Brief Paper

Finite gain stabilization of discrete-time linear systems subject to actuator saturation☆

Xiangyu Bao[a], Zongli Lin[a,*], Eduardo D. Sontag[b]

[a]*Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903, USA*
[b]*Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*

Abstract

It is shown that, for neutrally stable discrete-time linear systems subject to actuator saturation, finite gain l_p stabilization can be achieved by linear output feedback, for all $p \in (1, \infty]$. An explicit construction of the corresponding feedback laws is given. The feedback laws constructed also result in a closed-loop system that is globally asymptotically stable, and in an input-to-state estimate. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Input saturation; Discrete-time linear systems; Finite gain stability; Lyapunov functions

1. Introduction

In this paper, we consider the problem of global stabilization of a discrete-time linear system subject to actuator saturation:

$$\mathscr{P}: \begin{cases} x^+ = Ax + B\sigma(u + u_1), & x \in \mathbb{R}^n, \, u \in \mathbb{R}^m, \\ y = Cx + u_2, & y \in \mathbb{R}^r \end{cases} \tag{1}$$

(we use the notation x^+ to indicate a forward shift, that is, for a function x and an integer t, $x^+(t)$ is $x(t + 1)$), where $u_1 \in \mathbb{R}^m$ is the actuator disturbance, $u_2 \in \mathbb{R}^r$ is the sensor noise, and $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ represents actuator saturation, i.e., $\sigma(s) = [\sigma_1(s_1) \; \sigma_2(s_2) \; \cdots \; \sigma_m(s_m)]$ with $\sigma_i(s_i) = \text{sign}(s_i)\min\{1, |s_i|\}$, and the pair (A, B) is stabilizable. The problem of global asymptotic stabilization (internal stabilization) of this system has recently been solved using nonlinear state feedback laws and under the condition that all the eigenvalues of A are inside or on the unit circle (Yang, Sontag & Sussmann, 1997), and for neutrally stable open-loop system using linear state

feedback (Choi, 1999). Here, we are interested not only in closed-loop state space stability (internal stability), but also in stability with respect to both measurement and actuator noises. More specifically, we would like to construct a controller $\mathscr{C}$ so that the operator $(u_1, u_2) \mapsto (y_1, y_2)$ as defined by the following standard systems interconnection (see Fig. 1):

$$y_1 = \mathscr{P}(u_1 + y_2),$$

$$y_2 = \mathscr{C}(u_2 + y_1), \tag{2}$$

is well defined and finite gain stable.

We note that the disturbance u_1 we consider here is input additive and enters the system together with the control input u through the actuators. Simple examples show that the problem we consider does not always have a solution if the disturbance enters the system from outside the actuators.

The above problem was first studied for continuous-time systems. It was shown in Liu, Chitour and Sontag (1996) that, for neutrally stable open-loop systems, linear feedback laws can be used to achieve finite gain stability, with respect to every L_p-norm. For a neutrally stable system, all open loop poles are located in the closed left-half plane, with those on the $j\omega$-axis having Jordan blocks of size one. In the case that full state is available for feedback (i.e., $y_1 = x$ and $u_2 = 0$), it was shown in Lin, Saberi and Teel (1995) that if the external input signal is
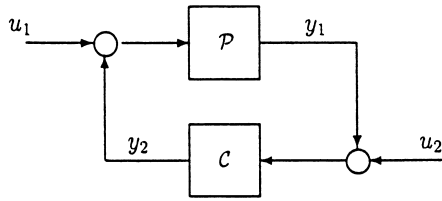

Fig. 1. Standard closed-loop connection.

uniformly bounded, then finite-gain L_p-stabilization and local asymptotic stabilization can always be achieved simultaneously by linear feedback, no matter where the poles of the open-loop system are. The uniform boundedness condition of Lin et al. (1995) was later removed by resorting to nonlinear feedback (Lin, 1997). Some other works related to the topic are Hou, Saberi and Lin (1997a), Chitour, Liu and Sontag (1995), Nguyen and Jabbari (1997), Saberi, Hou and Stoorvogel (1998), Suarez, Alvarez-Ramirez, Sznaier and Ibarra-Valdez (1997) and the references therein.

There are also several studies in the discrete-time setting, showing some of the continuous-time results carry over to discrete-time (for example, Hou, Saberi, Lin & Sannuti, 1997b; Yang et al., 1997) and some do not (for example, Hou et al., 1997a,b). In particular, Hou et al. (1997a,b) show that the results of Lin (1997) and Lin et al. (1995) on finite gain stabilization of continuous-time systems do not carry over to discrete-time systems. The objective of this paper is to show that the results of Liu et al. (1996), however, do carry over to discrete-time systems. More specifically, we show that, for neutrally stable discrete-time linear systems subject to actuator saturation, finite gain l_p stabilization can be achieved by linear output feedback for all $p \in (1, \infty]$. An explicit construction of the corresponding feedback laws is given. The feedback laws constructed also result in a closed-loop system that is globally asymptotically stable, and provide an input-to-state estimate. While many of the arguments used are conceptually similar to those used in the continuous-time case Liu et al. (1996), there are technical aspects that are very different and not totally obvious. For example, unlike in Liu et al. (1996), the feedback gain for the discrete-time case needs to be multiplied by a small factor, say κ, which causes the solution of a certain Lyapunov equation, and the subsequent estimation of the solution, to be dependent on κ (see Lemma 2). As another example, the difficulties in evaluating the difference of the non-quadratic Lyapunov function along the trajectories of the closed-loop system entail a careful estimation by Taylor expansion.

The remainder of the paper is organized as follows. Section 2 states the main results. Section 3 contains the proof of the results that were stated in Section 2. A brief concluding remark is given in Section 4.

2. Preliminary and problem statement

We first recall some notation. For a vector $X \in \mathbb{R}^\ell$, $|X|$ denotes the Euclidean norm of X, and for a matrix $X \in \mathbb{R}^{m \times n}$, the induced operator norm. For any $p \in [1, \infty)$, we write l_p^n for the set of all sequences $\{x(t)\}_{t=0}^\infty$, where $x \in \mathbb{R}^n$, such that $\sum_{t=0}^\infty |x(t)|^p < \infty$, and the l_p-norm of $x \in l_p^n$ is defined as $\|x\|_{l_p} = (\sum_{t=0}^\infty |x(t)|^p)^{1/p}$. We use l_∞^n to denote the set of all sequences $\{x(t)\}_{t=0}^\infty$, where $x \in \mathbb{R}^n$, such that $\sup_t |x(t)| < \infty$, and the l_∞-norm of $x \in l_\infty^n$ is defined as $\|x\|_{l_\infty} = \sup_t |x(t)|$.

The objective of this paper is to show the following result concerning the global asymptotic stabilization as well as l_p-stabilization of system $\mathcal{P}$, as given by (1), using linear output feedback.

Theorem 1. *Consider a system* (1). *Let A be neutrally stable, i.e., all the eigenvalues of A are inside or on the unit circle, with those on the unit circle having all Jordan blocks of size one. Also assume that (A, B) is stabilizable and (A, C) is detectable. Then, there exits a linear observer-based output feedback law of the form*

$$\hat{x}^+ = A\hat{x} + B\sigma(F\hat{x}) - L(y - C\hat{x}),$$

$$u = F\hat{x} \tag{3}$$

which has the following properties:

1. *It is finite gain l_p-stable for all $p \in (1, \infty]$, i.e., there exists a $\gamma_p > 0$ such that*

$$\|x\|_{l_p} \leq \gamma_p [\|u_1\|_{l_p} + \|u_2\|_{l_p}], \quad \forall u_1 \in l_p^m, u_2 \in l_p^r$$

$$and \ x(0) = 0, \ \hat{x}(0) = 0. \tag{4}$$

2. *In the absence of actuator and sensor noises u_1 and u_2, the equilibrium $(x, \hat{x}) = (0, 0)$ is globally asymptotically stable.*

Remark 1. We will in fact actually obtain the following stronger ISS-like property (see Sontag, 1998 and references there in):

$$\|(x, \hat{x})\|_{l_p} \leq \theta_p(|x(0)| + |\hat{x}(0)|) + \gamma_p[\|u_1\|_{l_p} + \|u_2\|_{l_p}], \tag{5}$$

where θ_p is a class-$\mathcal{K}$ function. Observe that the single estimate (5) encompasses both the gain estimate (4) and asymptotic stability. Obviously, (4) is the special case of (5) for zero initial states. On the other hand, when applied with arbitrary initial states but $u_1 = u_2 = 0$, there follows that $(x, \hat{x})$ is in l_p, which implies, in particular, that $(x(t), \hat{x}(t))$ must converge to zero as $t \to \infty$ (global attraction) and that $|(x(t), \hat{x}(t))|$ is bounded by $\theta_p(|x(0)| + |\hat{x}(0)|)$ (stability).

3. Proof of Theorem 1

The proof of Theorem 1 will follow readily from the following proposition, which we establish first.

Proposition 1. *Let A be orthogonal (i.e., $A'A = I$), and suppose that the pair (A, B) is controllable. Then, the system*

$$x^+ = Ax + B\sigma(-\kappa B'Ax + u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \quad (6)$$

is finite gain l_p-stable, $p \in (1, \infty]$, for sufficiently small $\kappa > 0$. Moreover, for each $p \in (1, \infty]$ there exist a real γ_p, a $\kappa^* \in (0, 1]$, and a class-$\mathcal{K}$ function θ_p such that, for all $\kappa \in (0, \kappa^*]$,

$$\|x\|_{l_p} \leq \gamma_p \|u\|_{l_p} + \theta_p(|x(0)|) \quad (7)$$

for all inputs $u \in l_p^m$ and all initial states $x(0)$.

To prove this proposition, we need to establish a few lemmas.

Lemma 1. *For any $p > l > 0$, there exist two scalars $M_1, M_2 > 0$ such that, for any two positive scalars ξ and ζ,*

$$\xi^{p-l}\zeta^l \leq M_1\xi^p + M_2\zeta^p \quad (8)$$

and consequently, for any $n > 0$ and $\kappa > 0$,

$$\xi^{p-l}\zeta^l \leq M_1\kappa^n\xi^p + \kappa^{n(l-p)/l}M_2\zeta^p. \quad (9)$$

Proof of Lemma 1. Let $h: \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $h(x) = x^{l/(p-l)}$, which is continuous and strictly increasing with $h(0) = 0$ and $h(\infty) = \infty$, and $k(x) = x^{(p-l)/l}$ be its pointwise inverse. Define

$$H(x) = \int_0^x h(v)\,dv = \frac{p-l}{p}x^{p/(p-l)} \quad (10)$$

and

$$K(x) = \int_0^x k(v)\,dv = \frac{l}{p}x^{p/l}. \quad (11)$$

Letting $a = \xi^{p-l}$ and $b = \zeta^l$, it follows from Young's inequality (Hardy, Littlewood & Polya, 1952), $ab \leq H(a) + K(b)$ for all $a, b \in \mathbb{R}^+$, that

$$\xi^{p-l}\zeta^l \leq \frac{p-l}{p}\xi^p + \frac{l}{p}\zeta^p = M_1\xi^p + M_2\zeta^p, \quad (12)$$

which also trivially implies (9). □

Lemma 2. *Let A and B be as given in Proposition* 1. *Then, for any $\kappa > 0$ such that $\kappa B'B < 2I$, $\bar{A}(\kappa) = A - \kappa BB'A$ is asymptotically stable. Moreover, let $P(\kappa)$ be the unique positive-definite solution to the Lyapunov equation,*

$$\bar{A}(\kappa)'P\bar{A}(\kappa) - P = -I. \quad (13)$$

Then, there exists a $\kappa^* > 0$ such that

$$\frac{\chi_1}{\kappa}I \leq P(\kappa) \leq \frac{\chi_2}{\kappa}I, \quad \forall \kappa \in (0, \kappa^*] \quad (14)$$

for some positive constants χ_1 and χ_2 independent of κ.

Proof of Lemma 2. The asymptotic stability of $\bar{A}$ follows from a simple Lyapunov/LaSalle argument (Choi, 1999). Let $\kappa_1^* > 0$ be such that $\kappa B'B < 2I$ for all $\kappa \in (0, \kappa_1^*]$. We recall that the solution to the Lyapunov equation (13) is given by

$$P(\kappa) = \sum_{k=0}^{\infty} (\bar{A}^k(\kappa))'\bar{A}^k(\kappa)$$

$$= \sum_{k=0}^{\infty} [(A - \kappa BB'A)']^k[(A - \kappa BB'A)]^k. \quad (15)$$

Using the fact that $AA' = I$, we have

$$(A - \kappa BB'A)'(A - \kappa BB'A)$$

$$= I - 2\kappa A'BB'A + \kappa^2 A'BB'BB'A$$

$$= I - \kappa A'B(2I - \kappa B'B)B'A. \quad (16)$$

Using now the fact that $\kappa B'B < 2I$ for $\kappa \in (0, \kappa_1^*]$, we know that there exists $\kappa_2^* \in (0, \kappa_1^*]$ such that

$$\tfrac{1}{2}I \leq (A - \kappa BB'A)'(A - \kappa BB'A) \leq I, \quad \forall \kappa \in (0, \kappa_2^*]. \quad (17)$$

Again using the fact that $A'A = I$, we verify in a straightforward way that

$$(A - \kappa BB'A)^n = A^n - \kappa C_{A,B}C'_{A,B}A^n + \kappa^2 M_1(\kappa), \quad (18)$$

where $M_1(k)$ is a polynomial matrix in κ of order $n - 2$, n being the order of the system (6), and

$$C_{A,B} = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

is the controllability matrix of the pair (A, B) and is of full rank. It then follows that

$$((A - \kappa BB'A)^n)'(A - \kappa BB'A)^n$$

$$= ((A^n)' - \kappa(A^n)'C_{A,B}C'_{A,B} + \kappa^2 M_1'(\kappa))$$

$$\quad (A^n - \kappa C_{A,B}C'_{A,B}A^n + \kappa^2 M_1(\kappa))$$

$$= I - 2\kappa(A^n)'C_{A,B}C'_{A,B}A^n + \kappa^2 M_2(\kappa), \quad (19)$$

where $M_2(\kappa)$ is a symmetric polynomial matrix in κ of order $2n - 2$. Since $C_{A,B}$ is of full rank, and because A is nonsingular, there exists a $\kappa^* \in (0, \kappa_2^*]$ such that

$$0 \leq I - \kappa M_1^0 I \leq ((A - \kappa BB'A)^n)'(A - \kappa BB'A)^n$$

$$\leq I - \kappa M_2^0 I < I, \quad \forall \kappa \in (0, \kappa^*] \quad (20)$$

for some constants $M_1^0, M_2^0 > 0$ independent of κ.

Using (17), (20) and the fact that $A'A = I$ in (15), we have that for all $\kappa \in (0, \kappa^*]$,

$$P(\kappa) \leq \sum_{i=0}^{n-1} [(A - \kappa BB'A)^i]'(A - \kappa BB'A)^i \sum_{k=0}^{\infty} (1 - \kappa M_2^0)^k I$$

$$\leq n \frac{1}{\kappa M_2^0} I = \frac{\chi_2}{\kappa} I \tag{21}$$

and

$$P(\kappa) \geq \sum_{i=0}^{n-1} [(A - \kappa BB'A)^i]'(A - \kappa BB'A)^i \sum_{k=0}^{\infty} (1 - \kappa M_1^0)^k I$$

$$\geq \left(\frac{1}{2}\right)^{n-1} n \frac{1}{\kappa M_1^0} I = \frac{\chi_1}{\kappa} I, \tag{22}$$

where $\chi_1 = n/2^{n-1} M_1^0$ and $\chi_2 = n/M_2^0$. $\quad \square$

Lemma 3. *Let $\bar{A}(\kappa)$ be as given in Proposition 1, $P(\kappa)$ as defined in Lemma 2, then for any $p \in (1, \infty)$, there exists a $\kappa^* > 0$ such that*

$$[x'\bar{A}'(\kappa)P(\kappa)\bar{A}(\kappa)x]^{p/2} - [x'P(\kappa)x]^{p/2}$$

$$\leq -\kappa^{(2-p)/2} \zeta |x|^p, \quad \kappa \in (0, \kappa^*], \tag{23}$$

where $\zeta > 0$ is some constant independent of κ.

Proof of Lemma 3. Inequality (23) holds trivially for $x = 0$. Hence, in what follows, we assume, without loss of generality, that $x \neq 0$.

For simplicity, we introduce from now the following notation:

$$\mu = x'\bar{A}'(\kappa)P(\kappa)\bar{A}(\kappa)x \tag{24}$$

(where x and κ will be clear from the context). By the definition of $P(\kappa)$, we have

$$\mu - x'P(\kappa)x = -x'x. \tag{25}$$

From Lemma 2, there exists a $\kappa_1^* > 0$ such that for all $\kappa \in (0, \kappa_1^*]$

$$\left| \frac{x'x}{x'P(\kappa)x} \right| \leq \frac{4}{5}, \quad \forall x \neq 0. \tag{26}$$

With (25) and (26), we can continue the proof using Taylor expansion with remainder,

$$[x'\bar{A}'(\kappa)P(\kappa)\bar{A}(\kappa)x]^{p/2} - [x'P(\kappa)x]^{p/2}$$

$$= [x'P(\kappa)x - x'x]^{p/2} - [x'P(\kappa)x]^{p/2}$$

$$\leq [x'P(\kappa)x]^{p/2}\left[1 - \frac{p}{2}\frac{x'x}{x'P(\kappa)x} + \delta\left(\frac{x'x}{x'P(\kappa)x}\right)^2\right]$$

$$\quad - [x'P(\kappa)x]^{p/2}$$

$$= -\frac{p}{2}[x'P(\kappa)x]^{(p-2)/2}|x|^2 + \delta[x'P(\kappa)x]^{(p-4)/2}|x|^4,$$

$$\kappa \in (0, \kappa_1^*], \tag{27}$$

where $\delta = \max_{|z| \leq 4/5}\{\frac{p}{8}|(p-2)(1+z)^{p/2-2}|\}$ is a constant independent of κ.

Again by Lemma 2, there exists a $\kappa^* \in (0, \kappa_1^*]$ such that

$$[\mu]^{p/2} - [x'P(\kappa)x]^{p/2} \leq -\kappa^{(2-p)/2}\zeta|x|^p, \quad \kappa \in (0, \kappa^*] \tag{28}$$

for some $\zeta > 0$ independent of κ. $\quad \square$

Lemma 4. *Let A and B be as given in Proposition 1. For any $l \in [1, \infty)$ and any $\kappa \in (0, 1]$,*

$$|\sigma(-\kappa B'Ax + u)|^l \leq 2^{l-1}\kappa^l|B|^l|x|^l + 2^{l-1}|u|^l. \tag{29}$$

Proof of Lemma 4. Since σ is a standard saturation function and $|A| = 1$, for any $l \geq 1$, we have

$$|\sigma(-\kappa B'Ax + u)|^l \leq (\kappa|B||x| + |u|)^l$$

$$\leq 2^{l-1}\kappa^l|B|^l|x|^l + 2^{l-1}|u|^l, \tag{30}$$

where the last inequality follows from Jensen's inequality applied to the convex function s^l:

$$(a+b)^l \leq \frac{1}{2}(2a)^l + \frac{1}{2}(2b)^l, \quad \forall a, b \geq 0. \quad \square$$

Lemma 5. *Let A and B be as given in Proposition 1. Pick any $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, any number $\eta \geq 3$, and any nonnegative real number l. Denote $\tilde{x} = -\kappa B'Ax + u$. Then, provided $|x| > \eta|B\sigma(\tilde{x})|$, we have*

$$|Ax + B\sigma(\tilde{x})|^l \leq |x|^l + l|x|^{l-2}x'A'B\sigma(\tilde{x})$$

$$\quad + M|x|^{l-2}|B\sigma(\tilde{x})|^2 \tag{31}$$

for some constant $M > 0$ which is independent of κ.

Proof of Lemma 5. We first note that, since $|x| > \eta|B\sigma(\tilde{x})| \geq 3|B\sigma(\tilde{x})|$,

$$\frac{|2x'A'B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2|}{|x|^2} \leq \frac{4}{5}. \tag{32}$$

Hence, using Taylor expansion with remainder, we have

$$|Ax + B\sigma(\tilde{x})|^l = [|x|^2 + 2x'A'B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2]^{l/2}$$

$$= |x|^l\left(1 + \frac{2x'A'B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2}\right)^{l/2}$$

$$\leq |x|^l\left[1 + \frac{l}{2}\frac{2x'A'B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2}\right.$$

$$\left. + \delta\left(\frac{2x'A'B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2}\right)^2\right]$$

$$\leq |x|^l + l|x|^{l-2}x'A'B\sigma(\tilde{x}) + \frac{l}{2}|x|^{l-2}|B\sigma(\tilde{x})|^2$$

$$\quad + \delta|x|^{l-4}((2 + 1/\eta)|x||B\sigma(\tilde{x})|)^2, \tag{33}$$

where $\delta = \max_{|z| \leq 4/5}\{\frac{1}{8}l|(l-2)(1+z)^{l/2-2}|\}$ is a constant independent of κ.

So we can see that the inequality (31) holds for $M = \frac{1}{2} + \delta(2 + 1/\eta)^2$.   □

We are now ready to prove Proposition 1.

Proof of Proposition 1. We separate the proof for $p \in (1, \infty)$ and for $p = \infty$.

Proof for $p \in (1, \infty)$: For clarity, let us repeat here the system equation (6)

$$x^+ = Ax + B\sigma(-\kappa B'Ax + u), \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m. \quad (34)$$

This may also be rewritten as

$$x^+ = \bar{A}(\kappa)x + B(-\tilde{x} + \sigma(\tilde{x}) + u), \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m, \quad (35)$$

where $\bar{A}(\kappa) = A - \kappa BB'A$, $\tilde{x} = -\kappa B'Ax + u$.

For this system, define the function V_1 as

$$V_1(x) = (x'P(\kappa)x)^{p/2}, \quad (36)$$

where $P(\kappa)$ is as given in Lemma 2. We next evaluate the increments $V(x^+(t)) - V(x(t))$, which we denote as "ΔV_1" for short, along any given trajectory of (35). It is convenient to treat separately the cases $|x| > \eta|B\sigma(\tilde{x})|$ and $|x| \leq \eta|B\sigma(\tilde{x})|$. Here $\eta \geq 3$ is a number to be specified soon.

Case 1: $|x| > \eta|B\sigma(\tilde{x})|$. Using the definition of V_1, we now give an upper bound on ΔV_1 along the trajectories of the system (35). To simplify the equations, we introduce the following notation:

$$v = 2\kappa x'A'P(\kappa)BB'Ax + 2x'A'P(\kappa)B\sigma(\tilde{x})$$
$$+ \sigma'(\tilde{x})B'P(\kappa)B\sigma(\tilde{x}),$$

in addition to μ as defined in Eq. (24). Thus,

$$\Delta V_1 = V_1^+ - V_1$$
$$= [(x^+)'P(\kappa)x^+]^{p/2} - [x'P(\kappa)x]^{p/2}$$
$$= [[\bar{A}(\kappa)x + \kappa BB'Ax + B\sigma(\tilde{x})]'P(\kappa)[\bar{A}(\kappa)x$$
$$+ \kappa BB'Ax + B\sigma(\tilde{x})]]^{p/2} - [x'P(\kappa)x]^{p/2}$$
$$= [\mu + 2x'A'P(\kappa)B(\kappa B'Ax + \sigma(\tilde{x}))$$
$$- \kappa^2 x'A'BB'P(\kappa)BB'Ax + \sigma'(\tilde{x})B'P(\kappa)B\sigma(\tilde{x})]^{p/2}$$
$$- [x'P(\kappa)x]^{p/2}$$
$$\leq [\mu + v]^{p/2} - [x'P(\kappa)x]^{p/2}$$
$$= [\mu]^{p/2}\left[1 + \frac{v}{\mu}\right]^{p/2} - [x'P(\kappa)x]^{p/2}. \quad (37)$$

By Lemma 2, there exist a $\kappa_1^* > 0$ and $\eta \geq 3$ independent of κ, such that for all $|x| > \eta|B\sigma(\tilde{x})|$

$$\left|\frac{v}{\mu}\right| \leq \frac{4}{5}, \quad \kappa \in (0, \kappa_1^*]. \quad (38)$$

To see this, let $\kappa_0^* > 0$ be such that (14) of Lemma 2 and (17) in the proof of Lemma 2 both hold for all $\kappa \in (0, \kappa_0^*]$. Then, for all $\kappa \in (0, \kappa_0^*]$, we have

$$|v| \leq \left[2\chi_2|B|^2 + \frac{2\chi_2}{\kappa\eta} + \frac{\chi_2}{\kappa\eta^2}\right]|x|^2 \quad (39)$$

and

$$|\mu| \geq \frac{\chi_1}{2\kappa}|x|^2, \quad (40)$$

from which it is clear that there exist κ_1^* and $\eta > 3$ such that (38) holds.

Next, we may use a Taylor expansion with remainder to continue the bounding of ΔV_1 as follows:

$$\Delta V_1 \leq [\mu]^{p/2}\left[1 + \frac{p}{2}\frac{v}{\mu} + \delta\left[\frac{v}{\mu}\right]^2\right] - [x'P(\kappa)x]^{p/2}, \quad (41)$$

where $\delta = \max_{|z| \leq 4/5}\{\frac{1}{8}p|(p-2)(1+z)^{p/2-2}|\}$ is a constant independent of κ.

By Lemmas 3 and 2, there exists $\kappa_2^* \in (0, \kappa_1^*]$ such that for any $\kappa \in (0, \kappa_2^*]$, we have

$$\Delta V_1 \leq -\kappa^{(2-p)/2}\zeta|x|^p + \frac{p}{2}[\mu]^{(p-2)/2}[v] + \delta[\mu]^{(p-4)/2}[v]^2$$

$$\leq -\kappa^{(2-p)/2}\zeta|x|^p + \psi_1\kappa^{(2-p)/2}|x|^{p-2}$$

$$\times [2|x||P(\kappa)B||\tilde{x} - \sigma(\tilde{x})| + 2|x||P(\kappa)B||u|$$

$$+ |P(\kappa)||B\sigma(\tilde{x})|^2] + \psi_2\kappa^{(4-p)/2}|x|^{p-4}$$

$$\times [2|x|^2|\kappa P(\kappa)||B|^2 + 2|P(\kappa)||x||B\sigma(\tilde{x})|]$$

$$+ |P(\kappa)||B\sigma(\tilde{x})|^2]^2, \quad (42)$$

where $\zeta > 0$ is as defined in Lemma 3, and $\psi_1, \psi_2 > 0$ are some constants independent of κ.

Before continuing, we digress to observe that

$$|\tilde{x} - \sigma(\tilde{x})| \leq \tilde{x}'\sigma(\tilde{x}). \quad (43)$$

Using (43), Lemmas 1 and 4, and the condition $|x| > \eta|B\sigma(\tilde{x})|$, we can show that there exists a $\kappa_3^* \in (0, \kappa_2^*]$ such that for all $\kappa \in (0, \kappa_3^*]$ the estimation of ΔV_1 can be now concluded as follows:

$$\Delta V_1 \leq -\kappa^{(2-p)/2}\zeta|x|^p + 2\psi_1\kappa^{(2-p)/2}|x|^{p-1}|P(\kappa)B|\tilde{x}'\sigma(\tilde{x})$$

$$+ M_{1a}\kappa^{(2-p)/2}\max\{\kappa, \kappa^{p-1}\}|x|^p + M_{2a}(\kappa)|u|^p, \quad (44)$$

where $M_{1a} > 0$, $M_{2a}(\kappa) > 0$ with M_{1a} independent of κ are defined in an obvious way. In deriving (44), we have also used the fact that $|x|^{p-2} < (|B\sigma(\tilde{x})|/\eta)^{p-2}$ for $p < 2$ and $B\sigma(\tilde{x}) \neq 0$.

Case 2: $|x| \leq \eta|B\sigma(\tilde{x})|$. By using Lemmas 2–4, ΔV_1 along the trajectories of (35) is bounded as follows:

$$\Delta V_1 = [(x^+)'P(\kappa)x^+]^{p/2} - [x'P(\kappa)x]^{p/2}$$

$$\leq |P(\kappa)|^{p/2}|Ax + B\sigma(\tilde{x})|^p - [x'P(\kappa)x]^{p/2} + [\mu]^{p/2}$$

$$\leq -\kappa^{(2-p)/2}\zeta|x|^p + \kappa^{-p/2}\chi_2^{p/2}(|x| + |B\sigma(\tilde{x})|)^p$$

$$\leq -\kappa^{(2-p)/2}\zeta|x|^p + (\eta + 1)^p\kappa^{-p/2}\chi_2^{p/2}|B\sigma(\tilde{x})|^p$$

$$\leq -\kappa^{(2-p)/2}\zeta|x|^p + M_{1b}\kappa^{-(p-2)/2}\kappa^{p-1}|x|^p$$

$$+ M_{2b}(\kappa)|u|^p, \quad \kappa \in (0, \kappa_3^*], \tag{45}$$

where $\chi_2 > 0$ and $\zeta > 0$ are as defined in Lemmas 2 and 3, respectively, and $M_{1b} > 0$, $M_{2b}(\kappa) > 0$ are constants with M_{1b} being independent of κ.

Summarizing, we may combine Case 1 with Case 2, to obtain

$$\Delta V_1 \leq$$

$$\begin{cases} -\kappa^{(2-p)/2}\zeta|x|^p + 2\psi_1\kappa^{(2-p)/2} |P(\kappa)B||x|^{p-1}\tilde{x}'\sigma(\tilde{x}) \\ \\ + M_1\kappa^{(2-p)/2}\max\{\kappa, \kappa^{p-1}\} |x|^p \\ \\ + M_2(\kappa)|u|^p, \quad \text{if } |x| > \eta|B\sigma(\tilde{x})|, \\ \\ -\kappa^{(2-p)/2}\zeta|x|^p + M_1\kappa^{(2-p)/2} \kappa^{p-1}|x|^p + M_2(\kappa)|u|^p \\ \\ \text{if } |x| \leq \eta|B\sigma(\tilde{x})|, \end{cases} \tag{46}$$

where

$$M_1 = \max\{M_{1a}, M_{1b}\}$$

and

$$M_2(\kappa) = \max\{M_{2a}(\kappa), M_{2b}(\kappa)\}.$$

For system (34), we next define another function

$$V_0(x) = |x|^{p+1}. \tag{47}$$

An estimation of its increments along the trajectories of (34) can also be carried out by separately considering each of the cases $|x| > \eta|B\sigma(\tilde{x})|$ and $|x| \leq \eta|B\sigma(\tilde{x})|$.

Case 1: $|x| \leq \eta|B\sigma(\tilde{x})|$. By Lemma 4, for any $\kappa \in (0, \kappa_3^*]$,

$$\Delta V_0 = |Ax + B\sigma(\tilde{x})|^{p+1} - |x|^{p+1} \leq |Ax + B\sigma(\tilde{x})|^{p+1}$$

$$\leq (|x| + |B\sigma(\tilde{x})|)^{p+1} \leq ((\eta + 1)|B\sigma(\tilde{x})|)^{p+1}$$

$$\leq \kappa N_{1a}|x|^p + N_{2a}|u|^p \tag{48}$$

for some positive constants N_{1a} and N_{2a} independent of κ. In deriving (48), we have used the fact that both σ and κ are bounded.

Case 2: $|x| > \eta|B\sigma(\tilde{x})|$. By Lemmas 5, 4 and 1, there exists $\kappa_4^* \in (0, \kappa_3^*]$ such that for any $\kappa \in (0, \kappa_4^*]$,

$$\Delta V_0 = |Ax + B\sigma(\tilde{x})|^{p+1} - |x|^{p+1},$$

$$\leq |x|^{p+1} + (p + 1)|x|^{p-1}x'A'B\sigma(\tilde{x})$$

$$+ N_{1b}|B\sigma(\tilde{x})|^2|x|^{p-1} - |x|^{p+1},$$

$$\leq -\frac{p+1}{\kappa}|x|^{p-1}\tilde{x}^T\sigma(\tilde{x}) + \kappa N_{1c}|x|^p + N_{2c}(\kappa)|u|^p, \tag{49}$$

where $N_{1c}, N_{1b} > 0$, $N_{2c}(\kappa) > 0$ are constants, and N_{1b}, N_{1c} are independent of κ. In deriving (49), the first inequality by Lemma 5, the second inequality is the consequence of the fact that σ is bounded and Lemmas 4 and 1.

Combining Case 1 with Case 2, we have, for any $\kappa \in (0, \kappa_4^*]$,

$$\Delta V_0 \leq \begin{cases} -\frac{p+1}{\kappa}|x|^{p-1}\tilde{x}'\sigma(\tilde{x}) \\ \quad + \kappa N_1|x|^p + N_2(\kappa)|u|^p, & \text{if } |x| > \eta|B\sigma(\tilde{x})|, \\ \kappa N_1|x|^p + N_2(\kappa)|u|^p, & \text{if } |x| \leq \eta|B\sigma(\tilde{x})|, \end{cases} \tag{50}$$

where

$$N_1 = \max\{N_{1a}, N_{1c}\}$$

and

$$N_2(\kappa) = \max\{N_{2a}, N_{2c}(\kappa)\}.$$

Finally, we define the following Lyapunov (or "storage") function:

$$V(x) = V_1(x) + \varpi V_0(x), \tag{51}$$

where

$$\varpi = \frac{2}{p+1}\kappa^{(4-p)/2}\psi_1|P(\kappa)B|.$$

It is straightforward to verify that there exists some $\kappa^* \in (0, \kappa_4^*]$ such that

$$\Delta V(x) \leq -\kappa^{(2-p)/2}\alpha|x|^p + \beta(\kappa)|u|^p, \quad \forall \kappa \in (0, \kappa^*] \tag{52}$$

for some $\alpha \in (0, \zeta)$ and $\beta(\kappa) > 0$.

Now consider an arbitrary initial state $x(0)$ and control u, and the ensuing trajectory x. Summing both sides of (52) from $t = 0$ to ∞ and using the fact that V is nonnegative, we conclude that

$$\kappa^{(2-p)/2}\alpha\|x\|_{l_p}^p \leq \beta(\kappa)\|u\|_{l_p}^p + \theta_{p0}(|x(0)|), \tag{53}$$

where $\theta_{p0}(r) = \varpi r^{p+1} + (\chi_2 r^2/\kappa)^{p/2}$. This implies that

$$\|x\|_{l_p} \leq \gamma_p\|u\|_{l_p} + \theta_p(|x(0)|), \tag{54}$$

where

$$\gamma_p = (\kappa^{(p-2)/2}\beta(\kappa)/\alpha)^{1/p},$$

and

$$\theta_p(r) = (\kappa^{(p-2)/2}\theta_{p0}(r)/\alpha)^{1/p}.$$

Proof for $p = \infty$. From (52) we get for $p = 2$,

$$\Delta V(x) \leq -\alpha|x|^2 + \beta(\kappa)\|u\|_{l_\infty}^2. \qquad (55)$$

Hence, $\Delta V(x)$ is negative outside the ball of radius $(\beta(\kappa)/\alpha)^{1/2}\|u\|_{l_\infty}$ centered at the origin, from which it follows that, for any state $x(t)$ in the trajectory:

$$V(x(t)) \leq \left(\frac{\varpi\beta^{3/2}(\kappa)}{\alpha^{3/2}}\|u\|_{l_\infty} + \frac{\chi_2\beta(\kappa)}{\alpha\kappa}\right)\|u\|_{l_\infty}^2$$
$$+ \theta_{\infty 0}(|x(0)|), \qquad (56)$$

where $\theta_{\infty 0}(r) = (\chi_2/\kappa)r^2 + \varpi r^3$. If $\|u\|_{l_\infty} \leq 1$, we have

$$\frac{\chi_1}{\kappa}|x(t)|^2 \leq x(t)'P(\kappa)x(t) \leq V(x(t)) \qquad (57)$$

and

$$V(x(t)) \leq \left(\frac{\varpi\beta^{3/2}(\kappa)}{\alpha^{3/2}} + \frac{\chi_2\beta(\kappa)}{\alpha\kappa}\right)\|u\|_{l_\infty}^2 + \theta_{\infty 0}(|x(0)|) \qquad (58)$$

which implies the following estimate for the entire trajectory:

$$\|x\|_{l_\infty} \leq \left\{\frac{\kappa\varpi\beta^{3/2}(\kappa)}{\alpha^{3/2}\chi_1} + \frac{\chi_2\beta(\kappa)}{\alpha\chi_1}\right\}^{1/2}\|u\|_{l_\infty} + \theta_{\infty 1}(|x(0)|), \qquad (59)$$

where $\theta_{\infty 1}(r) = (\kappa\theta_{\infty 0}(r)/\chi_1)^{1/2}$. If, instead, $\|u\|_{l_\infty} > 1$, we have

$$\varpi|x|^3 \leq V(x) \leq \left(\frac{\varpi\beta^{3/2}(\kappa)}{\alpha^{3/2}} + \frac{\chi_2\beta(\kappa)}{\alpha\kappa}\right)\|u\|_{l_\infty}^3$$
$$+ \theta_{\infty 1}(|x(0)|), \qquad (60)$$

from which we get that

$$\|x\|_{l_\infty} \leq \left(\frac{\beta^{3/2}(\kappa)}{\alpha^{3/2}\varpi} + \frac{\chi_2\beta(\kappa)}{\kappa\alpha\varpi}\right)^{1/3}\|u\|_{l_\infty} + \theta_{\infty 2}(|x(0)|), \qquad (61)$$

where $\theta_{\infty 2}(r) = (\theta_{\infty 0}(r)/\varpi)^{1/3}$. Letting

$$\gamma_\infty = \max\left\{\left\{\frac{\kappa\varpi\beta^{3/2}(\kappa)}{\alpha^{3/2}\chi_1} + \frac{\chi_2\beta(\kappa)}{\alpha\chi_1}\right\}^{\frac{1}{2}},\right.$$
$$\left.\left\{\frac{\beta^{3/2}(\kappa)}{\alpha^{3/2}\varpi} + \frac{\chi_2\beta(\kappa)}{\alpha\varpi\kappa}\right\}^{1/3}\right\}$$

and $\theta_\infty = \max\{\theta_{\infty 1}, \theta_{\infty 2}\}$, we have, finally, the required conclusion:

$$\|x\|_{l_\infty} \leq \gamma_\infty\|u\|_{l_\infty} + \theta_\infty(|x(0)|) \qquad (62)$$

for $p = \infty$ as well.

We are now ready to prove Theorem 1. $\quad\square$

Proof of Theorem 1. Without loss of generality, making a change of coordinates if required, we may assume that the system (1) has the following partitioned form:

$$x_1^+ = A_1x_1 + B_1\sigma(u + u_1),$$
$$x_0^+ = A_0x_0 + B_0\sigma(u + u_1), \qquad (63)$$
$$y = Cx + u_2.$$

where A_1 is orthogonal and A_0 is asymptotically stable, and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}.$$

We construct the output feedback law in the form of (3) with $F = [-\kappa B_1' A_1 \ 0]$, the matrix L being chosen such that $A + LC$ is asymptotically stable. Using this feedback, the closed-loop system is

$$\begin{cases} x_1^+ = A_1x_1 + B_1\sigma(-\kappa B_1'A_1\hat{x}_1 + u_1), \\ x_0^+ = A_0x_0 + B_0\sigma(-\kappa B_1'A_1\hat{x}_1 + u_1), \\ \hat{x}^+ = A\hat{x} + B\sigma(-\kappa B_1'A_1\hat{x}_1) - L(Cx - C\hat{x} + u_2). \end{cases} \qquad (64)$$

Let $e = [e_1' \ e_0']'$, where $e_1 = x_1 - \hat{x}_1$ and $e_0 = x_0 - \hat{x}_0$. Here we have partitioned $\hat{x} = [\hat{x}_1' \ \hat{x}_0']'$ accordingly. In the new states (x, e), (64) can be written as follows,

$$x_1^+ = A_1x_1 + B_1\sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1 + u_1),$$
$$x_0^+ = A_0x_0 + B_0\sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1 + u_1),$$
$$e^+ = (A + LC)e$$
$$\quad + B[\sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1 + u_1)$$
$$\quad - \sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1)] + Lu_2. \qquad (65)$$

Since σ is global Lipschitz with a Lipschitz constant 1,

$$|\sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1 + u_1)$$
$$\quad - \sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1)| \leq |u_1|. \qquad (66)$$

Noting that $A + LC$ is asymptotically stable and viewing

$$\sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1 + u_1)$$
$$\quad - \sigma(-\kappa B_1'A_1x_1 + \kappa B_1'A_1e_1) + Lu_2$$

as an l_p input to the e-subsystem, we have that, for some constant $\gamma_{pe} > 0$,

$$\|e\|_{l_p} \leq \gamma_{pe}(\|u_1\|_{l_p} + \|u_2\|_{l_p} + |e(0)|). \qquad (67)$$

Next, applying Proposition 1 to the x_1-subsystem, and viewing $\kappa B_1'A_1e_1 + u_1$ as an l_p input to this subsystem, we have,

$$\|x_1\|_{l_p} \leq \gamma_{p1}(\|u_1\|_{l_p} + \|u_2\|_{l_p} + |e(0)|) + \theta_{p1}(|x_1(0)|)$$

for some $\gamma_{p1} > 0$ and θ_{p1} of class $\mathcal{K}$.

On the other hand, viewing $\sigma(-\kappa B'_1 A_1 x_1 + \kappa B'_1 A_1 e_1 + u_1)$ as an l_p input to the x_0-subsystem, we have the estimate

$$\|x_0\|_{l_p} \le \gamma_{p0}(\|x_1\|_{l_p} + \|e\|_{l_p} + \|u_1\|_{l_p} + |x_0(0)|)$$

for some $\gamma_{p0} > 0$.

In conclusion, we have

$$\|x\|_{l_p} \le \|x_1\|_{l_p} + \|x_0\|_{l_p}$$
$$\le \gamma_p(\|u_1\|_{l_p} + \|u_2\|_{l_p}) + \varphi_p(|e(0)| + |x(0)|), \qquad (68)$$

where $\gamma_p > 0$ is some constant and φ_p is a suitable class-$\mathscr{K}$ function. Together with (67), and changing back to the original coordinates, we also conclude that an estimate like the one in (5) holds. $\square$

4. Conclusions

In this paper, we have established that a discrete-time, neutrally stable, stabilizable, and detectable linear system, when subject to actuator saturation, is finite gain l_p stabilizable by linear feedback, for any $p \in (1, \infty]$. A linear output feedback law which simultaneously achieves l_p stabilization and global asymptotic stabilization was constructed.

Acknowledgements

References

Choi, J. (1999). On the stabilization of linear discrete-time systems subject to input saturation. *Systems & Control Letters*, 36, 241–244.

Hardy, G. H., Littlewood, J. E., & Polya, G. (1952). *Inequalities*. Cambridge: Cambridge University Press.

Hou, P., Saberi, A., & Lin, Z. (1997a). On l_p-stabilization of strictly unstable discrete-time linear systems with saturating actuators. *Proceedings of the CDC* (pp. 4510–4515).

Hou, P., Saberi, A., Lin, Z., & Sannuti, P. (1997b). Simultaneous external and internal stabilization for continuous and discrete-time critically unstable linear systems with saturating actuators. *Proceedings of 1997 ACC* (pp. 1292–1296).

Lin, Z. (1997). H_∞-almost disturbance decoupling with internal stability for linear systems subject to input saturation. *IEEE Transactions on Automatic Control*, 42, 992–995.

Lin, Z., Saberi, A., & Teel, A. (1995). Simultaneously L_p — stabilization and internal stabilization of linear system subject to input saturation — state feedback case. *Systems & Control Letters*, 25, 219–226.

Liu, W., Chitour, Y., & Sontag, E. (1996). On finite gain stabilizability of linear systems subject to input saturation. *SIAM J. Control and Optimization*, 34, 1190–1219.

Chitour, Y., Liu, W., & Sontag, E. (1995). On the continuity and incremental-gain properties of certain saturated linear feedback loops. *International Journal of Robust and Nonlinear Control*, 5, 413–440.

Nguyen, T., & Jabbari, F. (1997). Output feedback controllers for disturbance attenuation with bounded control. *Proceedings of the 36th CDC* (pp. 177–182).

Saberi, A., Hou, P., & Stoorvogel, A. (1998). On simultaneous global external and global internal stabilization of critically unstable linear systems with saturating actuators. *Proceedings of 1998 ACC* (pp. 1463–1467).

Sontag, E. D. (1998). Comments on integral variants of ISS. *Systems & Control Letters*, 34, 93–100.

Suarez, R., Alvarez-Ramirez, J., Sznaier, M., & Ibarra-Valdez, C. (1997). L_2-disturbance attenuation for linear systems with bounded controls: an ARE-Based Approach. In S. Tarbouriech, and G. Garcia, *Control of uncertain systems with bounded inputs* (pp. 25–38). Lecture notes in control and information sciences, vol. 227. Berlin: Springer.

Yang, Y., Sontag, E. D., & Sussmann, H. J. (1997). Global stabilization of linear discrete-time systems with bounded feedback. *Systems & Control Letters*, 30, 273–281.

Xiangyu Bao received the B.S. degree in automatic control from Tsinghua University, Beijing, China, and the M.S. degree in control engineering from Beijing Institute of Control Engineering, Chinese Academy of Space Technology, Beijing, China, in 1993 and 1996, respectively. She is currently working toward her Ph.D. degree in electrical engineering at the University of Virginia, Charlottesville. Her research interests include nonlinear control theory and control of systems with saturating actuators.

Zongli Lin was born in Fuqing, Fujian, China on February 24, 1964. He received his B.S. degree in mathematics and computer science from Amoy University, Xiamen, China, in 1983, his Master of Engineering degree in automatic control from Chinese Academy of Space Technology, Beijing, China, in 1989, and his Ph.D. degree in electrical and computer engineering from Washington State University, Pullman, Washington, in May 1994. From July 1983 to July 1986, Dr. Lin worked as a control engineer at Chinese Academy of Space Technology. In January 1994, he joined the Department of Applied Mathematics and Statistics, State University of New York at Stony Brook as a visiting assistant professor, where he became an assistant professor in September 1994. Since July 1997, he has been an assistant professor in electrical engineering at University of Virginia. His current research interests include nonlinear control, robust control, and control of systems with saturating actuators. In these areas he has published several papers. He is also the author of the recent book, *Low Gain Feedback* (Springer-Verlag, London, 1998). Dr. Lin currently serves as an associate editor on the Conference Editorial Board of the IEEE Control Systems Society. He is the recipient of an ONR Young Investigator Award.

Eduardo Sontag's major current research interests lie in several areas of control theory and biologically-inspired mathematics (such as neural networks). He received his Licenciado degree from the Mathematics Department at the University of Buenos Aires in 1972, and his Ph.D. (Mathematics) under Rudolf E. Kalman at the Center for Mathematical Systems Theory, University of Florida, in 1976. Since 1977, Dr. Sontag has been with the Department of Mathematics at Rutgers, The State University of New Jersey, where he is currently Professor II of Mathematics as well as a Member of the Graduate Faculties of the Department of Computer Science and of the Department of Electrical and Computer Engineering. He is also the director of SYCON, the Rutgers Center for Systems and Control. Sontag has authored over two hundred and fifty journal and conference papers and book chapters in the above areas, as well as the books Topics in Artificial Intelligence (in Spanish, Buenos Aires: Prolam, 1972), Polynomial Response Maps (Berlin: Springer, 1979), and Mathematical Control Theory: Deterministic Finite Dimensional Systems (Texts in Applied Mathematics, Volume 6, Second Edition, New York: Springer, 1998). Among recent or upcoming major presentations are: a 75-min lecture at the AMS Short Course on Nonlinear Control (AMS Winter Meeting, San Antonio, Jan. 1999), a course on stabilization problems for nonlinear systems at the NATO Advanced Study Institute "Nonlinear Analysis, Differential Equations, and Control" (Montreal, August 1998), a keynote address to the British Applied Mathematics Colloquium'99 (Bath, April 1999), an hour talk at the Symposium on the Mathematical Theory of Networks and Systems (Padova, July 1998), a course on learning theory as part of the 1997 Cambridge University's Isaac Newton Institute Neural Networks and Machine Learning Programme, a 45-minute invited lecture at the 1994 International Congress of Mathematicians, a plenary talk at the 1995 SIAM Control Conference, a course on neurocontrol as part of the 1993 European Control Conference, an expository survey talk at the SPIE's 1997 International Symposium on Aerospace/Defense Sensing and Controls Conference, an hour talk at the AMS Summer Research Institute "Differential Geometry and Control" (Boulder, July 1997), two plenary talks at the 1995 Neural Information Processing Systems Conference, and plenaries at the 1992 IFAC Conference on Nonlinear Control and the 1993 Jerusalem Conference on Control Theory and Applications. Sontag is an Associate Editor for various journals, including: IEEE Transactions in Automatic Control, Control-Theory and Advanced Technology, SMAI-COCOV, Dynamics and Control, Journal of Computer and Systems Sciences, Neurocomputing, and Neural Computing Surveys (Board of Advisors), and a former Associate Editor for Neural Networks and for Systems and Control Letters. In addition, he is a co-founder and co-Managing Editor of the Springer journal MCSS (Mathematics of Control, Signals, and Systems). Sontag is an IEEE Fellow, and has been Program Director and Vice-Chair of the Activity Group in Control and Systems Theory of SIAM. He has been a member of several committees at SIAM and the AMS, and is a former Chair of the Committee on Human Rights of Mathematicians of the latter. He is listed in "Who's Who in Frontier Science and Technology" and in "American Men and Women of Science".