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ON THE CONTINUITY AND INCREMENTAL-GAIN PROPERTIES OF CERTAIN SATURATED LINEAR FEEDBACK LOOPS

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SUMMARY

This paper discusses various continuity and incremental-gain properties for neutrally stable linear systems under linear feedback subject to actuator saturation. The results complement our previous ones, which applied to the same class of problems and provided finite-gain stability.

KEY WORDS saturated-input linear systems; operator stability; finite incremental gain

1. INTRODUCTION

We continue in this paper the study, which we started in Reference 5, of operator stability properties for saturated-input linear systems. In the previous paper, we studied feedback systems of the form

$$\dot{x} = Ax + B\sigma(Fx + u) \qquad (\Sigma)$$

Here σ denotes a vector of saturation-type functions, each of which satisfies mild technical conditions that are recalled later (at this point, it suffices to say that all reasonable 'sigmoidal' maps such as $\sigma(x) = \tanh(x)$ and the standard saturation function $\sigma_0(t) = \mathrm{sign}(t)\min\{|t|, 1\}$ are included). The matrix A is assumed to be neutrally stable and one uses the standard passivity theory choice of feedback F that makes the origin of the unforced closed-loop system $\dot{x} = Ax + B\sigma(Fx)$ globally asymptotically stable. (For instance, if A has all eigenvalues in the imaginary axis and the pair (A, B) is controllable, $F = -B^{T}P$, where P is a positive-definite matrix satisfying $A^{T}P + PA = 0$.)

We proved in Reference 5 that this system is finite-L^p-gain stable, that is, the zero-initial state operator $F_{\sigma,p}$ mapping input functions $u(\cdot)$ to solutions $x(\cdot)$ is a well-defined and bounded operator from $L^p([0,\infty), \mathbb{R}^m)$ to $L^p([0,\infty), \mathbb{R}^n)$. The result is valid for each p in the range $[1,\infty]$. Estimates were provided of the operator norms, in particular giving for $p = 2$ an upper bound expressed in terms of the H^∞-norm of the same input-state map for the system in which the saturation σ is not present. We also dealt with partially observed states, generalizing the result to the case where an observer is inserted in the feedback construction. The assumption of neutral stability is critical: we also obtained examples showing that the double integrator cannot be stabilized in this operator sense by any linear feedback, contradicting what may be expected from the fact that such systems are globally asymptotically stabilizable in the state-space sense. (Recently, Lin, Saberi, and Teel[4] obtained related results, showing in particular that under the restriction that the input signals be bounded one can drop the stability assumption in obtaining finite-gain stability. See also References 7–9 for state-space stabilization of linear systems subject to saturation, under minimal conditions.)

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Finite-gain stability, studied in the abovementioned papers, means that the 'energy' of inputs is amplified by a bounded amount when passing through the system. Another property which is extremely important in the context of feedback systems analysis is that of *incrementally finite gain* ('ifg') stability. In mathematical terms, this latter property is the requirement that the operator $F_{\sigma,p}$ be globally Lipschitz. That is to say, if y_{nom} is the output produced in response to a nominal input u_{nom}, then a new input $u_{\text{nom}} + \Delta u$ produces an output whose energy differs from that of y_{nom} by at most a constant multiple of the energy of the increment Δu. This stronger notion measures sensitivity to input perturbations; for differentiable mappings, one would be asking that the derivative be bounded. In the context of stability, the usual formulations of the small-gain theorem involve ifg stability, because fg stability by itself is not sufficient in order to guarantee the existence and uniqueness of signals ('well-posedness') in a closed-loop system.[10] In the recent work,[3] it is shown how to generalize the gap metric, so successful in robustness analysis of linear systems, to the context of ifg stability of nonlinear systems. Even stronger properties may sometimes be needed; for instance, the work in Reference 2 requires what the author of that paper calls 'differential stability,' which means that ifg stability holds and $F_{\sigma,p}$ is Fréchet differentiable as well. Motivated by this, we ask here if stronger properties hold for the feedback configuration studied in Reference 5.

Our results can be summarized in informal terms as follows:

(1) The operator $F_{\sigma,p}$ is *continuous* if p is finite, but is not in general continuous for $p = \infty$ (uniform norm).
(2) $F_{\sigma,p}$ is *locally Lipschitz* under additional assumptions on the saturation (for p finite, a sufficient condition is that the components of σ be differentiable near the origin; for $p = \infty$ one asks in addition that they be differentiable everywhere, with positive derivative). A much stronger statement than the local Lipschitz property is established which we call 'semi-global Lipschitz' — a' incremental gains are shown to depend only on the norms of the controls; on the other hand, we also show by counter-example that these operators are not generally globally Lipschitz (so ifg stability does not hold).
(3) Assume that σ is continuously differentiable. For $p = \infty$, we show that $F_{\sigma,p}$ is Fréchet differentiable (under the assumption that σ' is always positive), but this may fail for finite p. In the latter case, however, we can prove that directional derivatives always exist.

The paper is organized as follows. First we review some needed facts from Reference 5, to be used in this paper. After that, we introduce our basic definitions and state the main results. Proofs of the positive statements are given first, and we close with counterexamples that justify the negative results.

2. PRELIMINARIES

In order to state our results, we need to first recall some definitions and basic results from Reference 5, including those of 'saturation function' and finite gain L^p-stability.

By a *saturation function* ('S-function' for short) we mean any $\sigma: \mathbb{R} \to \mathbb{R}$ which satisfies the following properties:

- σ is locally Lipschitz and bounded;
- $t\sigma(t) > 0$ if $t \neq 0$;
- $\liminf_{t\to 0} \dfrac{\sigma(t)}{t} > 0$, $\limsup_{t\to 0} \dfrac{\sigma(t)}{t} < \infty$; and
- $\liminf_{|t|\to\infty} |\sigma(t)| > 0$.

We remark that the locally Lipschitz assumption on σ is not really needed in establishing Theorem (FG) below. This purpose of this condition is only to guarantee that system (1) in Theorem (FG) has uniqueness of solutions for any input u.

All the interesting saturation functions found in usual systems models, including the standard saturation function $\sigma_0(t) = \operatorname{sign}(t)\min\{|t|, 1\}$ as well as the functions $\arctan(t)$ and $\tanh(t)$ are S-functions.

We say that σ is an $\mathbb{R}^n$*-valued S-function* if $\sigma = (\sigma_1, \dots, \sigma_n)^{\mathrm{T}}$, where each component σ_i is an S-function and

$$\sigma(x) \triangleq (\sigma_1(x_1), \dots, \sigma_n(x_n))^{\mathrm{T}}$$

for $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. (Here we use $(\dots)^{\mathrm{T}}$ to denote the transpose of the vector $(\dots)$.)

We now turn to the stability definitions. These can be introduced for any initialized control system

$$\dot{x} = f(x, u), \qquad x(0) = 0 \tag{Σ}$$

The state x and the control u take values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. With appropriate assumptions on f (for example $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is globally Lipschitz with respect to its argument (x, u)), the solution x of (Σ) corresponding to any input $u \in L^p([0,\infty), \mathbb{R}^m)$ for $1 \leqslant p \leqslant \infty$ is well defined for all $t \in [0, \infty)$. When defined for all $t \in [0, \infty)$, we denote this solution x, which is *a priori* just a locally absolutely continuous (l.a.c. for short) function, as $F_{(\Sigma)}(u)$.

In general, for any $1 \leqslant p \leqslant \infty$ and any vector function $x \in L^p([0,\infty), \mathbb{R}^n)$, we'll consider the standard L^p-norm

$$\|x\|_{L^p} \triangleq \left(\int_0^\infty \|x(t)\|^p \,\mathrm{d}t\right)^{1/p} \qquad (p < \infty)$$

$$\|x\|_{L^\infty} \triangleq \operatorname{ess\,sup}_{0 \leqslant t < \infty} \|x(t)\|$$

(For vectors in $\mathbb{R}^n$ we use Euclidean norm

$$\|\xi\| = \left(\sum_{i=1}^{n} \xi_i^2\right)^{1/2}$$

We use the same notation for matrices, that is, $\|S\|$ is the Frobenius norm equal to the square root of the sum of squares of entries, i.e. $\|S\| = \operatorname{Tr}(SS^{\mathrm{T}})^{1/2}$, where Tr denotes trace.)

We define the L^p*-gain* of a system (Σ) as the norm of the operator $F_{(\Sigma)}$ that maps inputs to solutions, assuming a zero initial state. That is, the L^p-gain of (Σ), to be denoted by G_p, is the infimum (possibly $+\infty$) of the numbers M so that

$$\|F_{(\Sigma)}(u)\|_{L^p} \leqslant M \|u\|_{L^p}$$

for all $u \in L^p([0, \infty), \mathbb{R}^m)$. (If $F_{(\Sigma)}(u)$ is undefined for any $u \in L^p([0,\infty), \mathbb{R}^m)$, we also write $G_p = \infty$.) When this number is finite, we say that the operator is *finite gain* L^p*-stable* (in more usual mathematical terms, it is a bounded operator).

The main result in Reference 5 concerns the finiteness of the L^p-gain of Σ for a specific class of input-saturated linear systems. We quote this result next.

Theorem (FG)

Let A, B be $n \times n$, $n \times m$ matrices respectively and let σ be an $\mathbb{R}^m$-valued S-function. Assume that A is neutrally stable. Then there exists an $m \times n$ matrix F such that the system

$$\begin{aligned} \dot{x} &= Ax + B\sigma(Fx + u) \\ x(0) &= 0 \end{aligned} \tag{1}$$

is finite gain L^p-stable for all $1 \leqslant p \leqslant \infty$.

By neutral stability, we mean as usual that the origin of the differential equation $\dot{x} = Ax$ is stable in the sense of Lyapunov (not necessarily asymptotically stable, of course; otherwise the result would be trivial from linear systems theory); equivalently, there is a symmetric positive definite matrix Q which provides a solution of the Lyapunov matrix inequality $A^{\mathrm{T}}Q + QA \leqslant 0$.

The results in this paper will refer to the specific feedback F that is found in the proof of the above-cited result. In order to understand the choice of F (which is the most natural choice of feedback to use in this context), we need to recall the preliminary steps in the proof of Theorem (FG). The first step consisted of the observation that one can assume without loss of generality that the pair (A, B) is controllable, because the trajectories lie in the controllability space $R(A, B)$. Next we applied a change of basis to reduce A to the block-diagonal form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{2}$$

where A_1 is an $r \times r$ Hurwitz matrix and A_2 is an $(n-r) \times (n-r)$ skew-symmetric matrix. (Recall that A is assumed to be neutrally stable.) Thus one only needs to obtain finite gain L^p-stability of the subsystem corresponding to A_2; then feeding back a function of these variables does not affect the finite gain L^p-stability of the first subsystem. Since A_2 is skew-symmetric and the pair (A_2, B_2) is controllable, the non-saturated closed-loop matrix $\tilde{A} \triangleq A_2 - B_2B_2^{\mathrm{T}}$ is Hurwitz. Therefore, the proof of Theorem (FG) is reduced to showing that the following system:

$$\dot{x} = Ax + B\sigma(-B^{\mathrm{T}}x + u), \; x(0) = 0 \tag{3}$$

is finite gain L^p-stable for every $1 \leqslant p \leqslant \infty$, provided that A is skew-symmetric and $\tilde{A} = A - BB^{\mathrm{T}}$ is Hurwitz. Thus, except for two coordinate changes (first to restrict to the controllability space and then to exhibit the above block structure), the F used in the proof of Theorem (FG) is $F = -B^{\mathrm{T}}$. This is the standard choice of feedback suggested by the passivity approach to control for a discussion and references see Reference 5.

(For completeness, we point out that, after these trivial preliminary steps, the proof of Theorem (FG) then centers upon the hard part, which consists of finding a suitable 'storage function' V_p and establishing for it a 'dissipation inequality'of the form

$$\frac{\mathrm{d}V_p(x(t))}{\mathrm{d}t} \leqslant -\|x(t)\|^p + \kappa_p \|u(t)\|^p \tag{4}$$

for $x = F_{(\Sigma)}(u)$, where now (Σ) is the system in equation (1) and $\kappa_p > 0$ is some constant. Surprisingly, a nonsmooth V_p is needed.)

In conclusion, we will denote by

$$\boxed{F_{\sigma,p} : L^p([0, \infty), \mathbb{R}^m) \to L^p([0, \infty), \mathbb{R}^n)}$$

the (nonlinear) input/state operator $F_{(\Sigma)}$ for system (1) when the feedback F is chosen as in the above discussion, for any fixed σ and any fixed p.

3. REGULARITY PROPERTIES OF $F_{\sigma,p}$

Now we can turn to the precise statement of the regularity properties of $F_{\sigma,p}$ such as continuity, incremental gains, differentiability, and so on, which we will study in this paper.

3.1. Statement of the incremental gain results

Recall that a $\mathcal{K}$-function g: $\mathbb{R}^+ \to \mathbb{R}^+$ is one that is continuous, strictly increasing, and satisfies $g(0)=0$.

Definition 1

The operator $F_{\sigma,p}$ satisfies the *generalized incremental gain* property (with respect to L^p) if

($\mathbf{GIG}_p$) there exists a $\mathcal{K}$-function g such that for all u, v in $L^p([0,\infty), \mathbb{R}^m)$,

$$\| F_{\sigma,p}(v) - F_{\sigma,p}(u)\|_{L^p} \leq g(\| v-u\|_{L^p})$$

It is obvious that $F_{\sigma,p}$ satisfies the $\mathbf{GIG}_p$ property if and only if it is *uniformly* continuous, i.e. iff for any given $\varepsilon>0$, there exists a $\delta>0$ such that $\|F_{\sigma,p}(u)-F_{\sigma,p}(v)\|_{L^p}\leq\varepsilon$ whenever $\|u-v\|_{L^p}\leq\delta$. Note that if g is linear, this is the standard 'finite incremental gain' property, or in mathematical terms, a global Lipschitz property.

It turns out that $\mathbf{GIG}_p$ is a very strong property. For most S-functions, even smooth ones, the operator $F_{\sigma,p}$ does not satisfy the $\mathbf{GIG}_p$ property. For general S-functions σ, $F_{\sigma,\infty}$ even fails to be continuous. However, for restricted classes of S-functions, more precisely the classes $\mathscr{C}_{(0)}$ and $\mathscr{C}^{1,+}$ defined below, $F_{\sigma,p}$ satisfies the following $\mathbf{SLP}_p$ property (*semiglobal Lipschitz property*):

($\mathbf{SLP}_p$) there exist $\mathcal{K}$-function g and a constant $c>0$ so that, for all u, v in $L^p([0,\infty), \mathbb{R}^m)$,

$$\| F_{\sigma,p}(v) - F_{\sigma,p}(u)\|_{L^p} \leq (c+g(\|u\|_{L^p}))\,\|v-u\|_{L^p}$$

This property clearly implies the continuity of $F_{\sigma,p}$.

The class $\mathscr{C}_{(0)}$ is defined as the class of functions σ: $\mathbb{R}\to\mathbb{R}$ which are globally Lipschitz, differentiable at 0 and satisfy

$$\lim_{\substack{t,s\to 0\\ t\neq s}} \frac{\sigma(t)-\sigma(s)}{t-s} = \sigma'(0) \tag{5}$$

An $\mathbb{R}^m$-valued S-function σ belongs to $\mathscr{C}_{(0)}$ if each of its components belongs to $\mathscr{C}_{(0)}$.

The class $\mathscr{C}^{1,+}$ is defined as the class of functions σ: $\mathbb{R}\to\mathbb{R}$ which are continuously differentiable and satisfy that σ' is everywhere positive. An $\mathbb{R}^m$-valued S-function σ belongs to $\mathscr{C}^{1,+}$ if each of its components belongs to $\mathscr{C}^{1,+}$.

The main results of this paper are summarized in the next theorem.

Theorem 1

Let σ be an $\mathbb{R}^m$-valued S-function and let $1 \leqslant p \leqslant \infty$. We have

(A) For each $1 \leqslant p < \infty$, the following conclusions hold:
- (i) $F_{\sigma,p}$ is continuous, but in general does not satisfy the $\mathbf{SLP}_p$ property.
- (ii) Assume that σ belongs to $\mathscr{C}_{(0)}$. Then $F_{\sigma,p}$ satisfies $\mathbf{SLP}_p$.
- (iii) Even for smooth nondecreasing saturation functions σ, $F_{\sigma,p}$ does not in general satisfy the $\mathbf{GIG}_p$ property.

(B) For $p = \infty$, the following conclusions hold:
- (i′) In general, $F_{\sigma,\infty}$ is not continuous.
- (ii′) Assume that each component of σ is nondecreasing. Then for $n = 1$, $F_{\sigma,\infty}$ is globally Lipschitz. If $n > 1$, even for $m = 1$ and σ nondecreasing, $F_{\sigma,\infty}$ need not be continuous.
- (iii′) Assume that σ belongs to $\mathscr{C}^{1,+}$. Then $F_{\sigma,\infty}$ satisfies $\mathbf{SLP}_\infty$.
- (iv′) Even for a smooth $\sigma \in \mathscr{C}^{1,+}$, $F_{\sigma,\infty}$ need not satisfy the $\mathbf{GIG}_\infty$ property.

3.2. *Statement of the differentiability results*

We can also discuss the differentiability properties of $F_{\sigma,p}$. First if σ is an $\mathbb{R}^m$-valued S-function, we say that σ is of class $\mathscr{C}^1$ if each component of σ is of class $\mathscr{C}^1$, i.e. continuously differentiable. We have:

Theorem 2

1. For $p = \infty$ and $\sigma \in \mathscr{C}^{1,+}$, $F_{\sigma,\mathrm{p}}$ is Fréchet-differentiable.
2. For $1 \leqslant p < \infty$ and σ of class $\mathscr{C}^1$ and globally Lipschitz, $F_{\sigma,p}$ is Gâteaux-differentiable.

We will give an example (Example 6 in Section 5) to show that $F_{\sigma,1}$ need not be Fréchet-differentiable even for smooth σ.

If $\sigma \in \mathscr{C}^{1,+}$ and $u,\ v \in L^\infty([0,\infty),\ \mathbb{R}^m)$, we will use $DF_{\sigma,\infty}(u)\ v$ to denote the differential of $F_{\sigma,\infty}$ at u applied to v. For each $1 \leqslant p < \infty$ and σ of class $\mathscr{C}^1$, we use $D_vF_{\sigma,p}(u)$ to denote the Gâteaux-differential of $F_{\sigma,p}$ at $u \in L^p([0,\infty),\ \mathbb{R}^m)$ in the direction v. It is well known that both $DF_{\sigma,\infty}(u)\ v$ and $D_vF_{\sigma,p}(u)$ are given by the linearization of (Σ) along the trajectory x of (Σ) corresponding to u (cf. Reference 6). In other words, $DF_{\sigma,\infty}(u)\ v$ and $D_vF_{\sigma,p}(u)$ are the respective solutions of the following time-varying initialized systems

$$(\Sigma^*(p,\sigma,u)) \qquad \dot{\xi} = A\xi + B\sigma'(\mathrm{F}\xi + u)(F\xi + v), \quad \xi(0) = 0 \tag{6}$$

where F is the $m \times n$ matrix given in the proof of Theorem (FG) and if $\sigma = (\sigma_1, \ldots, \sigma_m)$, $z \in \mathbb{R}^m$, then $\sigma'(z) = \operatorname{diag}(\sigma_1'(z_1), \ldots, \sigma_m'(z_m))$.

4. MAIN PROOFS

Now we prove the positive statements in Theorem 1 and Theorem 2. The negative results are covered in the next section.

When the context is clear, for simplicity, we will drop the indices σ, p or σ, ∞ and simply write F for $F_{\sigma,p}$ or $F_{\sigma,\infty}$, and use D_vF to denote $D_vF_{\sigma,p}(u)$ for $1 \leqslant p < \infty$.

4.1. Proof of (ii')

When $n=1$, system (1) is written as

$$\begin{aligned} \dot{x} &= -\alpha x + B\sigma(-B^{\mathrm{T}}x+u) \\ x(0) &= 0 \end{aligned} \qquad (\Sigma_\alpha)$$

where $A=-\alpha\leqslant 0$, $B=(b_1,\ldots,b_m)$ is a row vector, and $\sigma=(\sigma_1,\ldots,\sigma_m)^{\mathrm{T}}$.

Let $x=F(u)$. If v is also in $L^\infty([0,\infty),\mathbb{R}^m)$, define $y=F(v)$, $z=F(v)-F(u)$ and $h=v-u$. Let

$$D(t)=\operatorname{diag}(d_1(t),\ldots,d_m(t))$$

where

$$d_i(t)=\frac{\sigma_i(-b_iy(t)+v_i(t))-\sigma_i(-b_ix(t)+u_i(t))}{-b_iz(t)+h_i(t)} \quad \text{if } -b_iz(t)+h_i(t)\neq 0$$

and $d_i(t)=1$ if $-b_iz(t)+h_i(t)=0$. Since each σ_i is nondecreasing, $d_i(t)\geqslant 0$ a.e.

Now z satisfies

$$\begin{aligned} \dot{z}(t) &= -\alpha z(t) - BD(t)BDB^{\mathrm{T}}z(t) + BD(t)h(t) \\ z(0) &= 0 \end{aligned}$$

First assume that $\alpha>0$. Let $d(t)=BD(t)B^{\mathrm{T}}$. Then $d(t)\geqslant 0$. Let

$$\tau(t)=\int_0^t(\alpha+d(s))\,\mathrm{d}s$$

Then τ is strictly increasing on $[0,\infty)$, and onto $[0,\infty)$. Let $Z(s)=z(t)$ and

$$H(s)=\frac{BD(t)h(t)}{\alpha+d(t)}$$

if $\tau(t)=s$. Then Z satisfies

$$\begin{aligned} \frac{\mathrm{d}Z}{\mathrm{d}s} &= -Z+H \\ Z(0) &= 0 \end{aligned}$$

on $[0,\infty)$. Therefore the sup-norm $\|Z\|_{L^\infty[0,\infty)}$ of Z on $[0,\infty)$ is bounded by the sup-norm $\|H\|_{L^\infty[0,\infty)}$ of H on $[0,\infty)$. Then

$$\|z\|_{L^\infty}=\|Z\|_{L^p[0,\infty)}\leqslant\|H\|_{L^\infty[0,\infty)}$$

For each $0\leqslant\alpha<\infty$, let

$$C_\alpha=\sup\left[\frac{\left(\sum_{i=1}^m b_i^2\xi_i^2\right)^{1/2}}{\alpha+\sum_{i=1}^m b_i^2\xi_i}\,\Big|\,0<\xi_i\leqslant\|\sigma_i'\|_{L^\infty}(\leqslant\infty)\,i=1,\ldots,m\right]$$

Note that $0<C_\alpha\leqslant\infty$. Now for $\alpha>0$, we have $\|H\|_{L^\infty[0,\infty)}\leqslant C_\alpha\|h\|_{L^\infty}$. Therefore

$$\|F(v)-F(u)\|_{L^\infty}=\|z\|_{L^\infty}\leqslant C_\alpha\|v-u\|_{L^\infty}$$

This is also true for $\alpha=0$ because $C_\alpha\leqslant C_0$ and the trajectories of (Σ_α), for fixed u, v, converge to the trajectories of (Σ_0), uniformly on compact intervals as $\alpha\to 0^+$.

In the last section, we will show (Example 7) that the above constant C_α is in general the best possible Lipschitz constant.

4.2. *Proof of (iii')*

First observe that from the sketch of the proof of Theorem (FG) and the definition of $F_{\sigma,p}$ we may without loss of generality assume that A is skew-symmetric and (A,B) is controllable.

Take now any two u, v in $L^p([0,\infty), \mathbb{R}^m)$, where $1 \leq p \leq \infty$. We may also assume without loss of generality, when proving both of statements (ii) and (iii') on $\mathbf{SLP}_p$, that

$$\| v - u \|_{L^p} \leq \| u \|_{L^p} \tag{7}$$

Indeed, assume that for any constant C it would be the case that $\| v - u \|_{L^p} > C \| u \|_{L^p}$. Then, using finite gain L^p-stability, we have

$$\begin{aligned} \| F_{\sigma,p}(v) - F_{\sigma,p}(u) \|_{L^p} &\leq \| F_{\sigma,p}(v) \|_{L^p} + \| F_{\sigma,p}(u) \|_{L^p} \\ &\leq G_p(\| v \|_{L^p} + \| u \|_{L^p}) \leq G_p(\| v - u \|_{L^p} + 2 \| u \|_{L^p}) \\ &< \left(1 + \frac{2}{C}\right) G_p(\| v - u \|_{L^p} \end{aligned}$$

Thus the desired result holds and there would be nothing more to prove.

Since A is skew-symmetric, in this case system (1) is written as

$$\begin{aligned} \dot{x} &= Ax + B\sigma(-B^{\mathrm{T}}x + u) \\ x(0) &= 0 \end{aligned}$$

We will show that there exists a function κ: $\mathbb{R}_+ \to \mathbb{R}_+$ such that for any two functions $u, v \in L^\infty([0,\infty), \mathbb{R}^m)$, we have

$$\| F(u) - F(v) \|_{L^\infty} \leq \kappa(\| u \|_{L^\infty}) \| u - v \|_{L^\infty} \tag{8}$$

Once this is proved, the conclusion of (iii') would follow trivially, as follows. Consider the function

$$\tilde{g}(r) = \sup_{0 \leq s \leq r} \kappa(s) - \kappa(0)$$

Then $\tilde{g}(0) = 0$ and $\tilde{g}$ is increasing. Clearly we can take a $\mathcal{K}$-function g such that $\tilde{g}(r) \leq g(r)$ for all $r \in \mathbb{R}_+$. Then we have

$$\| F(u) - F(v) \|_{L^\infty} \leq (\kappa(0) + g(\| u \|_{L^\infty})) \| u - v \|_{L^\infty}$$

as desired. Now we show (8). Let $u, v \in L^\infty([0,\infty), \mathbb{R}^m)$. Let $h = v - u$, $x = F(u)$, $y = F(v)$, and $z = y - x$. We assume that (7) holds for u, v. Then z is the solution of

$$\begin{aligned} \dot{z} &= Az + B\{ \sigma(-B^{\mathrm{T}}y + v) - \sigma(-B^{\mathrm{T}}x + u) \\ z(0) &= 0 \end{aligned} \tag{9}$$

Write

$$\begin{aligned} \tilde{x} &= (\tilde{x}_1, \ldots, \tilde{x}_m)^{\mathrm{T}} = B^{\mathrm{T}}x \\ \tilde{y} &= (\tilde{t}_1, \ldots, \tilde{y}_m)^{\mathrm{T}} = B^{\mathrm{T}}y \\ \tilde{z} &= (\tilde{z}_1, \ldots, \tilde{z}_m)^{\mathrm{T}} = B^{\mathrm{T}}z \end{aligned}$$

For each $t \in [0, \infty)$, $1 \leq i \leq m$, by the mean-value theorem, there exists a

$$\xi_i(t) \in [\min\{\tilde{x}_i(t) + u_i(t), \tilde{y}_i(t) + v_i(\mathrm{t})\}, \max\{\tilde{x}_i(t) + u_i(t), -\tilde{y}_i(t) + v_i(t)\}]$$

such that

$$\sigma_i(-\tilde{y}_i(t) + v_i(t)) - \sigma_i(-\tilde{x}_i(t) + u_i(t)) = \sigma_i'(\xi_i(t))(-\tilde{z}_i(t) + \mathrm{h}_i(t))$$

Let

$$\begin{aligned} d_i(t) &= \sigma_i'(\xi_i(t)), \qquad \text{for } i = 1, \dots, m \\ D(t) &= \operatorname{diag}(d_1(t), \dots, d_m(t)) \end{aligned} \tag{10}$$

We may assume that each $d_i(t)$ is a measurable function, since whenever $\tilde{z}_i(t) \neq h_i(t)$ it is a quotient of two measurable functions and if $\tilde{z}_i(t) = h_i(t)$ any value can be chosen. Then (9) can be written as

$$\begin{aligned} \dot{z}(t)) &= Az(t) - BD(t)(B^{\mathrm{T}}z(t) - h(t)) \\ z(0)) &= 0 \end{aligned}$$

For $r > 0$, let $| - \tilde{e}(r) = 3(\| B \| G_\infty + 1)r$. Using (7) and the finite gain L^∞-stability, we have

$$\begin{aligned} |\xi_i(t)| &\leq \max\{|-\tilde{y}_i(t) + v_i(t)|, |-\tilde{x}_i(t) + u_i(t)|\} \\ &\leq |\tilde{y}_i(t) + v_i(t)| + |-\tilde{x}_i(t) + u_i(t)| \\ &\leq \|\tilde{y}\|_{L^\infty} + \|v\|_{L^\infty} + \|\tilde{x}\|_{L^\infty} + \|u\|_{L^\infty} \leq e(\|u\|_{L^\infty}) \end{aligned}$$

For each $r > 0$, let

$$m(r) = \min_{1 \leq i \leq m} \inf_{\xi \in [-e(r), e(r)]} \sigma_i'(\xi)$$

$$M(r) = \max_{1 \leq i \leq m} \sup_{\xi \in [-e(r), e(r)]} \sigma_i'(\xi)$$

Then we get

$$0 < m(\|u\|_{L^\infty}) \leq d_i(t) \leq M(\|u\|_{L^\infty}) < \infty$$

(Note that the positivity of $m(\|u\|_{L^\infty})$ and the finiteness of $M(\|u\|_{L^\infty})$ follow from the assumption that $\sigma \in \mathscr{C}^{1,+}$.) Now the existence of $\kappa: \mathbb{R}_+ \to \mathbb{R}_+$ such that (8) holds follows from the next lemma. (We state the lemma for arbitrary p, not just $p = \infty$, since it will be used again later.

Lemma 1

Let A be an $n \times n$ skew-symmetric matrix and B an $n \times m$ matrix. Assume that $\tilde{A} = A - BB^{\mathrm{T}}$ is Hurwitz. Let $D(t)$ be an $m \times m$ matrix with bounded measurable entries. Assume that there exists a constant $a > 0$ such that

$$D_S(t) \triangleq D(t) + D^{\mathrm{T}}(t) \geq aI \quad \text{for almost all } t \text{ in } [0, +\infty) \tag{11}$$

Then the following initialized system

$$\begin{aligned} \dot{x} &= (A - BD(t)B^{\mathrm{T}})x + u \\ x(0) &= 0 \end{aligned} \tag{Σ}$$

where $u \in L^p([0, +\infty), \mathbb{R}^n)$, is finite gain L^p-stable, and the L^p-gain depends only on p, a, A, B, and $b = \sup_{t \in [0, \infty)} \|D(t)\|$.

This lemma has been given in Reference 5. For the sake of completeness, we enclose the proof here.

Proof of Lemma 1. Fix a $1<p<\infty$ Since $\tilde{A}$ is Hurwitz, there exists a differentiable function V_p: $\mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$, we have

(1) $a_p\|x\|^p \leqslant V_p(x) \leqslant b_p\|x\|^p$
(2) $\|DV_p(x)\| \leqslant c_p\|x\|^{p-1}$
(3) $DV_p(x)\tilde{A}x \leqslant -\|x\|^p$

where a_p, b_p, c_p are some positive constants. We can choose V_p so that $\limsup_{p\to 1+} c_p = c_1 < \infty$. For any $1 \leqslant p < \infty$, let

$$\lambda_p = \frac{2[c_p(b+\sqrt{m})\|B\|]^2}{a}$$

For $1<p<\infty$, let

$$\tilde{V}_p(x) = \lambda_p \frac{\|x\|^p}{p} + 2V_p(x)$$

Then along the trajectories of $(\tilde{\Sigma})$, we have

$$\begin{aligned}
\dot{\tilde{V}}_n(x(t)) \leqslant{}& \lambda_p\|x(t)\|^{p-2}[-\; x^{\mathrm{T}}(t)BDs(t)B^{\mathrm{T}}x(t) + x^{\mathrm{T}}(t)u(t)]\\
&+2\|x(t)\|^{p-2}[-\|x(t)\|^2 + c_p\|x(t)\|\;\|B\|\;\|D(t)-I\|\;\|B^{\mathrm{T}}x(t)\| + c_p\|x(t)\|\;\|u(t)\|]\\
\leqslant{}& \|x(t)\|^{p-2}\Big[-\frac{a}{2}\lambda_p\|B^{\mathrm{T}}x(t)\|^2 + \lambda_2\|x(t)\|\,\|u(t)\| - 2\|x(t)\|^2\\
&+2c_p(b+\sqrt{m})\|B\|\;\|x(t)\|\;\|B^{\mathrm{T}}x(t)\| + 2c_p\|x(t)\|\;\|u(t)\|\Big]\\
={}& \|x(t)\|^{p-2}[-\|x(t)\|^2 - \{\|x(t)^2 - 2c_p(b+\sqrt{m})\|B\|\;\|x(t)\|\;\|B^{\mathrm{T}}x(t)\\
&+[c_p(b+\sqrt{m})\|B\|]^2\|B^{\mathrm{T}}x(t)\|^2\} + (\lambda_p + 2c_p)\|x(t)\|\,\|u(t)\|]
\end{aligned}$$

Letting $d_p = \lambda_p + 2c_p$, we obtain

$$\dot{\tilde{V}}_p(x(t)) \leqslant -\|x(t)\|^p + d_p\|x(t)\|^{p-1}\|u(t)\|$$

Integrating the above inequality from 0 to t, since $\tilde{V}_p \geqslant 0$, we get

$$\int_0^t \|x(s)\|^p\,\mathrm{d}s \leqslant d_p \int_0^t \|x(s)\|^{p-1}\|u(s)\|\,\mathrm{d}s \tag{12}$$

If we let $d_1 = \lambda_1 + 2c_1$, by the Lebesgue Dominated Convergence theorem (applied to any sequence $\{p^j\}_{j=1}^{\infty}$ decreasing to 1) we know that above inequality is also true for $p=1$.

Now applying Hölder's inequality to $\int_0^t \|x(s)\|^{p-1}\|u(s)\|\,\mathrm{d}s$ in (12) we get that

$$\|x\|_{L^p[0,t]} \leqslant d_p\|u\|_{L^p}$$

Letting $t \to p < \infty$ we conclude that $\|x\|_{L^p} \leqslant d_p\|u\|_{L^p}$. This finishes the proof of the lemma for $1 \leqslant p < \infty$.

For $p=\infty$, consider $\tilde{V}_2$. We have, along trajectories of $(\tilde{\Sigma})$,

$$\dot{\tilde{V}}_2(x(t) \leqslant -\|x(t)\|^2 + d_2\|x(t)\|\;\|u(t)\| \leqslant -\|x(t)\|(\|x(t)\| - d_2\|u\|_\infty). \tag{13}$$

Therefore,

$$\tilde{V}_2(x(t)) \leq \max_{\|\xi\| = d_2\|u\|_{L^\infty}} \tilde{V}_2(\xi) \leq \frac{\lambda_2 d_2^2}{2}\|u\| + 2b_2 d_2^2\|u\|_{L^\infty}^2 = d_2^2\left(\frac{\lambda_2}{2} + 2b_2\right)\|u\|_{L^\infty}^2$$

Since $\tilde{V}_2(x) \geq (\lambda_2/2)\|x\|^2$, we end up with

$$\|x\|_\infty \leq d_2\left(1 + \frac{4b_2}{\lambda_2}\right)^{1/2}\|u\|_{L^\infty} \qquad \square$$

Thus the proof of (iii′) is completed. It remains to show that the positive parts of (i) and (ii) hold. Let $1 \leq p < \infty$.

4.3. *Proof of* (i)

Let us fix a u in $L^p([0,\infty), \mathbb{R}^m)$. To prove the continuity of F, it is enough to show that for any sequence $\{u^j\}_{j=1}^\infty$ in $L^p([0,\infty), \mathbb{R}^m)$ such that $\lim_{j\to\infty}\|u^j - u\|_{L^p} = 0$, then $\lim_{j\to\infty}\|F(u^j) - F(u)\|_{L^p} = 0$. Let u^j and u be functions in $L^p([0,\infty), \mathbb{R}^m)$ such that $\|u^j - u\|_{L^p} \to 0$ as $j \to \infty$. Let $x = F(u)$ and $x^j = F(u^j)$. For any $T > 0$ and $j > 0$, we have

$$\|x^j - x\|_{L^p}^p \leq \int_0^T \|x^j(s) - x(s)\|^p\,ds + 2^p\int_0^\infty \|x^j(s) - x(s)\|^p\,ds + 2^p\int_0^\infty \|x^j(s)\|^p\,ds$$

Recalling (4) and integrating it from T to ∞ we have

$$\int_T^\infty \|x^j(s)\|^p\,ds \leq V_p(x^j(T)) + \kappa_p\int_T^\infty \|u^j(s)\|^p\,ds$$

$$\leq V_p(x^j(T)) + 2^p\kappa_p(\|u^j - u\|_{L^p}^p + \|u\|_{L^p[T,\infty)}^p)$$

Next we observe that $x(t) \to 0$ as $t \to \infty$. (This has been remarked in Reference 5 in a similar context. We only need to consider $\varphi_i(t) = x_i^p(t)$ for $i = 1, \ldots, m$. Then we know that φ_i is integrable on $[0,\infty)$. It is easily verified by a direct computation that $\dot{\varphi}_i$ is also in L^p on $[0,\infty)$. Therefore $\varphi_i(t) \to 0$ as $t \to \infty$ as claimed.)

Fix an $\varepsilon > 0$. There exists $T > 0$ such that

$$2^p\left(\int_T^\infty \|x(s)\|^p\,ds + 2V_p(x(T))\right) + 4^p\kappa_p\|u\|_{L^p[T,\infty)}^p \leq \varepsilon/2$$

For the fixed T, it can be proved that the x^j converge uniformly to x on $[0, T]$. Therefore there exists a $J > 0$ such that, if $j \geq J$, $V_p(x^j(T)) \leq 2V_p(x(T))$ and

$$\int_0^T \|x^j(s) - x(s)\|^p\,ds + 4^p\kappa_p\|u^j - u\|_{L^p}^p \leq \varepsilon/2$$

Therefore, when $j \geq J$, we have $\|x^j - x\|_{L^p}^p \leq \varepsilon$.

4.4. *Proof of* (ii)

As in the proof of (iii′), we again assume that A is skew-symmetric and (A, B) is controllable.

Assume that $\sigma = (\sigma_1, \ldots, \sigma_m)^{\mathrm{T}}$. Since A is skew-symmetric and (A, B) is controllable. The matrix $\tilde{A} = A - B\tilde{D}B^{\mathrm{T}}$ is Hurwitz, where $\tilde{D} = \mathrm{diag}(\sigma_1'(0), \ldots, \sigma_m'(0))$. Let $P > 0$ satisfy

$$P\tilde{A} + \tilde{A}^{\mathrm{T}}P = -I \tag{14}$$

Let $\lambda_{\max}$ and $\lambda_{\min}$ be respectively the largest and smallest eigenvalue of P and let

$$\beta = \frac{1}{4\sqrt{m}\,\|B\|\,\|PB\|}$$

Since σ belongs to $\mathscr{C}_{(0)}$, there exists an $\alpha > 0$ such that, for $|s| \leq \alpha$, $|t| \leq \alpha$ and $t \neq s$:

$$\left| \frac{\sigma_i(t) - \sigma_i(s)}{t - s} - \sigma_i'(0) \right| \leq \beta \quad \text{for } i = 1, \ldots, m$$

Fix u and v in $L^p([0,\infty), \mathbb{R}^m)$ for which (7) holds. As in the proof of (iii′), letting $x = F(u)$ and $y = F(v)$, then x, y satisfy

$$\begin{aligned} \dot{x} &= Ax + B\sigma(-B^{\mathrm{T}}x + u) \\ \dot{y} &= Ay + B\sigma(-B^{\mathrm{T}}y + v) \\ x(0) &= y(0) = 0 \end{aligned}$$

Write $z = y - x$, $h = v - \mathrm{u}$ and let $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_i$ denote respectively the ith component of $B^{\mathrm{T}}x$, $B^{\mathrm{T}}y$, $B^{\mathrm{T}}z$. We have

$$\begin{aligned} \dot{z} &= Az + BD(t)(-B^{\mathrm{T}}z + h) \\ z(0) &= 0 \end{aligned} \tag{15}$$

where

$$D(t) \triangleq \operatorname{diag}(d_1(t), \ldots, d_m(t))$$

$$d_i(t) \triangleq \frac{\sigma_i(-\tilde{y}_i(t) + v_i(t)) - \sigma(-\tilde{x}_i(t) + u_i(t))}{-\tilde{z}_i(t) + h_i(t)}$$

(If $\tilde{z}_i(t) - h_i(t) = 0$ we just let $d_i(t) = \sigma_i'(0)$.) Let $K > 0$ be a Lipschitz constant for σ (more precisely, let K be a Lipschitz constant for each component of σ). Then $\|d_i\|_{L^p} \leq K$. So $\|D(t)\| \leq \sqrt{m}K$.

Let

$$E = \bigcup_{i=1}^{m} \{t \mid |d_i(t) - \sigma_i'(0)| > \beta\}$$

Clearly

$$E = \subseteq \bigcup_{i=1}^{m} \left\{ \{t \mid |\tilde{x}_i(t) - u_i(t)| > \alpha\} \bigcup \{t \mid |\tilde{y}_i(t) - v_i(t)| > \alpha\} \right\}$$

Therefore, by Chebyshev's inequality we get

$$|E| \leq \tilde{C}(\|u\|_{L^p}^p + \|v\|_{L^p}^p)$$

for some constant $\tilde{C} > 0$ independent of u and v. Noticing (7) we have $|E| \leq C\|u\|_{L^p}^p$, where $C > 0$ is a constant independent of u, v.

If we let $V(z) = z^T P z$ for $z \in \mathbb{R}^n$, where P is defined in (14), we get along the trajectories of (15):

$$\begin{aligned} \dot{V}(z(t)) &= -\|z(t)\|^2 - 2z(t)^{\mathrm{T}}PB[(D(t) - \tilde{\mathrm{D}})B^{\mathrm{T}}z(t) - D(t)h(t)] \\ &\leq -[1 - 2\|B\|\,\|PB\|\,\|D(t) - \tilde{D}\|]\,\|z(t)\|^2 + 2\sqrt{m}K\|PB\|\,\|z(t)\|\,\|h(t)\| \end{aligned}$$

Therefore, along the trajectories of (15), V satisfies the differential inequality

$$\begin{aligned} \dot{V}(z(t)) &\leqslant 2\lambda(t)V(z(t)) + 2C_1 V^{1/2}(z(t))\|h(t)\| \\ V(0) &= 0 \end{aligned} \tag{16}$$

where

$$\lambda(t) = \begin{cases} C_2 & \text{if } t \in E \\ -C_3 & \text{if } t \notin E \end{cases}$$

and the constants C_1, C_2 and C_3 are respectively equal to

$$\frac{\sqrt{m}K\|PB\|}{\lambda_{\min}^{1/2}}, \qquad \frac{1+4\sqrt{m}K\|B\|\|PB\|}{\lambda_{\min}} \qquad \text{and} \qquad \frac{1}{2\lambda_{\max}}$$

Let $\Lambda(t) = \int_0^t \lambda(s)\,\mathrm{d}s$. From (16), if $W(t) = \mathrm{e}^{-2\Lambda(t)}V(z(t))$, we obtain

$$\dot{W}(t) \leqslant 2C_1 W^{1/2}(t)\mathrm{e}^{-\Lambda(t)}\|h(t)\|$$

and then

$$W^{1/2}(t) \leqslant C_1 \int_0^t \mathrm{e}^{-\Lambda(s)}\|h(s)\|\,\mathrm{d}s$$

which gives

$$V^{1/2}(z(t)) \leqslant C_1 \int_0^t \mathrm{e}^{(\Lambda(t)-\Lambda(s))}\|h(s)\|\,\mathrm{d}s$$

But for $z \in \mathbb{R}^n$, $V^{1/2}(z) \geqslant \lambda_{\min}^{1/2}\|z\|$, and if $t \geqslant s$, $\Lambda(t) - \Lambda(s) \leqslant (C_2 + C_3)|E| - C_3(t-s)$. Therefore, we have

$$\|z(t)\| \leqslant C_4 \int_0^t \mathrm{e}^{-C_3(t-s)}\|h(s)\|\,\mathrm{d}s$$

where

$$C_4 = \frac{C_1\mathrm{e}^{(C_2+C_3)|E|}}{\lambda_{\min}^{1/2}}\left(\frac{C_1\mathrm{e}^{C(C_2+C_3)\|u\|_{L^p}^p}}{\lambda_{\min}^{1/2}}\right)$$

We conclude from the previous inequality that

$$\|z\|_{L^p} \leqslant \Gamma(\|u\|_{L^p})\|\;\|_{L^p}$$

for some $\Gamma(\|u\|_{L^p}) > 0$.

4.5. *Proof of Theorem 2, Part 1*

Again we may assume that (A, B) is controllable. We first show conclusion 1 under the extra assumption that A is skew-symmetric.

Let $u, v \in L^\infty([0,\infty), \mathbb{R}^m)$, and let $DF_{\sigma,\infty}(u)\,v$ be the solution of (6). For simplicity, we write F for $F_{\sigma,\infty}$. Let $x = F(u)$, $y = F(v)$ and $w = F(v) - F(u) - DF(u)\,(v-u)$. Assume now that u is fixed. We want to show that

$$\|w\|_{L^\infty} = o(\|v-u\|_{L^\infty}) \qquad \text{as} \quad \|v - \mathrm{u}\|_{L^\infty} \to 0.$$

Let $h=v-u$ and $z=y-x$. Then from Theorem 1 part (iii′) we know that $\|z\|_{L^\infty}\leq C(\|u\|_{L^\infty})\|h\|_{L^\infty}$ for some constant $C(\|u\|_{L^\infty})>0$. Let $\tilde{x}=B^{\mathrm{T}}x$, $\tilde{y}=B^{\mathrm{T}}y$ and $\tilde{z}=B^{\mathrm{T}}z$. Again by the mean-value theorem, for each i, there exists a

$$\xi_i(t)\in[\min\{-\tilde{x}_i(t)+u_i(t),-\tilde{y}_i(t)+v_i(t)\},\max\{-\tilde{x}_i(t)+u_i(t),-\tilde{y}_i(t)+v_i(t)\}]$$

such that

$$\sigma_i(-\tilde{y}_i(t)+v_i(t))-\sigma_i(-\tilde{x}_i(t)+u_i(t))=\sigma_i'(\xi_i(t))(-\tilde{z}_i(t)+h_i(t)) \tag{18}$$

Let

$$\xi(t)=(\xi_1(t),\dots,\xi_m(t)),$$
$$D(t)=\sigma'(\xi(t))=\mathrm{diag}(\sigma_1'(\xi_l(t)),\dots,\sigma_m'(\xi_1(t)))$$

As in the proof of (iii′), $D(t)$ can be taken to be measurable. Then w satisfies

$$\dot{w}=[A-B\sigma'(-B^{\mathrm{T}}x(t)+u(t))B^{\mathrm{T}}]w+B\{[D(t)-\sigma'(-B^{\mathrm{T}}x(t)+u(t)][-B^{\mathrm{T}}z(t)+h(t)]\}$$

Since $\|-B^{\mathrm{T}}x+u\|_{L^\infty}$ is finite and σ is in $\mathscr{C}^{1,+}$, there exist $0<a\leq b<\infty$ such that $aI\leq\sigma'(-B^{\mathrm{T}}x(t)+u(t))\leq bI$ a.e. in $[0,\infty)$. Applying Lemma 1, we have

$$\begin{aligned}\|w\|_{L^\infty}&\leq\tilde{G}_\infty\|B\|\,\|[D-\sigma'(-B^{\mathrm{T}}x+u)](B^{\mathrm{T}}z-h)\|_{L^\infty}\\&\leq\tilde{G}_\infty\|B\|[C(\|u\|_{L^\infty})\|B\|+1])\,\|D-\sigma'(-B^{\mathrm{T}}x+u)\|_{L^\infty}\|h\|_{L^\infty}\end{aligned} \tag{19}$$

where $\tilde{G}_\infty$ is the L^∞-gain of the system $\dot{w}=[(A-B\sigma'(-B^{\mathrm{T}}x(t)+u(t))B^{\mathrm{T}}]w+\tilde{u}$, $w(0)=0$. By definition we have

$$\begin{aligned}&D(t)-\sigma'(-B^{\mathrm{T}}x(t)+\mathrm{u}(t))\\&\quad=\mathrm{diag}[\sigma_1'(\xi_1(t))-\sigma_1'(-\tilde{x}_l(t)+u_1((t)),\dots,\sigma_m'(\xi_m(t))-\sigma_m'(-\tilde{x}_m(t)+u_m(t))]\end{aligned}$$

But we know that

$$|\xi_i(t)-(-x_i(t)+u_i(t))|\leq|\tilde{z}_i(t)-h_i(t)|\leq(C(\|u\|_{L^\infty})+1)\|h\|_{L^\infty}$$

If we assume that $\|h\|_{L^\infty}\geq1$, then $|\xi_i(t)|$ is bounded. Let M be such that $|\xi_i(t)|\leq M$, $|\tilde{x}_i(t)-u_i(t)|\leq M$ if $\|h\|_{L^\infty}\geq1$ (M may depend on u). Let $K=\tilde{G}_\infty\|B\|(C(\|u\|_{L^\infty})\|B\|+1)$. Then by the uniform continuity of σ_i' on $[-M,M]$ we conclude that for any given $\varepsilon>$ there exists a $0<\delta<1$ such that $\|D-\sigma'(-B^{\mathrm{T}}x+u)\|_{L^\infty}\leq\varepsilon/K$ if $\|h\|_{L^\infty}\leq\delta$. So from (19) we conclude that $\|w\|_{L^\infty}\leq\varepsilon\|h\|_{L^\infty}$.

This was all assuming that A was skew-symmetric. Now consider a general (neutrally stable) A. Let $u,v\in L^\infty([0,\infty),\mathbb{R}^m)$ with u being fixed. Then we know that under a suitable change of coordinates, system (1) can be written as

$$\begin{aligned}\dot{x}_1&=A_1x_1+B_1\sigma(-B_2^{\mathrm{T}}x_2+u)\\\dot{x}_2&=A_2x_2+B_2\sigma(-B_2^{\mathrm{T}}x_2+u)\\x_1(0)&=0,\qquad x_2(0)=0\end{aligned} \tag{20}$$

where A_1 is Hurwitz and A_2 is skew-symmetric. Now (6) is written as

$$\begin{aligned}\dot{\xi}_1&=A_1\xi_1+B_1\sigma'(-B_2^{\mathrm{T}}x_2+u)(-B_2^{\mathrm{T}}\xi_2+v)\\\dot{\xi}_2&=A_2\xi_2+\mathrm{B}_2\sigma'(-B_2^{\mathrm{T}}x_2+u)(-B_2^{\mathrm{T}}\xi_2+v)\\\xi(0)&=0\end{aligned}$$

Again let $h = v - u$, $z = y - x$, $w = z - DF(u)h$. Write $w = (w_1, w_2)^{\mathrm{T}}$. Then the above proof implies that $\| w_2 \|_{L^\infty} = o(\| h \|_{L^\infty})$. We need to show that

$$\| w_1 \|_{L^\infty} = o(\| h \|_{L^\infty}) \tag{21}$$

Clearly w_1 satisfies

$$\begin{aligned}\dot{w}_1 = A_1 w_1 + B_1(\sigma'(\xi(t)) - \sigma'(-B_2^{\mathrm{T}} x_2(t) + u(t)))(-B_2^{\mathrm{T}} z_2(t) + h(t)) \\ - B_1 \sigma'(-B_2^{\mathrm{T}} x_2(t) + u(t) B_2^{\mathrm{T}} w_2(t)\end{aligned} \tag{22}$$

where ξ is defined similar to (18). Since $\| B_2^{\mathrm{T}} z_2 - h \|_{L^\infty} \leq \tilde{C}(\| u \|_{L^\infty}) \| h \|_{L^\infty}$ for some $\tilde{C}(\| u \|_{L^\infty}) > 0$, A_1 is Hurwitz, $\| \sigma'(\xi) - \sigma'(-B^{\mathrm{T}} x_2 + u) \|_{L^\infty} = o(1)$, and $\| B_1 \sigma'(-B_2^{\mathrm{T}} x_2 + u) B_2^{\mathrm{T}} w_2 \| = o(\| h \|_{L^\infty})$, (22) implies that (21) holds too. This finishes the first part of Theorem 2.

4.6. Proof of Theorem 2, Part 2

Let $1 \leq p < \infty$ and $u, v \in L^p([0, \infty), \mathbb{R}^m)$. As in establishing Part 1, we will first show conclusion 2 under the assumption that A is skew-symmetric. Let ν be a real number such that $0 < |\nu| \leq 1$. Write

$$x = F(u), \qquad y_\nu = F(u + \nu v), \qquad z_\nu = \frac{y_\nu - x}{\nu} \qquad \text{and} \qquad w_\nu = z_\nu - D_\nu F(u) \tag{23}$$

where $D_v F(u)$ is the solution of (6). We have to show that $\lim_{\nu \to 1} \| w_\nu \|_{L^p} = 0$. In the sequel, C_i will denote positive constants depending on A, B, σ, u, v and p.

Now z_ν satisfies

$$\dot{z} = Az + BD_\nu(t)(-B^{\mathrm{T}} z + v(t)) \tag{24}$$

with

$$D_\nu(t) = \mathrm{diag}[(\sigma_1'(\xi_{\nu,1}(t)), \ldots, \sigma_{\mathrm{V}}'(\xi_{\nu,m}(t))]$$

where the $\xi_{\nu,i}$ satisfy

$$\sigma_i(-\tilde{y}_{\nu,i}(t) + u_i(t) + \nu v_i(t)) - \sigma_i(-\tilde{x}_i(t) + u_i(t)) = \sigma_i'(\xi_{\nu,i}(t))(-\tilde{y}_{\nu,i}(t) + \tilde{x}_i(t) + \nu v_i(t))$$

Exactly by the same proof as in part (ii) of Theorem 1 (cf. (17)), we can prove that there exist constants C_1, $C_2 > 0$ (independent of $\nu \in [-1, 1]$) such that

$$\| z_\nu(t) \| \leq C_1 \int_0^t \mathrm{e}^{C_2(s-t)} \| v(s) \| \, \mathrm{d}s$$

Let $g(t) = C_1 \int_0^t \mathrm{e}^{C_2(s-t)} \| v(s) \| \, \mathrm{d}s$. Then $\| g \|_{L^q}$ is finite for any $p \leq q \leq \infty$. The above inequality can be written as

$$\| z_\nu(t) \| \leq g(t) \tag{25}$$

for all $t \in [0, \infty)$. So $\| z_\nu \|_{L^\infty} \leq \| g \|_{L^\infty}$. By the definition of z_ν we have

$$\| y_\nu - x \|_{L^\infty} \leq \| g \|_{L^\infty} | \nu | \tag{26}$$

Let $D = \mathrm{diag}(\sigma_+'(0), \ldots, \sigma_m'(0))$. Let $\tilde{A} = A - BDB^{\mathrm{T}}$. Then $\tilde{A}$ is Hurwitz. Let $P > 0$ satisfy

$$P\tilde{A} + \tilde{A}^{\mathrm{T}} P = -I$$

Define $\varepsilon_1 \triangleq 1/(4\sqrt{m} \| B \| \, \| PB \|)$. Let ε be such that $0 > \varepsilon \leq \varepsilon_1$. We show that $\| w_\nu \|_{L^p} \leq \varepsilon$ if ν is small enough.

Write $\bar{x} = B^{\mathrm{T}}x$. For any constant $M > 0$, let $H_M = \cup_{i=1}^{m} \{t \mid |ax_i(t) - u_i(t)| \geq M\}$. Then we know that there exists $C_3 > 0$ such that $|H_M| \leq C_3/M^p$. Let's fix an $M > 1$ large enough such that

$$\int_{H_M} \|z_v(t)\|^p \, \mathrm{d}t \leq \varepsilon (\text{cf. } (25)) \tag{27}$$

$$\int_{H_M} \|v(t)\|^p \, \mathrm{d}t \leq \varepsilon \tag{28}$$

By the continuity of the σ_i' there exists $0 < \varepsilon' \leq 1/2$ such that for $i = 1, \ldots, m$

$$|\sigma_i'(\xi) - \sigma_i'(\eta)| < \varepsilon \tag{29}$$

if $\xi, \eta \in [-2M, 2M]$ and $|\xi - \eta| > \varepsilon'$.

Let $E_i = \{t \mid |\bar{x}_i(t) - u_i(t)| > \varepsilon'\}$ and $E = \cup_{i=1}^{m} E_i$. By Chebyshev's inequality we have $|E| \leq C_4$ for some constant $C_4 > 0$.

Write $\bar{y}_\nu = (\bar{y}_{\nu,1}, \ldots, \bar{y}_{\nu,m})^{\mathrm{T}} = B^{\mathrm{T}} y_\nu$. Define

$$G_\nu = \bigcup_{i=1}^{m} \{t \mid |\bar{y}_{\nu,i}(t) - \bar{x}_i(t) - \nu x_i(t)| > \varepsilon'\}$$

Noticing (26), we have for $|\nu|$ small enough

$$G_\nu \subseteq \bigcup_{i=1}^{m} \{t \mid |\nu x_i(t)| > \varepsilon'/2\}$$

Therefore by Chebyshev's inequality again we have

$$|G_\nu| \leq \frac{2^p \nu^p}{(\varepsilon')^p} \sum_{i=1}^{m} \|v_i\|_{L^p}^p \tag{30}$$

Notice that if $t \notin H_M \cup G_\nu$, then

$$|\bar{y}_{\nu,i}(t) - u_i(t) - \nu v_i(t)| \leq |\bar{y}_{\nu,i}(t) - \bar{x}_i(t) - \nu v_i(t)| + |\bar{x}_i(t) - \mathrm{u}_i(t)| \leq \varepsilon' + M < 2M$$

Now w_ν satisfies

$$\dot{w}_\nu = [A - B\sigma'(-B^{\mathrm{T}}x(t) + u(t)B^{\mathrm{T}}]w_\nu + B[D_\nu(t) - \sigma'(-B^{\mathrm{T}}\mathrm{x}(t) + u(t))][-B^{\mathrm{T}}z_\nu(t) + v(t)] \tag{31}$$

If we let $V(\xi) = \xi^{\mathrm{T}} P \xi$ for $\xi \in \mathbb{R}^n$, we get

$$\begin{aligned} \dot{V}(w_\nu(t)) = -\|w_\nu(t)\|^2 - 2w_\nu^{\mathrm{T}}(t)PB[\sigma'(-B^{\mathrm{T}}x(t) + u(t)) - D]B^{\mathrm{T}}w_\nu(t) \\ + 2w_\nu^{\mathrm{T}}(t)PB[D_\nu(t) - \sigma'(-B^{\mathrm{T}}x(t) + u(t))][-B^{\mathrm{T}}z_\nu(t) + v(t)] \end{aligned} \tag{32}$$

Therefore, we conclude that there exist positive constants C_5, C_6, C_7 such that for almost all $t \in [0, \infty)/(E \cup H_M \cup G_\nu)$,

$$\dot{V}(w_\nu(t)) \leq -2C_5 V(w_\nu(t)) + 2C_6 \varepsilon V^{1/2}(w_\nu(t)) \| - B^{\mathrm{T}} z_\nu(t) + v(t)\|$$

for almost all $t \in E/(H_M \cup G_\nu)$,

$$\dot{V}(w_\nu(t)) \leq 2C_7 V(w_\nu(t)) + 2C_6 \varepsilon \mathrm{V}^{1/2}(w_\nu(t)) \| - B^{\mathrm{T}} \mathrm{z}_\nu(t) + v(t)\|$$

and for almost all $t \in H_M \cup G_\nu$,

$$\dot{V}(w_\nu(t)) \leq 2C_7 \mathrm{V}(w_\nu(t)) + 2C_6 V^{1/2}(w_\nu(t)) \| - B^{\mathrm{T}} \mathrm{z}_\nu(t) + v(t)\|$$

The previous inequalities imply

$$\dot{V}(w_\nu(t)) \leqslant 2\lambda_1(t)V(w_\nu(t)) + 2\lambda_2(t)V^{1/2}(w_\nu(t))\,\|-B^{\mathrm{T}}z_\nu(t) + v(t)\| \tag{33}$$

where

$$\lambda_1(t) = \begin{cases} -C_5 & \text{if } t \notin E \cup H_M \cup G_\nu, \\ C_7 & \text{if } t \in E \cup H_M \cup G_\nu, \end{cases} \quad \text{and} \quad \lambda_2(t) = \begin{cases} C_{6\varepsilon} & \text{if } t \notin H_M \cup G_\nu \\ C_6 & \text{if } t \in H_M \cup G_\nu \end{cases}$$

Similar to the proof of part (ii) in Theorem1 we obtain an inequality similar to (17), namely

$$\|w_\nu(t)\| \leqslant C_8 \int_0^t \mathrm{e}^{-C_9(t-\tau)}|\lambda_2(\tau)|\,\|(B^{\mathrm{T}}z_\nu(\tau) - v(\tau)\|\,\mathrm{d}\tau \tag{34}$$

for some constants C_8, $C_9 > 0$. Using (34) and the definition of λ_2, there exists $C_{10} > 0$ such that

$$\begin{aligned}\|w_\nu\|^p_{L^p} &\leqslant C_{10}\left[\varepsilon^p\|B^{\mathrm{T}}z_\nu - v\|^p_{L^p} + \int_{H_M\cup G_\nu}\|B^{\mathrm{T}}z_\nu(\tau) - v(\tau)\|^p\,\mathrm{d}\tau\right] \\ &\leqslant C_{10}\left[\varepsilon^p\|B^{\mathrm{T}}z_\nu - v\|^p_{L^p} + 2^p\|B\|^p\int_{H_M}\|z_\nu(\tau)\|^p\,\mathrm{d}\tau + 2^p\int_{H_M}\|v(\tau)\|^p\,\mathrm{d}\tau\right. \\ &\qquad \left. + \int_{G_\nu}\|B^{\mathrm{T}}z_\nu(\tau) - v(\tau)\|^p\,\mathrm{d}\tau\right]\end{aligned}$$

Noticing (25), (27) and (28) we get that

$$\|w_\nu\|^p_{L^p} \leqslant C_{10}\left(\varepsilon^p\|B^{\mathrm{T}}z_\nu - v\|^p_{L^p} + 2^p(\|B\|^p + 1)\varepsilon + 2^p\int_{G_\nu}(\|B\|^p g^p(\tau) + \|v(\tau)\|^p)\,\mathrm{d}\tau\right)$$

Now from (30) we get that the last integral in the right-hand side of the above inequality goes to 0 as $\nu \to 0$. Since ε is arbitrary, we have $\lim_{\nu\to 0}\|w_\nu\|_{L^p} = 0$.

Assume now that A is not skew-symmetric. Let $1 \leqslant p < \infty$, $u, v \in L^p([0,\infty), \mathbb{R}^m)$ and for $0 < |\nu| \leqslant 1$ use the functions introduced in (23). Write $w_\nu = (w_{\nu,1}, w_{\nu,2})^{\mathrm{T}}$. Then we need to show that

$$\lim_{\nu\to\infty}\|w_{\nu,1}\|_{L^p} = 0 \tag{35}$$

As in (22), $w_{\nu,1}$ satisfies

$$\begin{aligned}\dot{w}_{\nu,1} &= A_1 w_{\omega\nu,1} + B_1[D_{\nu,2}(t) - \sigma'(-B_2^{\mathrm{T}}x_2(t) + u(t))](-B_2^{\mathrm{T}}z_{\nu,2}(t) + v(t)) \\ &\quad - B_1\sigma'(-B_2^{\mathrm{T}}x_2(t) + u(t))B_2^{\mathrm{T}}w_{\nu,2}\end{aligned} \tag{36}$$

where $D_{\nu,2}(t)$ is defined in the same manner as $D_\nu(t)$, with the difference that $\tilde{x}_{\nu,i}$ and $\tilde{y}_{\nu,i}$ are now equal respectively to the ith component of $B_2{}^{\mathrm{T}}x_2$ and $B_2^{\mathrm{T}}y_{\nu,2}$. Fix $\varepsilon > 0$. Then taking into account the fact that A_1 is Hurwitz, $\lim_{\nu\to 0}\|w_{\nu,2}\|_{L^p} = 0$ and (25), we can get that there exists a constant $C > 0$ independent of ν such that

$$\|w_{\nu,1}\|^p_{L^p} \leqslant C\left[\|w_{\nu,2}\|^p_{L^p} + \varepsilon^p\|B_2^{\mathrm{T}}z_{\nu,2} - v\|^p_{L^p} + \int_{H_M\cup G_\nu}\|B_2^{\mathrm{T}}z_{\nu,2}(\tau) - v(\tau)\|^p\,\mathrm{d}\tau\right]$$

From the previous paragraph, the proof of conclusion 2 is now complete.

5. COUNTER-EXAMPLES

We now turn to the negative statements in Theorem 1. These are established by exhibiting counter-examples. More precisely, Examples 1, 2, 3, and 4 below refer respectively to (i'), (ii′), (iv′), and (i), while Example 5 refers to (iii). Finally, Example 6 refers to the fact that $F_{\sigma,1}$ need not be Fréchet-differentiable in general.

5.1. $F_{\sigma,\infty}$ *need not be continuous*

Example 1. Consider the one-dimensional initialized control system

$$\begin{aligned}\dot{x} &= -\sigma(x+u)\\ x(0) &= 0\end{aligned} \tag{37}$$

where the S-function σ verifies the following condition: there exists an $\alpha > 0$ such that for $\varepsilon = \pm 1$ and $|t| \leqslant \alpha$, we have

$$\sigma(\varepsilon + t) = \varepsilon - t$$

Let

$$h(t) = \begin{cases} -1 & \text{on } [2n, 2n+1), n \geqslant 0 \\ 1 & \text{on } [2n+1, 2n+2), n \geqslant 0 \end{cases}$$

and $x = -\int_0^t \sigma(h(s))\,\mathrm{d}s$. Then $\|x\|_{L^\infty} = 1$. Letting $u(t) = h(t) - x(t)$, we know that x is the solution of (37) corresponding to u. Clearly $\|u\|_{L^\infty} = 2$.

For $m > 1$, let $s_m = \alpha e^{-m}$ and

$$v_m(t) = h(t) + s_m e^t + \int_0^t \sigma(h(\tau) + s_m e^\tau)\,\mathrm{d}\tau$$

for $0 \leqslant t \leqslant m$ and $v_m(t) = u(t)$ for $t > m$. Let y_m be the solution of (37) corresponding to v_m. If we let $z_m = y_m - x$ and $h_m = v_m - u$, then on $[0, m]$ we have

$$z_m(t) = s_m(e^t - 1) \qquad \text{and} \qquad h_m(t) = s_m$$

Therefore, $\|h_m\|_{L^\infty} = s_m$ and $\|z_m\|_{L^\infty} \geqslant s_m(e^m - 1)$. Since $\lim_{m\to\infty} s_m = 0$ and $\|z_m\|_{L^\infty} \geqslant \alpha/2$ for m large enough, F is not continuous at u.

5.2. $F_{\sigma,\infty}$ *Need not be continuous even for nondecreasing* σ

Note that the example that follows could of course also be used to establish (i′), instead of using Example 1. However, it is far more complicated to analyse than the preceding one.

Example 2. We provide an example in which the input to state operator defined at the very end of Section 2, $F_{\sigma_0,\infty}$ (or F) corresponding to the system

$$\begin{aligned}\dot{x} &= Ax - b\sigma_0(x_2 + u)\\ x(0) &= 0\end{aligned}$$

is not continuous. The data are

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and σ_0 is the standard saturation function. The idea is to argue by contradiction: in a first step, we construct an l.a.c. curve $\bar{z}(t)$ such that

$$\lim_{t\to\infty} \|\bar{z}(t)\| = \infty \tag{38}$$

and a bounded input $\bar{h}$; in a second step, using the curve $\bar{z}(t)$ we exhibit another l.a.c. curve $x(t)$ such that there exists an input u for which

$$\|u\|_{L^\infty} < \infty, \qquad x = F(u)$$

Now define

$$z_s = \frac{F(u+sh) - F(u)}{s} \tag{39}$$

and assuming that F is continuous at u, we have $\lim_{s\to 0} sz_s = 0$. We show then that for $s>0$ small enough, $z_s \equiv \bar{z}$, which contradicts (38). Therefore, F cannot be continuous at u.

Let us start by the construction of $\bar{z}$. In the plane (z_1, z_2), define for $t \geq 0$ the l.a.c. curve $z(t)$ as follows:

(a) $z(0) = (\sqrt{2}, 0)$

(b) $\dot{z} = \begin{cases} Az + bz_2 & \text{if } |z_2| \leq 1 \\ Az & \text{if } |z_2| > 1 \end{cases}$

Since $z^{\mathrm{T}}\dot{z} \geq 0$, $\|z(t)\|$ is nondecreasing and in particular

$$\|z(t)\| > \|z(0)\| = \sqrt{2}$$

at the times $t>0$ for which $z(t)$ is defined.

Furthermore, one checks that in fact, the curve $z(t)$ is well defined for $t \geq 0$. (The times t so that $|z_2(t)| = 1$ are isolated because for each of them, there exists an open neighbourhood $(t-\tau, t+\tau)$ where $|\dot{z}_2| > 0$).

Now, by writing $z(t)$ in polar coordinates, if

$$\dot{\theta} = \frac{\dot{z}_2 z_1 - \dot{z}_1 z_2}{z_1^2 + z_2^2}$$

is the angular velocity, we have for all $t \geq 0$,

$$\tfrac{1}{2} \leq \dot{\theta} \leq \tfrac{3}{2}$$

The previous remarks imply that there exist two increasing sequences $(T_n)_{n\geq 0}$, $(T'_n)_{n\geq 0}$ and two positive numbers C_1, C_2 such that

(1) $T_0 = 0$, $T_n < T'_n < T_{n+1}$ for $n \geq 0$;

(2) $T'_n - T_n \sim_{n\to\infty} \dfrac{C_2}{n^{1/3}}$;

(3) $\lim_{n\to\infty} T_{n+1} - T'_n = \pi$;

(4) $|z_2| \leq 1$ on $[T_n, T'_n]$ and $|z_2| > 1$ on (T'_n, T_{n+1}) for $n \geq 0$;

(5) $\|z(t)\| \sim_{t\infty} C_1 t^{1/3}$;

(6) $\lim_{t\to\infty} \dot{\theta}(t) = 1$.

From this construction, we have $\lim_{t\to\infty} \|z(t)\| = \infty$. Let

$$d_0(t) = \begin{cases} 1 & \text{on } [T_n, T'_n] \\ 0 & \text{on } (T'_n, T_{n+1}),\ n \geq 0 \end{cases}$$

$$h_0(t) = \begin{cases} -2z_2(t) & \text{on } [T_n, T'_n] \\ 0 & \text{on } (T'_n, T_{n+1}),\ n \geq 0 \end{cases}$$

Then z is the solution of the system (E_0)

$$\dot{z} = Az - d_0(t)b(z_2 + h_0(t))$$
$$z(0) = z_0$$

Let (E_1) be the planar system

$$\dot{x} = Ax - b\sigma_0[f(x,t)]$$
$$x(0) = x_0$$

where A, b are defined as above, $x_0 \neq 0$ is small and

$$f(x,t) = \begin{cases} 0 & \text{if } t \in [T_n, T_{n'}] \\ 2 & \text{if } t \in (T_n', T_{n+1}), |x_1| > 2 \text{ and } x_2 \geq 0 \\ -2 & \text{if} \in (T'_n, T_{n+1}), |x_1| > 2 \text{ and } x_2 < 0 \\ 2 & \text{if } t \in (T'_n, T_{n+1}) \text{ and } |x_1| \leq 2 \end{cases}$$

Once again, one checks that the previous curve is well defined for $t \geq 0$. Note that if $\|x(t)\| \geq 3\sqrt{2}$, then $1/2 \leq \dot{\theta} \leq 3/2$ and $\dot{x}x^{\mathrm{T}}(t) \geq 0$ if and only if $t \in (T'_n, T_{n+1})$ for some $n \geq 0$, $|x_1(t)| \leq 2$ and $x_2(t) < 0$.

Let us show that $\|x\|_{L^\infty} < \infty$. Fix an integer n_0 such that for all $n \geq n_0$

$$\frac{9\pi}{10} \leq T_{n+1} - T'_n \leq \frac{11\pi}{10}$$

Then writing (E_1) in polar coordinates gives

$$\dot{r} = -\sin\theta\sigma_0(t(x,t)), \qquad |\dot{\theta} - 1| \leq \frac{1}{r} \tag{40}$$

Consider the positive constant $r_0 = 15$ and the first time $t_0 \geq T_{n_0}$ such that $\|x(t_0)\| = r(t_0) \geq r_0$. Define also

$$r_1 = \sup_{t \leq t_0} \|x(t)\| \geq r_0$$

If such t_0 does not exist then we are done. Otherwise, in the worst case, there exist $T' < T''$ in (T'_{n_0}, T_{n_0+1}) such that for $t \in [T', T'']$

$$x_1(T') = -2, x_1(T'') = 2, |x_1(t)| \leq 2 \qquad \text{and} \qquad x_2(t) \leq 0$$

A direct computation shows that one of the two following cases occurs:

(1) $\|x(T_{n_0+1})\| \leq r_0$, or
(2) for $n \geq n_0$, $\|x(T_n)\|$ is decreasing and

$$\sup_{t_0 \leq t \leq T_n} \|x(t)\| \leq 2r_1$$

as long as $\|x(T_n)\| \geq r_0$

Therefore, if the $n \geq n_0$ for which (2) is satisfied are unbounded, we are again done. Otherwise, there exists an $n_1 \geq n_0$ for which $\|x(T_{n_1})\| \leq r_0$. Once again, consider the first time $t_1 \geq n_1$ such that

$$\|x(T'_{n_1}) = \|x(T_{n_1})\| = r(T'_{n_1}) \geq r_0$$

If t_1 does not exist, the assertion is proved. Otherwise one shows that $\|x(T_{n_1+1})\| \leq r_0$ and

$$\sup_{T_{n_1} \leq t \leq T_{n_1+1}} \|x(t)\| \leq 2r_0$$

By repeating this argument if necessary, it follows that

$$\|x\|_{L^\infty} \leq 2r_1 < \infty \qquad \text{and} \qquad \|u_0\|_{L^\infty} \leq \|x\|_{L^\infty} + 2$$

Furthermore, there exist $t_0' \geq 0$ and two bounded inputs u_1 and h_1 such that:

(i) the solution of

$$\begin{aligned} \dot{x} &= Ax - b\sigma_0(x_2 + u_1) \\ x(0) &= 0 \end{aligned}$$

reaches x_0 in time t_0' and $|x_2(t) + u_1(t)| \leq 1/2$ for $t \in [0, t_0']$;

(ii) the solution of

$$\begin{aligned} \dot{z} &= Az - b(z_2 + \mathrm{h}_1) \\ z(0) &= 0 \end{aligned}$$

reaches $(\sqrt{2}, 0)$ in time t_0'.

Concatenate u_1 and $u_0 \triangleq f(x(t), t) - x_2(t)$, h_1 and h_0 to respectively the bounded inputs u and $\tilde{h}$. For x the solution of

$$\begin{aligned} \dot{x} &= Ax - b\sigma_0(x_2 + u) \\ x(0) &= 0 \end{aligned}$$

we get $\|x\|_{L^\infty} < \infty$. Note that for all $t \geq 0$, $|x_2(\mathrm{t}) + u(\mathrm{t})| \leq 1/2$ or $|x_2(t) + u(\mathrm{t})| = 2$. Then, $\tilde{z}$, the solution of

$$\begin{aligned} \dot{z} &= Az - d(t)b(z_2 + h) \\ z(0) &= 0 \end{aligned}$$

where

$$d(t) = \begin{cases} 1 & \text{on } [0, t_0) \\ d_0(t - t_0) & \text{on } [t_0, \infty) \end{cases}$$

is in fact the solution of

$$\begin{aligned} (E_2)\ \dot{z} &= Az - \sigma_0'(x_2 + \mathrm{u})b(z_2 + h) \\ z(0) &= 0 \end{aligned}$$

where $\sigma_0'(\cdot)$ stands for the derivative of σ_0 with respect to its argument.

Suppose that F is continuous at u. For $0 < s < 1$, define z_s by (38) and note that it is the solution of

$$\dot{z} = Az - \frac{\sigma_0(x_2 + u + s(z_2 + h)) - \sigma_0(x_2 + u)}{s(z_2 + h)}\, b(z_2 + h)$$

$$z(0) = 0$$

Since F is continuous at u,

$$|s(z_{s,2}(t) + h(\mathrm{t}))| \leq 1/4$$

for s small enough, and then z_s is the solution of (E_2) and then $z_s \equiv \tilde{z}$.

Therefore, for $s>0$ and small enough, $\lim_{t\to\infty} s\|z_s(t)\| = \infty$, which contradicts the continuity of F at u.

5.3. *Failure of* $\mathbf{GIG}_\infty$*, even for* $\sigma \in \mathscr{C}^{1,+}$

Example 3. The next example serves to show that $\mathbf{GIG}_\infty$ does not hold for all $\sigma \in \mathscr{C}^{1,+}$. Pick any element σ of $\mathscr{C}^{1,+}$ (i.e., σ is continuously differentiable and $\sigma'>0$) such that in addition, σ is smooth, $\sigma(t)=t$ for $t\in[-\alpha_0, \alpha_0]$, where $\alpha_0>0$ is a positive constant, $\sigma''>0$ on $(-\infty, -\alpha_0)$ and $\sigma''<0$ on (α_0, ∞). Let A and b be as in the previous example. Thus, the system that we consider now is essentially the same as in Example 2, with the only difference that the standard saturation function σ_0 is now replaced by any σ which satisfies the above properties.

If $F_{\sigma,\infty}=F$ is the corresponding input-to-state operator, we will prove that for any α, $\beta>0$, there exist u, v in $L^\infty([0,\infty), \mathbb{R})$ such that:

$$\|v-u\|_{L^\infty}\le\alpha \qquad \text{and} \qquad \|F(v)-F(u)\|_{L^\infty}\ge\beta$$

The strategy is to construct two l.a.c. curves $z(t)$ and $h(t)$ such that

$$\lim_{t\to\infty}\|z(t)\| = \infty, \qquad \|h\|_{L^\infty}\le\alpha$$

and then two other l.a.c. curves $x(t)$ and $y(t)$ such that:

(a) $y(t)-x(t)=z(t)$ for $t\in[0,t_1']$ where t_1' is chosen so that $\|z(t_1')\|\ge\beta$;
(b) there exist two essentially bounded inputs u, v such that $x=F(u)$, $y=F(v)$, $v(t)-u(t)=h(t)$ for $t\in[0,t_1']$ and $v(t)=u(t)=0$ for $t\ge t_1'$.

Consider α, $\beta>0$. We choose a $z_0=(\bar z, 0)\ne 0$ so that $|\bar z|\le 1$ and there exist $t_*\ge 0$ and a bounded input $\tilde h$ for which

(i) $\|\tilde h\|_{L^\infty}\le\alpha$;
(ii) the solution $\zeta=(\zeta_1,\zeta_2)$ of

$$\dot\zeta = A\zeta-\sigma(\zeta_2(t)+\tilde h(t)),$$
$$\zeta(0)=0$$

reaches z_0 at time t_* and $\zeta_2(t)+\tilde h(t)\ne 0$ a.e. in $[0,t_*)$.

Define the constants $\alpha'>0$ and r_0 such that

$$\alpha' = \min\left(1, \frac{\alpha}{2}, \frac{\|z_0\|}{2}\right)$$

$$r_0 = \frac{2^8\pi^3}{\alpha'^3}$$

For $\varepsilon>0$ in $(0,1)$ and for $t\ge t_*$ consider $z_\varepsilon(t)$ the l.a.c. curve defined by

(a) $z_\varepsilon(t_*)=z_0$;

(b) $\dot z_\varepsilon = \begin{cases} Az_\varepsilon + bz_{\varepsilon,2} & \text{if } |z_{\varepsilon,2}|\le\alpha' \\ Az_\varepsilon + b\varepsilon z_{\varepsilon,2} & \text{if } |z_{\varepsilon,2}|>\alpha' \end{cases}$

(Note that for a given $\varepsilon \in (0,1)$, z_ε need not be defined for all $t \geqslant t_*$.) It is easy to see by using polar coordinates, that if we choose $\varepsilon_0 > 0$ small enough, then there exists $t_0 \geqslant t_*$ such that

$$r(t_0) = \| z_{\varepsilon_0}(t_0)\| > 2r_0, \qquad z_{\varepsilon_0,2}(t_0) > 0 \qquad \text{and} \qquad z_{\varepsilon_0,2}(t_0) = -\alpha$$

Now, for $t \geqslant t_0$ consider the l.a.c. curve $z'(t)$ defined by

(a) $z'(t_0) = z_\varepsilon(t_0)$;

(b) $$\dot{z}' = \begin{cases} Az' + bz_2' & \text{if } |z_2'| \leqslant \alpha' \\ Az' - b\dfrac{z_2'}{\|z_2'\|^4} & \text{if } \|z_2' > \alpha' \end{cases}$$

Write $z'(t)$ and $\dot{z}'(t)$ in polar coordinates $r(t)$, $\theta(t)$ and define for $n \geqslant 1$

$$r_n = r(t_{n-1}) \text{ with } z_1'(t_{n-1}) > 0 \qquad \text{and} \qquad z_2'(t_{n-1}) = -\alpha'$$

By an induction argument, we show that

(1) the sequences $(t_n)_{n \geqslant 0}, (r_n)_{n \geqslant 1}$ are well-defined,
(2) $\forall n \geqslant 1$, $r(t) \geqslant r_n$ if $t \geqslant t_{n+1}$,
(3) there exists a constant $C > 0$ depending only on r_0 such that for all $n \geqslant 0$,

$$r_{n+1}^4 \geqslant r_n^4 + Cr_n$$

Therefore $\lim_{t \to \infty} r(t) = \infty$.

Concatenate ζ, z_{ε_0} and z' to obtain an l.a.c. curve $z(t)$ from $[0, \infty)$ to $\mathbb{R}^2$. Then, define for $t \geqslant 0$, the functions $d(t)$, $h(t)$ by

$$d(t) = \begin{cases} \dfrac{\sigma(\zeta_2(t) + \tilde{h}(t)}{\zeta_2(t) + \tilde{h}(t)} & \text{a.e. in } [0, t_*) \\ 1 & \text{if } t \geqslant t_*, |z_2(t)| \leqslant \alpha' \\ \varepsilon_0 & \text{if } t_* \leqslant t \leqslant t_0 \text{ and } |z_2(t)| > \alpha' \\ \dfrac{1}{\|z(t)\|^4} & \text{if } t \geqslant t_0 \text{ and } |z_2(t)| > \alpha' \end{cases}$$

and

$$h(t) = \begin{cases} \tilde{h}(t) & \text{on } [0, t_*) \\ -2z_2(t) & \text{if } t \geqslant t_*, |z_2(t)| \leqslant \alpha' \\ 0 & \text{if } t \geqslant t_*, |z_2(t)| > \alpha' \end{cases}$$

We observe that z is actually the solution of

$$\dot{z} = Az - d(t)b[z_2 + h(t)]$$
$$z(0) = 0$$

and therefore we get that $\|h\|_{L^\infty} \leqslant \alpha$ and $\lim_{t \to \infty} \|z(t)\| = \infty$.

In order to construct x and y as we want, it is enough to know $\xi(t) \triangleq x_2(t) + u(t)$, because of the following formula:

$$x(t) = \mathrm{e}^{At}\int_0^t \mathrm{e}^{-As} b\sigma[\xi(s)]\, \mathrm{d}s \tag{41}$$

This is simply done by considering the equation in ξ

$$\frac{\int_{\xi}^{\xi+z_2(t)+h(t)} \sigma'}{z_2(t)+h(t)} = d(t) \tag{42}$$

with $z_2(t)+h(t)\neq 0$ and $\sigma' 0(\xi)=d(t)$ if $z_2(t)+h(t)=0$. The times $t\geq t_*$ for which $z_2(t)+h(\mathrm{t})=0$ are isolated and then z_2+h is a piecewise continuous function. Therefore, for the times $t\geq t_*$ such that $|z_2(t)|\leq\alpha$ it is enough to set $\xi=0$ in (42) and for the other times $t\geq t_*$, we can choose a piecewise continuous selection for ξ in (42).

Therefore, if t_1 is picked so that $\|z(t_1)\|\geq\beta$ (without loss of generality we can suppose $t_1\geq t_0$), we construct x on $[0,t_*)$ by taking $x=0$, then on $[t_*,\tau_1)$, by using the solution of (42) in (41) and finally $u=0$ for $t\geq \mathrm{t}_1$. As for y, it is defined by $x+z$ on $[0,\mathrm{t}_1)$ and $v=0$ for $t\geq t_1$. To conclude the construction, notice that

$$v(t)-u(t)=\begin{cases} h(t) & \text{on } [0,t_1) \\ 0 & \text{on } [t_1,\infty) \end{cases}$$

5.4. *Failure of* $\mathbf{SLP}_p$ *for arbitrary* σ

Example 4. Next, we deal with counter examples to property $\mathbf{SLP}_p$. Let $\sigma(t)=\int_0^t\sigma'(s)\,\mathrm{d}s$, where σ' is an even function and for $t>0$ is given by

$$\sigma'(t)=\begin{cases} 0 & \text{if } t\geq 1 \\ 1 & \text{if } t\in\left[\dfrac{1}{(n+1)^2}+\dfrac{1}{(n+1)^4},\dfrac{1}{n^2}\right),\ n\geq 1 \\ -1 & \text{if } t\in\left[\dfrac{1}{(n+1)^2},\dfrac{1}{(n+1)^2}+\dfrac{1}{n^2}\right),\ n\geq 1 \end{cases}$$

It is easy to verify that σ is an S-function and obviously σ does not belong to $\mathscr{C}_{(0)}$. Consider the system

$$\begin{aligned} \dot{x} &= -\sigma(x+u) \\ x(0) &= 0 \end{aligned} \tag{43}$$

Let

$$h(t)=\begin{cases} 0 & \text{on } [0,1] \\ u_n & \text{on } [n,n+1/2),\ n\geq 1 \\ -u_n & \text{on } [n+1/2,n+1) \end{cases}$$

with

$$u_n=\frac{1}{(n+1)^2}+\frac{1}{2(n+1)^4}$$

for $n\geq 1$. Let $x(t)=-\int_0^t\sigma(h(s))\,\mathrm{d}s$. Then, if we let $u=h-x$, x is the solution of (43). Clearly for all $1\leq p<\infty$, $\|x\|_{L^p}<\infty$, $\|u\|_{L^p}<\infty$.

For $m\geq 1$, define

$$s_m=\frac{e^{-m}}{2(m+1)^4}$$

and let $w_m = -\int_0^t \sigma(g(s))\,ds$, where

$$g(t) = \begin{cases} 0 & \text{if } t \in [0, 1] \\ h(t) + s_m e^t & \text{if } t \in [1, m] \\ 0 & \text{if } t \in (m, \infty) \end{cases}$$

Now let $v_m = g(t) - w_m(t)$ on $[0, m]$ and $v_m = u$ for $t > m$. Let y_m be the solution (43) corresponding to v_m. Then, if $z_m = y_m - x$ and $h_m = v_m - u$, we have on $[1, m]$:

$$z_m(t) = s_m(e^t - e) \qquad \text{and} \qquad h_m(t) = s_m e.$$

Therefore, for m large enough, $\|h_m\|_{L^p} \leq m^{1/p} s_m e$ and $\|z_m\|_{L^p} \geq s_m [\int_1^m (e^s - e)^p\,ds]^{1/p}$. Let

$$a_m = \left[\int_0^{m-1} (e^s - e)^p\,ds\right]^{1/p}$$

We get

$$\frac{\|y_m - x\|_{L^p}}{\|v_m - u\|_{L^p}} \geq m^{-1/p} a_m$$

which goes to ∞ as $m \to \infty$. Therefore F does not satisfy $\mathbf{SLP}_p$ at u.

5.5. *Failure of* $\mathbf{GIG}_p$, *even for nondecreasing* σ

Example 5. Let σ be an S-function that satisfies the following condition. There exists a $\delta > 0$ such that $\sigma(t) = t$ if $|t| \leq \delta$ and $\sigma(t) = \operatorname{sign}(t)$ if $|t| \geq 1 + \delta$. For example, σ could even be the standard saturation function. Consider the one-dimensional system

$$\begin{aligned} \dot{x} &= -\sigma(x + u) \\ x(0) &= 0 \end{aligned} \tag{44}$$

Let $1 \leq p < \infty$ be a real number. Let $a > 1 + \delta$, $0 < \varepsilon < \delta$ be two real numbers. Take two inputs $u, v \in L^p([0, \infty), \mathbb{R})$ as follows:

$$\begin{aligned} &u(t) = v(t) = -t - 1 - \delta, && \text{for } 0 \leq t \leq a \\ &u(t) = -a,\ v(t) = -\varepsilon(t - a) - a - \varepsilon, && \text{for } a > t \leq a + 1 \\ &u(t) = v(t) = 0, && \text{if } t > a + 1 \end{aligned}$$

Let x, y be the solutions of (44) corresponding to u, v respectively. Then we have for $a \leq t \leq a + 1$,

$$x(t) = a,\ y(t) = a + \varepsilon(t - a)$$

and

$$x(t) = 2a + 1 - t,\ y(t) = 2a + \varepsilon + 1 - t$$

for $a + 1 \leq t \leq 2a - \delta$. Therefore

$$\int_0^\infty |y(s) - x(s)|^p\,ds > \int_{a+1}^{2a-\delta} |y(s) - x(s)|^p\,ds = \varepsilon^p(a - 1 - \delta)$$

So, $\|y - x\|_{L^p} \geq \varepsilon(a - 1 - \delta)^{1/p}$. On the other hand,

$$\|v - u\|_{L^p} = \varepsilon\left(\int_a^{a+1} |t - a + 1|^p\,dt\right)^{1/p} = \varepsilon\left(\frac{2^{p+1} - 1}{p + 1}\right)^{1/p}$$

Noticing that a and ε could be almost arbitrary, we have shown that for any $\alpha, \beta > 0$, there exist u, v in $L^p([0,\infty), \mathbb{R})$ such that

$$\|v - u_{L^p}\| \leq \alpha \qquad \text{and} \qquad \|F(v) - F(u)_{L^p}\| \geq \beta$$

5.6. *Nondifferentiability of* $F_{\sigma,1}$

Example 6. Consider a saturation function σ continuously differentiable, linear in a neighbourhood of 0 (i.e. $\sigma(t) = t$ for $|t|$ small enough) and the control system

$$\begin{aligned} \dot{x} &= -\sigma(x+u) \\ x(0) &= 0 \end{aligned} \tag{45}$$

Fix an $\alpha > 1$ and let $K = \|\sigma\|_{L^p}$. Take $u = 0$ and consider the sequence $\{u^j\}_{j=1}^{\infty}$ of inputs defined by

$$u^j(t) = \begin{cases} j & \text{on } [0, j^{-\alpha}] \\ 0 & \text{on } (j^{-\alpha}, \infty) \end{cases}$$

Then $\|u^j{}_{L^1} = j^{1-\alpha} \to 0$ as $j \to \infty$. Note that for any $v \in L^1([0,\infty), \mathbb{R})$, $D_v F(0)$ is the solution of

$$\dot{z} = -z + v, \qquad z(0) = 0$$

Then for j large enough and $t \geq j^{-\alpha}$, we have

$$|F(u^j)(t)| \leq K j^{-\alpha} e^{-t+j^{-\alpha}} \qquad \text{and} \qquad D_{u^j}F(0)(t) = j(1 - e^{-j^{l\alpha}}) e^{-t+j^{-\alpha}}$$

Then

$$\frac{\|F(u^j) - D_{u^j}F(0)\|_{L^1}}{\|u^j\|_{L^1}} \geq 1/2$$

for j large enough. Therefore, $F_{\sigma,1}$ is not Fréchet-differentiable at 0.

5.7. *One last example*

The last example shows that the Lipschitz constant C obtained in the proof of part (ii′), Theorem 1 is the best possible one. We keep the notations used in the proof of Theorem 1.

Example 7. Let σ be a saturation function. Consider the system

$$\begin{aligned} \dot{x} &= -\alpha x + B\sigma(-B^{\mathrm{T}}x + u) \\ x(0) &= 0 \end{aligned} \tag{46}$$

With no loss of generality, we can assume that $b_k > 0$ for $k = 1, \ldots, m$. Fix $\varepsilon > 0$ and an $\omega = (\omega_1, \ldots, \omega_m)$ such that

$$0 < \omega_k < \|\sigma'_k\|_{L^\infty}(\leq \infty), \qquad k = 1, \ldots, m$$

and

$$C_\alpha - \varepsilon \leq \frac{\left(\sum_{k=1}^{m} b_k^2 \omega_k^2\right)^{1/2}}{\alpha + \sum_{k=1}^{m} b_k^2 \omega_k} < C_\alpha$$

We can pick $\varepsilon > 0$ small enough so that

$$\alpha + \sum_{k=1}^{m} b_k^2 \omega_k - \varepsilon > 0$$

Let $\xi_0 = (\xi_1, \ldots, \xi_m)$ be chosen so that for $k = 1, \ldots, m$, σ_k is differentiable at ξ_k and

$$|\sigma_k'(\xi_k) - \omega_k| \leqslant \varepsilon/2$$

Define $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_m)$. Furthermore, there exists an $\varepsilon > 0$ such that if $\|\xi - \xi_0\| < \varepsilon'$, then

$$\left\| \frac{\sigma(\xi) - \sigma(\xi_0)}{(\xi - \xi_0)} - \sigma'(\xi_0) \right\| < \varepsilon/2$$

For $\alpha > 0$, let x_α be the solution of the system

$$\dot{x} = -\alpha x + B\sigma(\xi_0)$$
$$x(0) = 0$$

If we let $u_\alpha(t) = \xi_0 + B^{\mathrm{T}} x_\alpha(t)$, then x_α is the solution of (46) corresponding to u_α.

Pick a constant $h_0 > 0$ so that $h_0(1 + \|B\| C_\alpha) < \varepsilon'$. Let

$$h = \frac{h_0}{\left(\sum_{k=1}^{m} b_k^2 \omega_k^2\right)^{1/2}} (b_1\omega_1, \ldots, b_m\omega_m)$$

and $v_\alpha = \xi_0 + B^{\mathrm{T}} x_\alpha + h$. Let y_α be the solution of (46) corresponding to v_α. Let $z_\alpha = y_\alpha\ x_\alpha$. Then z_α satisfies

$$\dot{z} = -\alpha z - d(t)z + BD(t)h$$
$$z(0) = 0$$

where

$$D(t) = \frac{\sigma(\xi_0 + B^{\mathrm{T}} z_\alpha + h) - \sigma(\xi_0)}{B^{\mathrm{T}} z_\alpha + h}$$

and $d(t) = BD(t)\mathrm{B}^{\mathrm{T}}$.

By the choice of h_0 and the fact that $F_{\sigma,\infty}$ is globally Lipschitz, we have $\|B^{\mathrm{T}} z_\alpha + h_{L^\infty} \leqslant \|B\| C_\alpha h_0 + h_0 < \varepsilon'$. Then, for $t \geqslant 0$, we get

$$\|D(t) - \Omega\| \leqslant \varepsilon$$

In that case, $\|x_\alpha\|_{L^\infty}$, $\|y_\alpha\|_{L^\infty} < \infty$ and a direct computation shows that there exists a positive constant μ (independent of ε) such that

$$\liminf_{t \to \infty} \left| \frac{z_\alpha(t)}{h_0} \right| \geqslant C_\alpha - \mu\varepsilon$$

Letting $\varepsilon \to 0$, we prove that C_α is the desired Lipschitz constant.

Now assume that $\alpha = 0$. For $T > 0$, define $x_T(t) = B\sigma(\xi_0)t$ on $[0, T]$ and as the solution of

$$\dot{x} = B\sigma(-B^{\mathrm{T}} x),\ x(T) = B\sigma(\xi_0)T$$

for $t \geqslant T$. Then consider y_T defined on $[0, T]$ as the solution of (46) corresponding to $v = \xi_0 + B^{\mathrm{T}} x_T + h$ and for $t \geqslant T$, defined as the solution of

$$\dot{x} = B\sigma(-B^{\mathrm{T}}x),\ x(T) = y_T(T)$$

Finally set $z_T = y_T - x_T$. If we define the input $h_T = h$ for $t \in [0, T]$ and 0 for $t \geqslant T$, we obtain for T large enough, there exists a positive constant μ (independent of ε) such that

$$\frac{\| z_T \|_{L^\infty}}{\| h_T \|_{L^\infty}} \geqslant C_0 - (\mu + 1)\varepsilon$$

As in the case $\alpha > 0$, we conclude that C_0 is the desired Lipschitz constant.

One can also notice that $C_\alpha \to C_0$ as $\alpha \to 0^+$ and under the extra assumption that $\| \sigma_k' \|_{L^\infty} = \infty$ for some integer k, C_α is independent of $\alpha \geqslant 0$ and is equal to $(\inf_{k=1,\dots,m} | b_k |)^{-1}$.

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