

$$\cos \theta_1 \ldots \cos \theta_n \leqslant \left(\cos \frac{\pi}{n} \right)^n. \tag{8}$$

The angles θ_i all lie in $[0, \pi/2]$, for each $i = 2, \ldots, n$, since each system is OSP; thus $\cos \theta_i \geqslant 0$ for all such i. However, it is possible that $\cos \theta_1 < 0$, since all that is known is that $\langle u + y_0, y_1 \rangle_T \geqslant 0$, not that $\langle y_0, y_1 \rangle_T \geqslant 0$. But if $\cos \theta_1 < 0$, then (8) is true because the left-hand side is $\leqslant 0$ and the right-hand side is positive. So, in order to prove (8), we may assume from now on that all $\theta_i \in [0, \pi/2]$.

We prove, more generally, this fact about Hilbert spaces: suppose given vectors $v_0, v_1, \ldots, v_n$ such that $\langle v_i, v_{i+1} \rangle \geqslant 0$, and $v_0 = -v_n$. Let $\theta_i \in [0, \pi/2]$ be the angle between v_{i-1} and v_i. Then (8) holds. Intuitively, the property that the start

and end vector are at angle π means that the consecutive vectors cannot be too close in angle, and therefore at least some of the angles must be large, and hence have small cosine, and the largest possible value is achieved when all angles are the same.

To prove this general fact, without loss of generality, we may assume that all the v_i are unit vectors (since only angles matter). Notice that $\sum_i \theta_i \geqslant \pi$. This is because, for any three unit vectors, $\theta(u, v) + \theta(v, w) \geqslant \theta(u, w)$, since we can view the angle as the geodesic distance in a sphere, and apply the triangle inequality; inductively applied starting from v_0, we get that $\sum_i \theta_i \geqslant \theta(v_0, v_n) = \pi$. Now, we have also this algebraic fact:

$$\cos \theta_1 \ldots \cos \theta_n \leqslant \left(\cos \frac{\theta_1 + \cdots + \theta_n}{n} \right)^n,$$

which follows by noticing that the function $f(x) = -\ln \cos x$ is convex for $x \in [0, \pi/2]$, applying Jensen's inequality to obtain $f(\sum_i \theta_i/n) \leqslant (1/n)\sum_i f(\theta_i)$, and taking exponentials. Together with $\sum_i \theta_i \geqslant \pi$, using that $\pi/n \leqslant (\theta_1 + \cdots + \theta_n)/n \leqslant \pi/2 < \pi$ (recall that each $\theta_i \in [0, \pi/2]$), and using that $\cos$ decreases on $[0, \pi]$, we conclude:

$$\left(\cos \frac{\theta_1 + \cdots + \theta_n}{n} \right)^n \leqslant \left(\cos \frac{\pi}{n} \right)^n.$$

This completes the proof of the Theorem.

To prove the Corollary, we provide a standard argument, as done e.g. in [9, Theorem 33]. Pick any initial state x_0 and consider the solution $x(\cdot)$ of the closed-loop system $\dot{x} = f(x, u - h(x))$ with input 0 and $x(0) = x_0$. Zero-reachability means that there is some finite-time input $u_0 : [0, T] \to \mathbb{R}$ such that, if $z_0(\cdot)$ solves the closed-loop equations $\dot{z} = f(z, u - h(z))$ with initial state $z_0(0) = 0$ and this input u_0 on the interval $[0, T]$, then $z_0(T) = x_0$. Consider now the input u obtained by the formula $u(t) = u_0(t)$ for $t \leqslant T$ and $u(t) \equiv 0$ for $t > T$, and let $z(\cdot)$ be the solution with initial state $z(0) = 0$ and this input u; by causality, $z(t) = z_0(t)$ for $t \leqslant T$, and hence $z(T) = x_0 = x(0)$, from which it follows that $z(t + T) = x(t)$ for all $t \geqslant 0$. Showing $x(t) \to 0$ as $t \to \infty$ is the same as showing $z(t) \to 0$ as $t \to \infty$. Let y_i be the outputs of the subsystems when using input u (and zero initial state). Since $u \in L^2$ and $\|y\| \leqslant c\|u\| < \infty$, we have that $y_i \in L^2$ for each of the intermediate outputs. Since each subsystem is input to state L^2-stable, meaning that L^2 inputs (and zero initial state) produces L^2 state trajectories, we have that the complete state z is in L^2. Finally, as z is a trajectory of a semiflow in finite dimensions, we must have that $z(t) \to 0$, by a Barbălat's Lemma type of argument (see e.g. [3]).

Finally, we review in the present context a weaker version that applies when $n = 2$, basically part of the statement of the classical Passivity Theorem. Suppose that the first system is OSP but the second system is only known to be passive, in the sense that no estimate $\|y_2\|_T^2 \leqslant \gamma_2 \langle y_1, y_2 \rangle_T$ may hold, but

we do know that $\langle y_1, y_2 \rangle_T \geqslant 0$ for all $T > 0$. Then, $y_0 = -y_2$ implies that:

$$
\begin{aligned}
\|y_1\|_T^2 &\leqslant \gamma_1 \langle u + y_0, y_1 \rangle_T \\
&= \gamma_1 \langle u, y_1 \rangle_T - \gamma_1 \langle y_2, y_1 \rangle_T \leqslant \gamma_1 \langle u, y_1 \rangle_T
\end{aligned}
$$

and so the system *with output* y_1 is OSP, and in particular, L^2 stable. If, in addition, the second system is also L^2 stable, then stability to y_2 holds as well.

5. Linear systems

The condition that a system be OSP is of course a restrictive one, but the concept of OSP system is thoroughly well-studied, and examples of passive systems abound, especially, but not only, for linear systems. We collect here some facts, mostly well-known, regarding the linear case.

For a stable linear system with transfer function $G(s)$, the secant gain can be characterized as the smallest γ such that

$$|G(i\omega)|^2 \leqslant \gamma \operatorname{Re} G(i\omega) \quad \forall \omega \in \mathbb{R}. \tag{9}$$

A proof is as follows. First of all, squaring the expression below and expanding $\langle y - (\gamma/2), y - (\gamma/2) \rangle_T$, one easily sees that the definition of OSP system is equivalent to the requirement that

$$\|y - (\gamma/2)u\|_T \leqslant (\gamma/2)\|u\|_T \tag{10}$$

for all i/o pairs and all T, which means γ_s is the smallest number such that the L^2-induced norm of $u \mapsto y - (\gamma/2)u$ is $\leqslant \gamma/2$. For linear systems, induced L^2-induced norm corresponds to H_∞ gain, that is to say, γ_s is the smallest number so that $\sup_{\omega \in \mathbb{R}} |G(i\omega) - (\gamma/2)| \leqslant \gamma/2$. Writing $|G(i\omega) - (\gamma/2)|^2 = (G(i\omega) - (\gamma/2))(\overline{G(i\omega)} - (\gamma/2))$ and expanding, one has (9).

An equivalent formulation of (9) is via the following analog of the estimate (5):

$$|G(i\omega)| \leqslant \gamma \cos \theta(G(i\omega)) \quad \forall \omega \in \mathbb{R}, \tag{11}$$

where we are denoting now by $\theta(\mu)$ the argument of a complex number μ. Since G is analytic on $\operatorname{Re} \lambda \geqslant 0$ (stability), the maximum modulus principle for analytic functions implies that same estimate is obtained when maximizing not merely over $\lambda = i\omega$ purely imaginary, but also over all complex numbers with nonnegative real part.

If we write $G(s) = p(s)/q(s)$ as a quotient of two polynomials, condition (9) can be also written as

$$|p(i\omega)|^2 \leqslant \gamma \operatorname{Re} [p(i\omega)\overline{q(i\omega)}].$$

For example, for a one-dimensional system $\dot{x} = -\alpha x + \beta u$ with output $y = x$, the transfer function is $\beta/(s + \alpha)$, so that $p(i\omega) = \beta$ and $\operatorname{Re} [p(i\omega)\overline{q(i\omega)}] = \alpha\beta$ for any ω, from which it follows that $\gamma_s = \beta/\alpha$, and the classical result is obtained. On the other hand, as is well-known for OSP systems, $G(s)$ must have relative degree at most one (the condition $\operatorname{Re} G(i\omega) \geqslant 0$

is otherwise violated). Therefore, cascades, as studied here, of two or more such one-dimensional systems are not OSP themselves.

For linear systems, a sufficient condition for a system to be OSP is that its transfer function $G(s)$ be strictly positive real (SPR), meaning that $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$, or equivalently (see e.g. [5, Lemma 10.1]) that it be stable (all poles have negative real part) and satisfy $\operatorname{Re} G(i\omega) > 0$ for all $\omega \in \mathbb{R}$ and $\lim_{\omega \to \infty} \omega^2 \operatorname{Re} G(i\omega) > 0$. (Note that our transfer functions are strictly proper, by definition, since we are considering state-space systems with no direct i/o term; for nonstrictly proper transfer functions, the condition is slightly different.) This provides a large class of examples; for instance, any transfer function of the form $(s + \alpha)/(s^2 + as + b)$ with $b > 0$ and $0 < a < 2\sqrt{b}$ is SPR if and only if $0 < \alpha < a$ [5, Exercise 10.1]. That SPR implies OSP can be proved using the Kalman–Yakubovich–Popov (KYP) Lemma. The converse implication does not hold: $s/(s^2 + s + 1)$ is not SPR, since it fails the test just quoted with $(a = b = 1, \ \alpha = \frac{1}{2})$ or just by noting that there is an imaginary axis zero, since $\operatorname{Re} G(0) = 0$, but it is OSP, since $|p(i\omega)|/\operatorname{Re}[p(i\omega)\overline{q(i\omega)}] \equiv 1 < \infty$.

More generally, for not necessarily linear systems, if there exists some nonnegative definite smooth function V on states with the property that, for some $\gamma > 0$,

$$\nabla V(x) \cdot f(x, u) \leqslant - y^2 + \gamma u y$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y = h(x)$, then the system is OSP. Indeed, integrating along solutions corresponding to $x(0) = 0$, and using that V is nonnegative definite (so that $V(0) = 0$ and $V(x(T)) \geqslant 0$), one has that

$$0 \leqslant V(x(T)) - V(0)$$
$$\leqslant -\int_0^T y(s)^2 \, \mathrm{d}s + \gamma \int_0^T u(s) y(s) \, \mathrm{d}s$$

and thus $\|y\|_T^2 \leqslant \gamma \langle u, y \rangle_T$ as claimed. This property can be checked by means of nonlinear versions of the KYP Lemma, see e.g. [5,13].

Yet another way of stating the estimate (4) is in terms of integral quadratic constraints (IQC's), cf. [7]: one may equivalently write "$w^T M w \geqslant 0$" in L^2 for i/o pairs $w = (u, y)'$ and where

$$M = \begin{pmatrix} 0 & \gamma/2 \\ \gamma/2 & -1 \end{pmatrix}.$$

The powerful tools for analysis of IQC's, based on LMI's, as developed by Megretski and Rantzer and others, should thus be useful for the study of secant gains. (We wish to thank R. Sepulchre for suggesting this reformulation.)

We pointed out that the induced L^2 gain γ_∞ is upper bounded by the secant gain γ_s. In general, one has the strict inequality $\gamma_\infty < \gamma_s$. For example consider the linear system
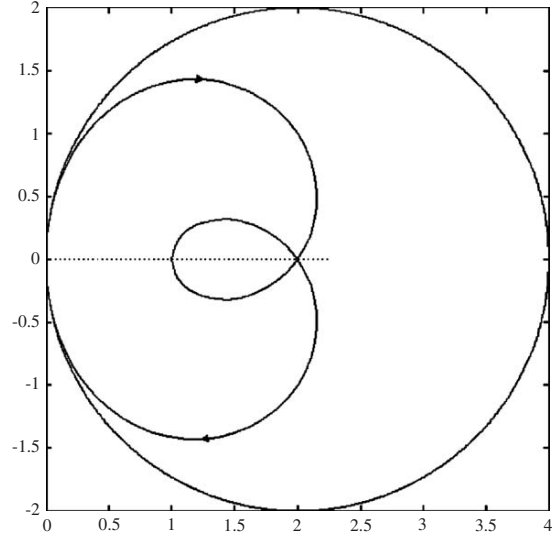

Fig. 2. Nyquist plot for $(2s+1)/(s^2+s+1)$ and circle $|s-2| \leqslant 2$, $\gamma_s = 4$, $\gamma_\infty \approx 2.25$.

with transfer function

$$G(s) = \frac{2s + 1}{s^2 + s + 1}.$$

This is a scalar multiple of $(s + 1/2)/(s^2 + s + 1)$, so it is SPR by the criterion mentioned earlier, and hence OSP. Explicitly

$$\gamma_s = \sup_{\omega \in \mathbb{R}} \frac{|p(i\omega)|^2}{\operatorname{Re}[p(i\omega)\overline{q(i\omega)}]} = \sup_{\omega \in \mathbb{R}} \frac{1 + 4\omega^2}{1 + \omega^2} = 4$$

and

$$\gamma_\infty = \sup_{\omega \in \mathbb{R}} \left| \frac{1 + 2i\omega}{1 - i\omega - \omega^2} \right| = \sup_{\omega \in \mathbb{R}} \sqrt{\frac{1 + 4\omega^2}{1 - \omega^2 + \omega^4}}$$
$$= \sqrt{2 + (2/3)\sqrt{21}} \approx 2.25 < 4$$

(the maximum value is achieved at $\omega = 1/2\sqrt{\sqrt{21} - 1}$). Graphically, we can see these conclusions from Fig. 2, which shows that the smallest circle of the form $|s - \gamma/2| \leqslant \gamma/2$ which contains the Nyquist plot must have $\gamma = 4$ (circle shown), so that this is the value of γ_s, but $\gamma_\infty \approx 2.25$ because the plot fits in a circle (not shown) centered at the origin with radius ≈ 2.25.

To conclude, let us provide a direct proof of the main theorem in the linear case. This proof, when specialized to linear one-dimensional systems, is basically the same as the proof given in [11]. We assume a unity negative feedback about the cascade in Fig. 1, where each system has transfer function $G_i(s)$ and secant gain γ_i. As remarked earlier, this means that an estimate $|G_i(\lambda)| \leqslant \gamma_i \cos \theta(G_i(\lambda))$ as in (11) holds for every λ with

real part $\geqslant 0$. If the closed-loop were not to be stable, then there would exist a pole λ with real part $\geqslant 0$ for $G/(1+G)$, where $G = G_1 \ldots G_n$ is the open-loop system. For any such λ

$$G_1(\lambda)G_2(\lambda) \ldots G_n(\lambda) = -1 \qquad (12)$$

from which we conclude, writing $\theta_i = \theta(G_i(\lambda))$, that $\sum_i \theta_i$ is a multiple of π. Moreover, taking absolute values in (12) and using $|G_i(\lambda)| \leqslant \gamma_i \cos \theta_i$, we also have that

$$1 \leqslant \gamma_1 \gamma_2 \ldots \gamma_n \prod_{i=1}^{n} \cos \theta_i$$

and so $\gamma_1 \ldots \gamma_n \geqslant (\sec \pi/n)^n$, again using the convexity of $-\ln \cos x$. Thus, no such poles can exist, if the hypothesis of the theorem holds.

6. Conclusions and remarks

We presented a result that extends the classical secant condition from cascades of one-dimensional systems to cascades of arbitrary OSP linear or nonlinear systems. As discussed, our proof does not even require state-space representations, and is therefore valid in great generality, including systems defined entirely through operators on signal spaces.

After this paper was submitted, Murat Arcak and the author found (see [1]) an alternative approach, based upon Lyapunov functions and Popov Criterion techniques. Although limited to finite dimensional continuous-time systems, this other approach is very powerful and it helps tighten up estimates for many examples, such as the inhibitory feedback loop with Michaelis–Menten kinetics mentioned in this paper.

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