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Brief paper

Stability certification of large scale stochastic systems using dissipativity[☆]

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ABSTRACT

In this paper, we analyse the stability of large-scale nonlinear stochastic systems, represented as an interconnection of lower-order stochastic subsystems. Stochastic stability in probability and noise-to-state stability are addressed, and sufficient conditions for the latter are provided. The method proposed proves network stability by using appropriate stochastic passivity properties of its subsystems, and the structure of its interactions. Stability properties are established by the diagonal stability of a dissipativity matrix, which incorporates information about the passivity properties of the systems and their interconnection. Next, we derive equilibrium-independent conditions for the verification of the relevant passivity properties of the subsystems. Finally, we illustrate the proposed approach on a class of biological reaction networks.

1. Introduction

The analysis of nonlinear systems becomes intractable as the dimension of the state space increases. It is imperative to develop approaches that decompose this analysis into smaller subproblems. In this paper, we represent large-scale nonlinear stochastic systems as an interconnection of lower-order stochastic dynamical subsystems. We then certify stability based on appropriate stochastic passivity properties of the subsystems and the structure of their interactions.

Previous studies have shown the effectiveness of this approach for deterministic models of biological networks, Arcak and Sontag (2006, 2008). In Arcak and Sontag (2006), global asymptotic stability of a cyclic interconnection structure is established from the diagonal stability of a dissipativity matrix that incorporates information about the passivity properties of the subsystems and the interconnection structure of the network. The results are extended in Arcak and Sontag (2008) to a more general interconnection structure. Both Arcak and Sontag (2006, 2008) exploit output strict passivity (OSP) properties and corresponding storage functions of the subsystems, and construct a composite Lyapunov function for the interconnection using these storage functions.

In biochemical reactions, deterministic models may be inadequate, particularly when the copy numbers of the species are small. Stochasticity appears as external noise (due to cell-to-cell variability of external signals) and as intrinsic noise (since chemical reactions depend on random motion). While external noise can be incorporated in noise-driven deterministic models, *i.e.* stochastic differential equations (SDEs), internal noise is accounted for by a Chemical Master Equation models (CME). Under appropriate assumptions, Gillespie (2000), it is common to perform a diffusion approximation of the CME, leading to the Chemical Langevin Equation (CLE), which is a particular type of SDE. Thus, both internal and external noise can be treated jointly with SDEs.

We study large-scale nonlinear stochastic models described by SDEs. We extend the passivity approach in Arcak and Sontag (2006, 2008) to the stochastic framework, by using the expansion of the definitions of passivity introduced in Florchinger (1999). We prove stability in probability for an interconnection of stochastic OSP (sOSP) subsystems, if an appropriate diagonal stability condition holds for a dissipativity matrix similar to the one in Arcak and Sontag (2006). Early references, such as Michel (1975), Michel and Rasmussen (1976), constructed composite Lyapunov functions for stochastic stability. However, as is common in the classical large-scale systems literature, these references restrict the magnitude of the coupling terms without regard to their sign structure. The

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passivity-based approach in the present paper takes advantage of the negative feedback loops in the network to obtain less restrictive stability criteria.

We next investigate the notion of Noise-to-State stability (NSS), defined in Krstic and Deng (1998), which is a stochastic counterpart of deterministic input-to-state stability, Sontag (1989). NSS implies that if there exists a bound in the noise variance, the state of a NSS system is also bounded in probability. This notion is less restrictive than stochastic stability in the sense that it accommodates systems with nonvanishing noise at the equilibrium, and unknown noise intensity. First, we provide a new sufficient condition for NSS that is easy to verify. We then introduce a new input–output definition that combines NSS and OSP properties, referred to as NSS⊕OSP. We show that the interconnected system is NSS if the diagonal stability of a similar dissipativity matrix is ascertained.

Since passivity properties are defined in reference to the equilibrium, which depends on the full network, the verification of the sOSP and NSS⊕OSP properties of the subsystems is a difficulty encountered in the methodology presented. In Arcak and Sontag (2006), equilibrium-independent results for the verification of OSP properties are provided. In this paper, we provide an extension for stochastic systems. We derive equilibrium-independent verification of stochastic passivity properties, which are not considered in the classical literature.

In Section 2, we provide the necessary notation and definitions, and derive sufficient conditions for NSS. The main results for stochastic stability of interconnected systems are presented in Section 3, where stability in probability and noise-to-state stability are achieved. In Section 4, we focus on the input/output passivity properties of the systems, by deriving equilibrium-independent conditions that guarantee sOSP and NSS⊕OSP. Finally, in Section 5, we illustrate the application of the results obtained to classes of biological reaction networks.

2. Preliminaries

Consider the following nonlinear stochastic system

$$dx = f(x)dt + l(x)\Sigma dw \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t)$ is an r-dimensional independent standard Wiener process, $\Sigma = \{\sigma_{ij}\}$ is an $r \times r$ non-negative-definite matrix, and σ_{ij} represents the intensity with which the jth source of uncertainty influences the ith state. Assume that the vector field and matrix function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $l : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitz continuous.

For the notions of stochastic stability and passivity, defined in Sections 2.1 and 2.2, we assume $\Sigma = I$ because $l(x)$ can be redefined to incorporate the constant Σ. Moreover, we also assume $f(0) = 0$ and $l(0) = 0$, so that $x(t) \equiv 0$ is a solution for the system. However, for the notion of noise-to-state stability, defined in Section 2.3, where Σ is treated as an unknown, the assumption $\Sigma = I$ is dropped, and also $l(x)$ is not necessarily required to be vanishing at the origin ($l(0) \neq 0$).

Notation and definitions. For a matrix $A \in \mathbb{R}^{p\times q}$, the *Frobenius Norm*, $|\cdot|_F : \mathbb{R}^{p\times q} \to \mathbb{R}_{\geq 0}$, is defined as $|A|_F = \sqrt{\mathrm{Tr}\{A^T A\}} = \sqrt{\sum_{i=1}^p \sum_{j=1}^q |a_{ij}|^2}$. A scalar continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is *class* $\mathcal{K}_\infty$ if, in addition, $\lim_{s\to\infty} \alpha(s) = \infty$. A scalar continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *class* $\mathcal{KL}$ if, for each fixed t, function $\beta(\cdot, t)$ is class $\mathcal{K}$ and, for each fixed s, function $\beta(s, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. Given functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$, the expression $a(s) = O(b(s))$ as $s \to \infty$ means that $\exists M > 0 \in \mathbb{R}$ and $\exists x_0 \in \mathbb{R}$ such that $|a(x)| \leq M|b(x)|\ \forall x > x_0$. Analogously, the expression $a(s) = o(b(s))$ as $s \to \infty$ means that $\forall M > 0 \in \mathbb{R}$, $\exists x_0 \in \mathbb{R}$ such that $|a(x)| \leq M|b(x)|\ \forall x > x_0$, or equivalently, $\lim_{s\to\infty} |a(s)/b(s)| = 0$. Given a continuous function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, we denote by:

$$\bar{f}(x) = \sup_{|s|\leq|x|} f(s) \quad \text{and} \quad \underline{f}(x) = \inf_{|s|\geq|x|} f(s). \tag{2}$$

Clearly, $\bar{f}$ and $\underline{f}$ are nondecreasing functions. Note that, $\underline{f}^2(x) : x \to \underline{f}(x)\underline{f}(x)$, and so $\overline{f^2}(x) : x \to \overline{f(x)f(x)}$.

2.1. Stochastic stability

An extensive coverage of stochastic stability and stochastic Lyapunov theorems exists in the literature, Hasminskii (1980); Kushner (1967). In what follows, we refer to Deng, Krstic, and Williams (2001) where a notation based on class $\mathcal{K}$ functions is used, instead of the classical $\epsilon - \delta$.

Definition 2.1. The equilibrium $x = 0$ of system (1) is:

(i) Globally Stable in Probability if $\forall\epsilon > 0, \exists\gamma \in \mathcal{K}$ s.t.

$$P\{|x(t)| \leq \gamma(|x_0|)\} \geq 1 - \epsilon, \quad \forall t \geq 0, \forall x_0 \in \mathbb{R}^n. \tag{3}$$

(ii) Globally Asymptotically Stable in Probability if it is globally stable in probability and

$$P\left\{\lim_{t\to\infty} |x(t)| = 0\right\} = 1, \quad \forall x_0 \in \mathbb{R}^n. \tag{4}$$

Proposition 2.2. *For system (1), with $\Sigma = I$, suppose there exists a $\mathcal{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions α_1, α_2, and a continuous nonnegative function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that for all $x \in \mathbb{R}^n$,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{5}$$

$$\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} Tr\left\{l(x)^T \frac{\partial^2 V}{\partial x^2} l(x)\right\} \leq -S(x). \tag{6}$$

Then, the equilibrium $x = 0$ is globally stable in probability. If S is a positive definite function, the equilibrium $x = 0$ is globally asymptotically stable in probability.

2.2. Stochastic passivity and output strict passivity

Consider now the controlled stochastic nonlinear system

$$\begin{cases} dx = (f(x) + g(x)u)dt + l(x)\Sigma dw \\ y = h(x) \end{cases} \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^m$ is the output.[2]

Definition 2.3. The system (7), with $\Sigma = I$, is said to be stochastic passive, Florchinger (1999), if there exists a $\mathcal{C}^2$ positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that $\forall x \in \mathbb{R}^n$, and $\forall u \in \mathbb{R}^m$,

$$\begin{aligned} \mathcal{L}V(x) &= \frac{\partial V}{\partial x}(f(x) + g(x)u) + \frac{1}{2}\mathrm{Tr}\left\{l(x)^T \frac{\partial^2 V}{\partial x^2} l(x)\right\} \\ &\leq h(x)^T u - S(x) \end{aligned} \tag{8}$$

[2] The control input u may be seen as a function of t that satisfies appropriate regularity popeties so as to obtain existence and uniqueness of solutions. However, we do not need to specify these regularity properties in this paper, since the only place where inputs are used is in defining passivity and other stability properties. These properties are defined in terms of algebraic inequalities involving Lyapunov-like functions and only pointwise values of x and u, so that regularity of $u(t)$ as a function of t is not relevant. On the other hand, when interconnecting several systems, $u(t)$ becomes a function of the subsystems' state variables, and the closed-loop system is assumed to satisfy the conditions assumed for (1).

where $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a positive semidefinite function. It is said to be stochastic strictly passive, Lin and Lin (2009), if the function S can be picked positive definite, and stochastic output strictly passive (sOSP) if $S(x) = \frac{1}{\gamma}h(x)^T h(x)$, for some constant $\gamma > 0$, which we refer to as a "gain".

When $l(x) \equiv 0$, Definition 2.3 recovers the deterministic notions of passivity, strict passivity, and output strict passivity (OSP). From Dynkin's formula, Dynkin (1965), we conclude that the notion of sOSP is similar, in terms of expectation, to the notion of OSP, since

$$E\int_0^t y(s)^T u(s)ds \geq E\int_0^t \mathcal{L}\mathcal{V}(x(s))ds = E(V(x(t))) - V(x(0)) \geq -V(x(0)). \tag{9}$$

Note that with $u \equiv 0$, Eq. (8) implies asymptotic stability in probability of the sOSP system. Further results have been provided in the stochastic framework relating passivity with stability and feedback stabilization, see Florchinger (1999); Lin and Lin (2009).

2.3. Noise-to-state stability

In this section, we discuss systems that may have nonvanishing noise ($l(0) \neq 0$) and unknown noise intensity Σ. We use the notion of noise-to-state stability (NSS), which guarantees that for any noise covariance there exists a probability bound on the system's state, Krstic and Deng (1998). To accommodate unknown noise intensity, we drop the assumption $\Sigma = I$ used in the previous sections.

Definition 2.4 (*Adapted*[3] *from Deng et al. (2001)*)**.** For the nonlinear stochastic system (1), suppose there exists a $\mathcal{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and ρ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{10}$$

and, for all nonnegative definite matrices $\Sigma \in \mathbb{R}^{r\times r}$,

$$\mathcal{L}V(x,\Sigma) \triangleq \frac{\partial V}{\partial x}f(x) + \frac{1}{2}\mathrm{Tr}\left\{\Sigma^T l(x)^T \frac{\partial^2 V}{\partial x^2} l(x)\Sigma\right\} \leq -\alpha_3(|x|) + \rho(|\Sigma\Sigma^T|_F). \tag{11}$$

Then, the system is said to be noise-to-state stable (NSS) and $V(x)$ is called a noise-to-state Lyapunov function.

In the special case that $\alpha_3(|x|) \geq cV(x)$ for some constant $c > 0$, the following inequality holds from Deng et al. (2001, Thm 4.1):

$$E[V(x(t))] \leq e^{-ct}V(x_0) + c^{-1}\rho(|\Sigma\Sigma^T|_F). \tag{12}$$

Therefore, from the Markov inequality,[4] and from inequality (10) it is easy to conclude that, for any $\epsilon > 0$, there exists a $\mathcal{KL}$ function β, and a $\mathcal{K}_\infty$ function δ, such that:

$$P\{|x| < \beta(|x_0|, t) + \delta(|\Sigma\Sigma^T|)\} \geq 1 - \epsilon \quad \forall t \geq 0. \tag{13}$$

This shows that the state of the system is bounded in probability. In the next proposition we derive an easy to verify sufficient condition for NSS.

Proposition 2.5. *The nonlinear stochastic system (1) is noise-to-state stable if there exists a $\mathcal{C}^2$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying (10), a continuous strictly increasing function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a class $\mathcal{K}_\infty$ function α, such that:*

$$\frac{\partial V(x)}{\partial x}f(x) \leq -\alpha(|x|), \qquad \left|l(x)^T \frac{\partial^2 V(x)}{\partial x^2} l(x)\right|_F \leq \eta(|x|) \tag{14}$$

and

$$\eta(s) = o(\alpha(s)) \quad \text{as } s \to \infty. \tag{15}$$

Proof. From the assumptions, $\mathcal{L}V(x,\Sigma)$ satisfies

$$\begin{aligned}\mathcal{L}V(x,\Sigma) &= \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\mathrm{Tr}\left\{\Sigma^T l(x)^T \frac{\partial^2 V(x)}{\partial x^2} l(x)\Sigma\right\}\\ &= \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\mathrm{Tr}\left\{\Sigma\Sigma^T l(x)^T \frac{\partial^2 V(x)}{\partial x^2} l(x)\right\}\\ &\leq \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}|\Sigma\Sigma^T|_F\left|l(x)^T \frac{\partial^2 V(x)}{\partial x^2} l(x)\right|_F\\ &\leq -\alpha(|x|) + \eta(|x|)|z|\end{aligned}$$

where $z = \frac{1}{2}|\Sigma\Sigma^T|_F$, and the first inequality follows from the matrix Cauchy–Schwarz inequality, Magnus and Neudecker (1999) (*i.e.*, for any two real matrices A, B of the same order, $(\mathrm{Tr}\{A^T B\})^2 \leq \mathrm{Tr}\{A^T A\}\mathrm{Tr}\{B^T B\}$).

If $\lim_{s\to\infty}\eta(s) = c < \infty$, the proof is straightforward, since $\mathcal{L}V \leq -\alpha(|x|) + c|z|$. When $\lim_{s\to\infty}\eta(s) = \infty$, define $\tilde{\eta}(\cdot) = \eta(\cdot) - \eta(0)$. Clearly, $\tilde{\eta}$ is a class $\mathcal{K}_\infty$ function. Therefore, $\mathcal{L}V(x,\Sigma)$ is upper bounded by

$$\begin{aligned}\mathcal{L}V(x,\Sigma) &\leq -\alpha(|x|) + \eta(0)|z| + \tilde{\eta}(|x|)|z|\\ &\leq -\alpha(|x|) + (\eta(0) + \theta^{-1}(|z|))|z| + \tilde{\eta}(|x|)\theta(\tilde{\eta}(|x|))\end{aligned}$$

where θ is a class $\mathcal{K}_\infty$ function to be selected.

Now, let $q_0(s) = \inf_{r\geq s}\frac{\alpha(r)}{\tilde{\eta}(r)}$ for $s > 0$. Since α and $\tilde{\eta}$ are positive and continuous, $q_0(s)$ is well-defined and continuous for $s > 0$. Moreover, by construction, $q_0(s)$ is non-decreasing and positive. From the definition of $\tilde{\eta}$, $\lim_{s\to\infty}\frac{\alpha(s)}{\tilde{\eta}(s)} = \lim_{s\to\infty}\frac{\alpha(s)}{\eta(s)-\eta(0)} \geq \lim_{s\to\infty}\frac{\alpha(s)}{\eta(s)}$. Given that $\eta(s) = o(\alpha(s))$ as $s \to \infty$, $\lim_{s\to\infty}\frac{\alpha(s)}{\tilde{\eta}(s)} = \infty$, which means that $q_0(s) \to \infty$ as $s \to \infty$.

Since $q_0(s)$ is non-decreasing and goes to infinity with s, there exists a class $\mathcal{K}_\infty$ function $q(s)$ s.t. $q(s) \leq q_0(s)\ \forall s > 0$. Since $q_0(s) = \inf_{r\geq s}\frac{\alpha(r)}{\tilde{\eta}(r)} \leq \frac{\alpha(s)}{\tilde{\eta}(s)}\forall s > 0$, then $q(s)\tilde{\eta}(s) \leq \alpha(s)\forall s \in [0,\infty)$.

Finally, choose $\theta \in \mathcal{K}_\infty$ to be $\theta(\cdot) = \frac{1}{2}q(\tilde{\eta}^{-1}(\cdot))$, so that $\theta(\tilde{\eta}(\cdot)) = q(\cdot)/2$. The inequality becomes:

$$\begin{aligned}\mathcal{L}V(x,\Sigma) &\leq -\frac{1}{2}\alpha(|x|) + |z|(\theta^{-1}(|z|) + \eta(0))\\ &= -\frac{1}{2}\alpha(|x|) + \rho(|\Sigma\Sigma^T|_F),\end{aligned} \tag{16}$$

with $\mathcal{K}_\infty$ function $\rho(s) = \frac{1}{2}s(\theta^{-1}(\frac{1}{2}s) + \eta(0))$. The system is thus noise-to-state stable as in (11). □

The proposition above provides a new tool that simplifies the verification of NSS. Note that in the definition of NSS, the second term in condition (11) has Σ and x coupled, while Proposition 2.5 is only dependent on x.

3. Stochastic and noise-to-state stability of interconnected systems

Consider an interconnection of stochastic dynamical *"subsystems"* H_i, $i = 1,\ldots,N$, given by

$$H_i : \begin{cases} dx_i = (f_i(x_i) + g_i(x_i)u_i)dt + l_i(x_i)\Sigma_i dw_i\\ y_i = h_i(x_i)\end{cases} \tag{17}$$

[3] In Deng et al. (2001), inequality (11) is equivalently stated as: $|x| \geq \rho(|\Sigma\Sigma^T|_F) \Rightarrow \mathcal{L}V(x,\Sigma) \leq -\alpha(x)$, where $\rho, \alpha \in \mathcal{K}_\infty$.

[4] The Markov inequality states that for any random variable X and any constant $a > 0$, $P(|X| \geq a) \leq E|X|/a$.

where, for each subsystem H_i, $x_i \in \mathbb{R}^{n_i}$ is the state vector, $u_i \in \mathbb{R}^m$ the input, $y_i \in \mathbb{R}^m$ the output, and Σ_i a $r_i \times r_i$ nonnegative definite matrix. The coupling of the subsystems is described by:

$$u = (K \otimes I_m)y \tag{18}$$

where $u = [u_1^T, \dots, u_N^T]^T$, $y = [y_1^T, \dots, y_N^T]^T$, and $K \in \mathbb{R}^{N\times N}$. In the same manner, let $x = [x_1^T, \dots, x_N^T]^T$. Furthermore, assume $f_i(0) = 0$ and $h_i(0) = 0$, so that the resulting interconnected system has an equilibrium at the origin.

3.1. Stochastic stability of interconnected systems

In the following theorem, we give a matrix condition that guarantees stochastic stability for an interconnection of sOSP subsystems.

Theorem 3.1. *For the interconnected system described in (17)–(18), with $\Sigma_i = I_{r_i} i = 1, \dots, N$, assume that each dynamical subsystem H_i is stochastic output strictly passive, as in Definition 2.3, with gain γ_i, and storage function satisfying (5). If there exists a diagonal matrix $D = diag\{d_1, \dots, d_N\} > 0$ such that:*

$$D(K - \Gamma) + (K - \Gamma)^T D \le 0 \tag{19}$$

where $\Gamma = diag(\gamma_1^{-1}, \dots, \gamma_N^{-1})$, then the interconnected system is globally stable in probability.

Proof. Let $V(x) = \sum_{i=1}^N d_i V_i(x_i)$, where V_i is as in (8) for each H_i. Then,

$$\begin{aligned}\mathcal{L}V(x) &= \sum_{i=1}^N d_i \frac{\partial V_i}{\partial x_i}(f_i(x_i) + g_i(x_i)u_i) + \frac{1}{2} d_i \Sigma_i^T l_i(x_i)^T \frac{\partial^2 V_i}{\partial x_i^2} l_i(x_i)\Sigma_i^T \\ &\le \sum_{i=1}^N -\frac{1}{\gamma_i} d_i y_i^T y_i + d_i y_i^T u_i \\ &= y^T[(D(K-\Gamma) + (K-\Gamma)^T D) \otimes I_m]y.\end{aligned}$$

Thus, from assumption (19), $\mathcal{L}V(x) \le 0 \forall x \in \mathbb{R}^{n_1 + \cdots n_N}$, the origin is globally stable in probability. □

3.2. Noise-to-state stability of interconnected systems

In this section, we deal with an interconnection of stochastic subsystems with unknown noise intensity. We show that (19) guarantees NSS for an interconnection of subsystems which satisfy the following property.

Definition 3.2. The stochastic dynamical system (7) is called NSS⊕OSP if it has a storage function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying (10) and:

$$\begin{aligned}&\frac{\partial V}{\partial x} f(x) \le -\alpha(|x|) - \frac{1}{\gamma} h(x)^T h(x), \\ &\frac{\partial V}{\partial x} g(x) = h(x)^T \quad \text{and} \quad \left| l(x)^T \frac{\partial^2 V}{\partial x^2} l(x) \right|_F \le \eta(|x|)\end{aligned} \tag{20}$$

where $\gamma > 0$ is a constant, referred to as "gain", $\alpha \in \mathcal{K}_\infty$, and $\eta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a strictly increasing continuous function, satisfying $\eta(s) = o(\alpha(s))$ as $s \to \infty$.

To see why we refer to this property as NSS⊕OSP, note that the first and third conditions in (20) guarantee NSS for the uncontrolled system (*cf.*, Proposition 2.5). Likewise, the first and second conditions imply OSP for the deterministic part of the system.

Theorem 3.3. *For the interconnected system described in (17)–(18), assume that each dynamical subsystem H_i is NSS ⊕ OSP, as in Definition 3.2, with gain γ_i. If there exists a diagonal matrix $D = diag\{d_1, \dots, d_N\} > 0$ satisfying inequality (19), where $\Gamma = diag(\gamma_1^{-1}, \dots, \gamma_N^{-1})$, then the interconnected system is noise-to-state stable.*

Proof. Let $V(x) = \sum d_i V_i(x_i)$, then

$$\begin{aligned}\mathcal{L}V(x, \Sigma) &= \sum_{i=1}^N d_i \mathcal{L}V_i \\ &\le \sum_{i=1}^N d_i \left(-\alpha_i(|x_i|) - \frac{1}{\gamma_i} h_i(x_i)^T h_i(x_i)\right) \\ &\quad + \sum_{i=1}^N d_i \left(h_i(x_i)^T u_i + \frac{1}{2}\eta_i(|x_i|)|\Sigma_i \Sigma_i^T|_F\right) \\ &= \sum_{i=1}^N d_i \left(-\alpha_i(|x_i|) + \frac{1}{2}\eta_i(|x_i|)|\Sigma_i \Sigma_i^T|_F\right) \\ &\quad + \frac{1}{2} y^T[(D(K-\Gamma) + (K-\Gamma)^T D) \otimes I_m]y \\ &\le \sum_{i=1}^N d_i(-\alpha_i(|x_i|) + \eta_i(|x_i|)|z_i|)\end{aligned} \tag{21}$$

where $z_i = \frac{1}{2}|\Sigma_i \Sigma_i|_F$, and the last inequality follows from assumptions on D, K, and Γ.

Let $J, I \subset \{1, \dots, N\}$ be such that $J = \{j \in \{1, \dots, N\} | \lim_{s\to\infty} \eta_j(s) = c_j < \infty\}$ and $I = J^c = \{i \in \{1, \dots, N\} | \lim_{s\to\infty} \eta_i(s) = \infty\}$. Since $\eta_i(s) = o(\alpha_i(s))$ as $s \to \infty$, there exist $\theta_i \in \mathcal{K}_\infty$, $i \in I$, such that the next inequality follows as in the derivations leading to (16):

$$\begin{aligned}\mathcal{L}V &\le -\frac{1}{2}\sum_{i=1}^N d_i \alpha_i(|x_i|) + \sum_{j\in J} d_j c_j |z_j| \\ &\quad + \sum_{i \in I} d_i |z_i| (\theta_i^{-1}(|z_i|) + \eta_i(0)).\end{aligned}$$

Let

$$\tilde{\alpha}(r) = \min_{|x|\ge r} \sum_{i=1}^N d_i \alpha_i(|x_i|).$$

Clearly, $\tilde{\alpha}(|x|) \le \sum d_i \alpha_i(|x_i|)$, $\tilde{\alpha}$ is nondecreasing, and $\tilde{\alpha}(0) = 0$. As $|x| \to \infty$, $|x_i| \to \infty$ for at least one i, which implies that $\tilde{\alpha}(r) \to \infty$. Therefore, $\exists \alpha \in \mathcal{K}_\infty$ s.t. $\alpha(r) \le \tilde{\alpha}(r)\ \forall r \in \mathbb{R}_{\ge 0}$.

Now, choose

$$\rho_i(s) = \begin{cases} \dfrac{c_i}{2} s, & i \in J \\ \left(\theta_i^{-1}\left(\dfrac{1}{2}s\right) + \eta_i(0)\right)\dfrac{1}{2}s, & i \in I \end{cases}$$

where, since $|z_i| \le |z|$ and $\theta_i^{-1} \in \mathcal{K}_\infty$, we know that $\rho_i(|z|) \ge \rho_i(|z_i|)$. Let $\rho(|z|) = \sum_{i=1}^N d_i \rho_i(|z|) \ge \sum_{i=1}^N d_i \rho_i(|z_i|)$, and $\rho \in \mathcal{K}_\infty$. The inequality becomes, $\mathcal{L}V \le -\frac{1}{2}\alpha(|x|) + \rho(|\Sigma\Sigma^T|_F)$, and thus the interconnected system is noise-to-state stable as in (11). □

Theorems 3.1 and 3.3 divide the problem of certifying stability properties of a large scale nonlinear system into two tractable steps. Step one is to identify dissipativity properties, such as sOSP or NSS⊕OSP, as an abstraction of the detailed dynamical model of the subsystems. Step two is to establish the feasibility of the linear matrix inequality (19), for which various computational methods exist, Boyd, Ghaoui, Feron, and Balakrishnan (1994); Grant and Boyd (2011). Tools for the verification of dissipation properties, for step one, are provided in the next section.

4. Verifying sOSP and NSS⊕OSP

4.1. Sufficient conditions for one-dimensional systems with nonzero equilibrium

We now present conditions that guarantee the sOSP and the NSS⊕OSP properties for a scalar stochastic dynamical system. The goal is to apply such results to interconnected systems as in the previous section. As a simplifying assumption, we assume $g(x) = g$ is a constant, and without loss of generality $g = 1$ because the interconnection matrix K can be modified to incorporate a different value of g.

Consider again H_i, one of the nonlinear stochastic subsystems defined in (17), with $x_i, y_i, u_i \in \mathbb{R}$, $g_i(x_i) \equiv 1$, and where the inputs u_i are given by the feedback relation in (18). We drop the assumption that $f_i(0) = 0$, $h_i(0) = 0$, and assume instead that the deterministic part of (17) has an unique equilibrium point at x^*. This means that

$$0 = f_i(x_i^*) + u_i^*,$$

where u_i^* is the ith entry of $u^* = Ky^*$. By taking the coordinate change $\tilde{(.)} = (.) - (.)^*$, we obtain:

$$\begin{cases} d\tilde{x}_i = (\tilde{f}_i(\tilde{x}_i) + \tilde{u}_i)dt + \tilde{l}_i(\tilde{x}_i)\Sigma_i dw_i \\ \tilde{y}_i = \tilde{h}_i(\tilde{x}_i) \end{cases} \tag{22}$$

where $\tilde{h}_i(\tilde{x}_i) = h_i(\tilde{x}_i + x_i^*) - h_i(x_i^*)$, $\tilde{l}_i(\tilde{x}_i) = l_i(\tilde{x}_i + x_i^*)$, $\tilde{f}_i(\tilde{x}_i) = f_i(\tilde{x}_i + x_i^*) - f_i(x_i^*)$, and, hence, $\tilde{f}_i(0) = 0$ and $\tilde{h}_i(0) = 0$. In what follows, we drop the subscript i to simplify the notation.

Corollary 4.1. ***(sOSP)*** *For each stochastic subsystem in (22), with $x, y, u \in \mathbb{R}$, $\Sigma = 1$, $l(x^*) = 0$, and h differentiable, assume that $\forall x \neq x^*$:*

(A1) *$(x - x^*)(h(x) - h(x^*)) > 0$ and $(x - x^*)(f(x) - f(x^*)) < 0$;*
(A2) *There exists a constant $\gamma > 0$ such that*

$$\frac{f(x) - f(x^*)}{h(x) - h(x^*)} + \frac{1}{2}h'(x)\left(\frac{\|l(x)\|_2}{h(x) - h(x^*)}\right)^2 \leq -\frac{1}{\gamma}, \tag{23}$$

where $h'(x) := \frac{\partial h}{\partial x}$.
Then, the system is stochastic output strictly passive.

Proof. Let

$$V(x) = \int_0^{x-x^*} h(s) - h(x^*)ds \tag{24}$$

which is positive definite, from (A1). Therefore, $\frac{\partial V}{\partial x} = h(x) - h(x^*)$ and $\frac{\partial^2 V}{\partial x^2} = \frac{\partial h(x)}{\partial x}$. Hence, from assumption (A2), inequality (8) holds, and the system is sOSP. □

Corollary 4.2 (*NSS⊕OSP*)**.** *Consider a stochastic subsystem as described in (22), with $x, y, u \in \mathbb{R}$, $g(x) = 1$, and $h \in \mathcal{C}^1$. Assume the following holds:*

(B1) *$(x - x^*)(h(x) - h(x^*)) > 0$ and*
$(x - x^*)(f(x) - f(x^*)) < 0$, $\forall x \neq x^*$;
(B2) *There exists a constant $\hat{\gamma} > 0$ such that*

$$\frac{f(x) - f(x^*)}{h(x) - h(x^*)} \leq -\frac{1}{\hat{\gamma}} \quad \forall x \neq x^*; \tag{25}$$

(B3) *$l(\cdot)$, $h(\cdot)$, and $f(.)$ are such that as $|x| \to \infty$, $|(h(x) - h(x^*))(f(x) - f(x^*))| \to \infty$, and that $\forall i, j = 1, \ldots, r$, as $|x| \to \infty$*

$$|h'(x)l_i(x)l_j(x)| = o(|(h(x) - h(x^*))(f(x) - f(x^*))|). \tag{26}$$

Then, the system is NSS ⊕ OSP for any $\gamma > \hat{\gamma}$.

Proof. Without loss of generality, assume that $x^* = 0$. Let $V(x)$ be given as in (24), so that the equality condition in (20) holds, and $\frac{\partial V}{\partial x}f + \frac{1}{\hat{\gamma}}h^T h = hf + \frac{1}{\hat{\gamma}}h^2$. Choose some constant $\gamma > \hat{\gamma}$ so that, from assumption (B2),

$$hf + \frac{1}{\gamma}h^2 < 0 \quad \forall x \neq 0. \tag{27}$$

Moreover, from (B2) and (B1), we know that $hf + \frac{1}{\gamma}h^2 = \frac{\hat{\gamma}+(\gamma-\hat{\gamma})}{\gamma}$ $hf + \frac{1}{\gamma}h^2 \leq \frac{\gamma-\hat{\gamma}}{\gamma}hf \leq 0$, *i.e.*,

$$\left|hf + \frac{1}{\gamma}h^2\right| \geq \frac{\gamma - \hat{\gamma}}{\gamma}|hf|. \tag{28}$$

Then, $|hf + \frac{1}{\gamma}h^2| \geq \frac{\gamma-\hat{\gamma}}{\gamma}|hf|$, and from assumption (B3), $\forall i, j = 1, \ldots r$,

$$|h'(x)l_i(x)l_j(x)| = o(|h(x)f(x) + \frac{1}{\gamma}h^2(x)|) \quad \text{as } |x| \to \infty. \tag{29}$$

Using relation (29) we will show that there exist functions α and η as defined in (20) such that $\eta(s) = o(\alpha(s))$ as $s \to \infty$. The following lemmas construct such functions.

Lemma 4.3. *Consider a continuous function $m : \mathbb{R} \to \mathbb{R}$ such that $m(x) < 0 \forall x \neq 0$, and $m(x) \to -\infty$ as $|x| \to \infty$. Then, there exists a $\mathcal{K}_\infty$ function α such that $m(x) \leq -\alpha(|x|)$ and $|m(x)| = O(\alpha(|x|))$ as $|x| \to \infty$.*

Lemma 4.4. *Consider a continuous function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$. There exists a strictly increasing function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $f(x) \leq \eta(|x|)$ and $\eta(|x|) = O(f(x))$ as $|x| \to \infty$.*

From Lemma 4.4, let $\eta(|x|) \geq \sum_{i,j=1}^r |h'(x)l_i(x)l_j(x)| \geq |h'(x)| \sqrt{\sum_{i,j=1}^r |l_i(x)l_j(x)|^2} = |l(x)^T \frac{\partial^2 V}{\partial x^2} l(x)|_F$, and $\eta(|x|) = O(\sum_{i,j=1}^r |h'(x)l_i(x)l_j(x)|)$ as $|x| \to \infty$. Likewise, we can choose α from Lemma 4.3 with $m(x) = h(x)f(x) + \frac{1}{\gamma}h(x)^2$. Note that function $hf + \frac{1}{\gamma}h^2$ satisfies conditions of Lemma 4.3, from (27)–(28) and the assumption that $|hf| \to \infty$. Finally, from (29) and properties of α and η we conclude that, as $|x| \to \infty$,

$$\begin{aligned} \eta(|x|) &= O\left(\sum_{i,j=1}^r |h'(x)|\,|l_i(x)l_j(x)|\right) \\ &= o\left(\left|h(x)f(x) + \frac{1}{\gamma}h^2(x)\right|\right) = o(\alpha(|x|)). \end{aligned}$$

The system is thus NSS⊕OSP, with constant $\gamma > \hat{\gamma}$, as in Definition 3.2. □

4.2. Sufficient conditions for one-dimensional systems with unknown equilibrium

The equilibrium point of an interconnected system becomes harder to determine as the system dimension increases. The following results give conditions for sOSP and NSS⊕OSP which are equilibrium-independent.

Corollary 4.5. ***(sOSP)*** *For the stochastic subsystem in (22), with $x, y, u \in \mathbb{R}$, $\Sigma = 1$, and $l(x^*) = 0$, assume that h, f, and l are differentiable and satisfy the following, for all $x \in \mathbb{R}$:*

(A1*) *f is strictly decreasing, h is strictly increasing;*

(A2*) *There exist constants* $a, b_k > 0, k = 1, \ldots, r$, *s.t.*

$$\frac{\partial h(x)}{\partial x} \in [0, a] \quad \text{and} \quad \left|\frac{\partial l_k(x)}{\partial x}\right| \leq \sqrt{b_k}\frac{\partial h(x)}{\partial x}, \tag{30}$$

and there exists a constant $\gamma > 0$ *such that*

$$\frac{\partial f(x)}{\partial x} \leq \left(-\frac{1}{\gamma} - \frac{1}{2}ab\right)\frac{\partial h(x)}{\partial x}, \tag{31}$$

where $b = \sum_{k=1}^{r} b_k$.
Then, the system is stochastic output strictly passive.

Proof. Given inequality (30), and the fact that $l_k(x^*) = 0$, we have $|l_k(x)| \leq \sqrt{b_k}|h(x) - h(x^*)|$. Thus, $\|l(x)\|_2^2 = \sum_{k=1}^{r} l_k(x)^2 \leq b(h(x) - h(x^*))^2$. Moreover, from (31), and since $\frac{\partial h(x)}{\partial x} \in [0, a]$, it is easy to see from Corollary 4.1 that the system is sOSP. □

Corollary 4.6 (*NSS⊕OSP*). *Consider a stochastic subsystem described by (22), with* $x, y, u \in \mathbb{R}$, $g(x) = 1$, $h \in \mathcal{C}^1$, *and where f is differentiable. Assume the following holds:*

(B1*) *f is strictly decreasing, h is strictly increasing;*
(B2*) *There exists a constant* $\hat{\gamma} > 0$ *s.t.*

$$\frac{\partial f(x)}{\partial x} \leq -\frac{1}{\hat{\gamma}}\frac{\partial h(x)}{\partial x}; \tag{32}$$

(B3*) $l(\cdot)$, $h(\cdot)$, *and* $f(.)$ *are such that* $|h(x)f(x)| \to \infty$ *as* $|x| \to \infty$ *and* $\forall i, j = 1, \ldots, r$

$$\overline{|h'(x)l_i(x)l_j(x)|} = o(\underline{|h(x)f(x)|}) \text{ as } |x| \to \infty. \tag{33}$$

Then, the system is NSS ⊕ OSP for any $\gamma > \hat{\gamma}$.

Proof. It is clear that (B1*)–(B2*) imply (B1)–(B2) in Corollary 4.2. Since $h(x^*)$ and $f(x^*)$ are constants, and since h and f are strictly monotone functions, we conclude that $|h(x)f(x)| \to \infty$ as $|x| \to \infty$ implies that $|\tilde{h}(\tilde{x})\tilde{f}(\tilde{x})| \to \infty$ as $|\tilde{x}| \to \infty$. Furthermore, such conditions imply that $\underline{|h(x)f(x)|} = O(\underline{|(h(x) - h(x^*))(f(x) - f(x^*))|})$. Hence, assumption (B3*) implies (B3) in Corollary 4.2. The subsystem is thus NSS⊕OSP with constant $\gamma > \hat{\gamma}$. □

5. Application to biological reaction networks

Chemical reactions are dependent on random thermal motion, and are inherently stochastic. Stochastic models are described by a Markov jump process $X(t)$, where $X_i(t)$ represents the number of species i at time t. This process is usually defined by the Chemical Master Equation (CME), a system of coupled ordinary differential equations describing the probability transition function of every reaction over time, Kampen (2007). However, since the CME involves, in most cases, an infinite-dimensional probability transition vector, it is computationally expensive to obtain the exact solution. The Chemical Langevin Equation (CLE), replaces the large dimensional CME with a small stochastic differential equation (SDE) that is easier to compute. The solution of such an equation is now a continuous random process instead of the discrete Markov jump process $X(t)$, and thus, the solutions are not exactly the same. Nonetheless, one can derive a CLE,[5] from the CME, such that the solution provides an approximation of $X(t)$ when the system is sufficiently large, Gillespie (2000); Khanin and Higham (2007). This approximation is particularly useful for system sizes that are not so large that stochastic effects are averaged out.

[5] This SDE is to be interpreted in the Itô sense, because, under appropriate assumptions, Kurtz (1978), a density dependent Markov Chain can be approximated by an Itô diffusion process.

Below we study a class of SDEs, that can be seen as an interconnection of stochastic subsystems as described by (17)–(18):

$$\begin{cases} dx_i = (-c_i x_i + u_i)dt + \sqrt{c_i|x_i|}\sigma_{i1}dw_{i1} \\ \quad + \sum_{j=1}^{N}\sqrt{k_{ij}|y_j|}\sigma_{i1(j+1)}dw_{i(j+1)} \\ y_i = h_i(x_i) \\ u = Ky, \end{cases} \tag{34}$$

where $x_i, y_i, u_i \in \mathbb{R}$, and dw_{ij} are independent. The structure of these equations is motivated by the Chemical Langevin Equation. Since the regularity assumptions impose local Lipschitz continuity, there is a technical issue that arises from the square root terms of the CLE. We may view the results as applying to a slightly perturbed system with nonlinearities $\sqrt{\epsilon + c_i|x_i|}$, and similarly for y, where $0 < \epsilon \ll 1$. In this class, for each subsystem i, the vector $l_i : \mathbb{R}^N \to \mathbb{R}^{1\times(N+1)}$ depends not only on x_i but also on other entries of x. However, when there exists $l_i^u : \mathbb{R} \to \mathbb{R}^{1\times(N+1)}$ so that

$$l_i(x) \leq l_i^u(x_i) \quad \forall x \in \mathbb{R}^N,$$

where the inequality is elementwise, a result similar to Theorem 3.3 and Corollary 4.2 holds by using $l_i^u(x_i)$ instead of $l_i(x)$. The proof follows similarly since

$$\begin{aligned} \left|l_i(x)^T\frac{\partial^2 V_i}{\partial x_i^2}l_i(x)\right|_F &= \sqrt{\left|\frac{\partial^2 V}{\partial x^2}\right|^2 \sum_{j=1}^{N+1}\sum_{k=1}^{N+1}|l_{ij}(x)l_{ik}(x)|^2} \\ &\leq \left|l_i^u(x_i)^T\frac{\partial^2 V_i}{\partial x_i^2}l_i^u(x_i)\right|_F. \end{aligned}$$

Proposition 5.1. *For a stochastic system as described in (34), assume that each* h_i *is:* (i) *strictly increasing;* (ii) *upper and lower bounded;* (iii) *has bounded derivative; and* (iv) *does not converge to zero at infinity. Then, each subsystem* $i = 1, \ldots, N$ *is NSS ⊕ OSP.*

Proof. It is sufficient to show that conditions in Corollary 4.6 hold. Since function h_i is strictly increasing, and $f_i = -c_i x_i$ is linearly decreasing, (B1*) holds. Moreover, since h_i has bounded derivative, we know that $0 \leq h_i'(x_i) \leq a_i$ for some $a_i > 0$. Therefore, for $\hat{\gamma}_i \geq \frac{a_i}{c_i}$, assumption (B2*) holds because:

$$\frac{\partial f_i}{\partial x_i} = -c_i \leq -\frac{1}{\hat{\gamma}_i}a_i \leq -\frac{1}{\hat{\gamma}_i}\frac{\partial h_i}{\partial x_i}.$$

Clearly, $|h_i(x_i)f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. Since every h_i is bounded, then there exists some constant $b_{ij} \geq 0$ such that the corresponding diffusion coefficients $\sqrt{k_{ij}y_j} \leq b_{ij}$. Note that

$$\begin{aligned} l_i(x) &= [\sqrt{c_i|x_i|}, \sqrt{k_{i1}|h_1(x_1)|}, \ldots, \sqrt{k_{iN}|h_N(x_N)|}]^T \\ &\leq l_i^u(x_i) = [\sqrt{c_i|x_i|}, b_{i1}, \ldots, b_{i,N}]^T. \end{aligned}$$

We thus need to show that l_i^u verifies (33). Since $h_i'(x_i) \leq a_i$, for $j, k = 2, \ldots, N+1$ we obtain $\overline{h_i'(x_i)l_{ij}^u(x_i)l_{ik}^u(x_i)} \leq a_i b_{ij} b_{ik} = o(x_i) = O(\underline{|h_i(x_i)f_i(x_i)|})$ as $|x_i| \to \infty$, and also, $\overline{h_i'(x_i)l_{i1}^u(x_i)l_{ik}^u(x_i)} \leq a_i b_{ik}\sqrt{c_i|x_i|} = o(x_i)$ as $|x_i| \to \infty$. Additionally, $\lim_{|x_i|\to\infty} h_i'(x_i) = 0$, because h_i is a strictly increasing and bounded. Therefore, $\lim_{|x_i|\to\infty} |\overline{h_i'(x_i)l_{11}^u(x_i)^2}/x_i| = \lim_{|x_i|\to\infty} h_i'(x_i)c_i = 0$, which implies that $\overline{h_i'(x_i)l_{11}^u(x_i)^2} = o(x_i)$ as $|x_i| \to \infty$. Since (B3*) also holds, the system is NSS⊕OSP. □

The conditions imposed on h_i in Proposition 5.1 are satisfied by standard activation models in enzyme kinetics, such as Hill equations of the form $h(s) = \frac{k_1 s^p}{1+s^p}$. Likewise, inhibition terms, such as $h(s) = \frac{k_1}{1+s^p}$, can be encompassed by Proposition 5.1, by defining $\hat{h}(s) = -h(s)$, and incorporating the negative sign in the interconnection matrix K.

As a special case of (34), consider a cycle of three genes, each repressing the expression of the next one in the cycle, as in Elowitz and Leibler (2000):

$$\begin{aligned} dx_1 &= \left(-c_1x_1 + k_{32} + \frac{k_{31}}{1+x_3^p}\right)dt + \sqrt{c_1x_1}\,\Sigma_{11}dw_{11} \\ &\quad + \sqrt{k_{32}}\,\Sigma_{12}dw_{12} + \sqrt{\frac{k_{31}}{1+x_3^p}}\,\Sigma_{13}dw_{13} \\ dx_2 &= \left(-c_2x_2 + k_{12} + \frac{k_{11}}{1+x_1^p}\right)dt + \sqrt{c_2x_2}\,\Sigma_{21}dw_{21} \\ &\quad + \sqrt{k_{12}}\,\Sigma_{22}dw_{22} + \sqrt{\frac{k_{11}}{1+x_1^p}}\,\Sigma_{23}dw_{23} \\ dx_3 &= \left(-c_3x_3 + k_{22} + \frac{k_{21}}{1+x_2^p}\right)dt + \sqrt{c_3x_3}\,\Sigma_{31}dw_{31} \\ &\quad + \sqrt{k_{22}}\,\Sigma_{32}dw_{32} + \sqrt{\frac{k_{21}}{1+x_2^p}}\,\Sigma_{33}dw_{33} \end{aligned} \tag{35}$$

where c_i and k_{jl}, for $i, j, l = 1, 2, 3$, are positive constants, and dw_{ij}'s are independent standard Brownian processes. Although, biologically, the system variables only make physical sense in the positive quadrant, we view the system as evolving on $\mathbb{R}^3$. Therefore, we let $h_i(x_i) = -k_{i2} - \frac{k_{i1}}{1+x_i^p}$ for $x_i \geq 0$, and define it to be $h_i(x_i) = -h_i(-x_i) + 2h_i(0)$ for $x_i < 0$, so that the system is well-defined for negative values of x_i. Then, it can be written as in (17)–(18), for $x \in \mathbb{R}^3$:

$$\begin{cases} dx_i = (-c_ix_i + u_i)dt + \sqrt{c_i|x_i|}\,\Sigma_{i1}dw_{i1} \\ \quad +\sqrt{k_{i2}}\,\Sigma_{i2}dw_{i2} + \sqrt{u_i}\,\Sigma_{i3}dw_{i3}, \quad i = 1, 2, 3, \\ y_i = h_i(x_i) \end{cases}$$

where

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

From the proof of Proposition 5.1, each subsystem i is NSS$\oplus$OSP with $\gamma_i > a_i/c_i$, where $a_i \geq \max_{s\in\mathbb{R}_{\geq 0}} h_i'(s) = \frac{k_{i1}}{4p}(p-1)^{\frac{p-1}{p}}(p+1)^{\frac{p+1}{p}}$. In order to conclude NSS for the interconnected system, we need to verify the matrix inequality (19). Let $\Gamma = diag\{\gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1}\}$, and note that

$$(K - \Gamma) = \begin{bmatrix} -\gamma_1^{-1} & 0 & -1 \\ -1 & -\gamma_2^{-1} & 0 \\ 0 & -1 & -\gamma_3^{-1} \end{bmatrix}.$$

For matrices of this cyclic form, it was shown in Arcak and Sontag (2006) that a diagonal matrix $D > 0$ satisfying $D(K - \Gamma) + (K - \Gamma)^TD < 0$ exists if and only if $\gamma_1\gamma_2\gamma_3 < \sec(\frac{\pi}{3})^3 = 8$. Thus, we conclude that the interconnected system (35) is NSS if

$$\frac{k_{11}k_{21}k_{31}}{c_1c_2c_3} < \frac{8 \cdot 4^3}{(p-1)^{3\frac{p-1}{p}}(p+1)^{3\frac{p+1}{p}}}. \tag{36}$$

We simulated the system with two different sets of parameters and several noise levels (*i.e.*, $|\Sigma\Sigma^T|_F$). Fig. 1 shows the behavior of a system that is not NSS. In the absence of noise ($\Sigma_{ij} = 0$), the system converges to a steady-state oscillation, and therefore it is not asymptotically stable. Note that NSS implies asymptotic stability of the deterministic part of the system (*cf.* (11) with $\Sigma_{ij} = 0$). For the second case, we selected a set of parameters that satisfy the condition for NSS derived in (36). Indeed, in Fig. 2 we see that the system is asymptotically stable when $\Sigma_{ij} = 0$ and, in the presence of noise, its sample paths are bounded in probability (as seen from the 99% confidence level plots).

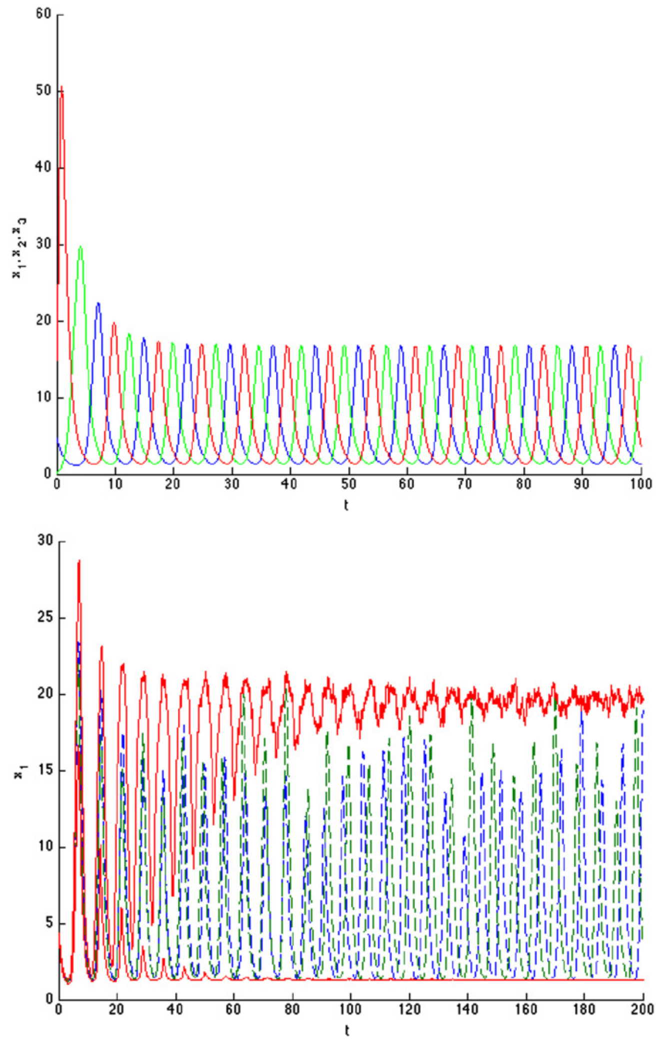

Fig. 1. Simulation of system (35) with parameters $p = 4$, $c_i = 1$, $k_{i1} = 100$, $k_{i2} = 1$, for $i = \{1, 2, 3\}$, and $(x_1(0), x_2(0), x_3(0)) = (5, 0, 5)$. (Top) Plots of $x_1(t)$ (blue), $x_2(t)$ (green), and $x_3(t)$ (red) for $\Sigma_{ij} = 0$. (Bottom) Plots of two sample paths of $x_1(t)$ (dashed) and 99% confidence levels of 2000 samples paths of $x_1(t)$ (line), for $\Sigma_{ij} = 0.05$.

6. Conclusions

We provided a technique to certify stability in probability and NSS using passivity of subsystems and a matrix diagonal stability condition. This technique is different from the classical large-scale literature, since it takes into account the sign structure of the interconnection. Moreover, we demonstrated that, under appropriate assumptions, SOSP and NSS$\oplus$OSP can be easily verifiable by equilibrium independent conditions. Literature results typically rely on the calculation of such equilibrium, which can be intractable as the dimension of the system increases. The NSS notion is appropriate for the stability analysis of biological reaction networks, as it admits systems with nonvanishing noise and unknown noise intensity.

Appendix

Proof of Lemma 4.4. Let (a_i, b_i) be the intervals where $\bar{f}$ is not increasing. Choose a constant $M > 0$. For each interval i, let ϵ_i be such that $0 < \epsilon_i < a_{i+1} - b_i$, $\bar{f}(b_i + \epsilon_i) - \bar{f}(a_i) \leq M$, and such that the intersection between the function $\bar{f}$ and the straight

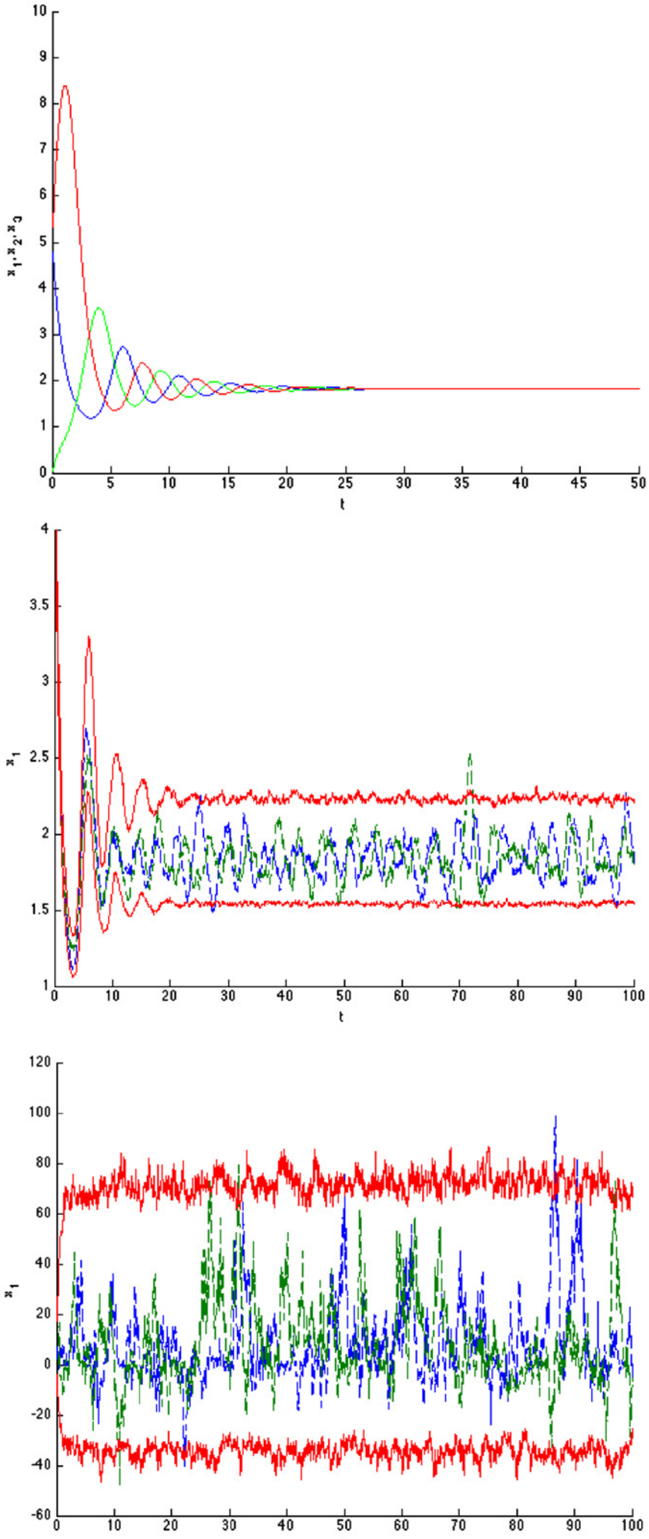

Fig. 2. Behavior of system (35) with parameters: $p = 4, c_i = 1, k_{i1} = 10, k_{i2} = 1$ for $i = \{1, 2, 3\}$, and $(x_1(0), x_2(0), x_3(0)) = (5, 0, 5)$. (Top) Plots of $x_1(t)$ (blue), $x_2(t)$ (green), and $x_3(t)$ (red) for $\Sigma_{ij} = 0$. (Middle–Bottom) Plots of two sample paths of $x_1(t)$ (dashed), and 99% confidence levels of 2000 sample paths of $x_1(t)$ (line), for $\Sigma_{ij} = 0.05$ and $\Sigma_{ij} = 5$, respectively.

line passing through $(a_i, \bar{f}(a_i))$ and $(b_i + \epsilon_i, \bar{f}(b_i + \epsilon_i))$ is empty for $a_i < x < b_i + \epsilon_i$. Note that, from continuity assumption of $\bar{f}$, such ϵ_i is guaranteed to exist. Define

$$\tilde{\eta}(x) = \begin{cases} \dfrac{\bar{f}(b_i + \epsilon_i) - \bar{f}(a_i)}{b_i + \epsilon_i - a_i}(x - a_i) + \bar{f}(a_i), \\ \quad x \in [a_i, b_i + \epsilon_i) \\ \bar{f}(x), \quad \text{otherwise}. \end{cases}$$

If $\bar{f}$ is such that there exists $b_i = \infty$ (*i.e.*, $\bar{f} = C \forall x \geq a_i$), let $\tilde{\eta} = \bar{f}(a_i) + M - e^{-(x-a_i)^2}$ for $x \geq a_i$.

Set $\eta = a\tilde{\eta}$. Then, $\eta = a\tilde{\eta} \geq a\bar{f} \geq a\underline{f}$ and also $\tilde{\eta} - \bar{f} \leq M$. Therefore, if we select a constant $c > 0$ s.t. $\underline{f} \geq c$ for $x > x_0$, and

$$\frac{\eta}{\underline{f}} \leq a\left(\frac{M}{\bar{f}} + 1\right) \leq a\left(\frac{M}{c} + 1\right) \quad \text{for } x > x_0.$$

Thus, $\eta = O(\underline{f})$. $\square$

Proof of Lemma 4.3. Since $m(x) < 0\,\forall x \neq 0$ and $m(x) \to -\infty$ as $|x| \to \infty$, then $\underline{|m(x)|} > 0 \forall x \neq 0$ and $\underline{|m(x)|} \to \infty$ as $|x| \to \infty$. Hence, there exists a $\mathcal{K}_\infty$ function α such that $\alpha \leq \underline{|m|}$. Using a construction similar to the proof of Lemma 4.4, we can define the $\mathcal{K}_\infty$ function α such that $\alpha \leq \underline{|m|} \leq |m|$, and $\underline{|m|} - \alpha \leq M$, for some constant $M > 0$. Therefore, $\underline{|m|} = O(\alpha)$, concluding the proof. $\square$

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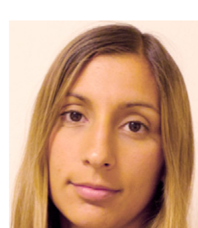

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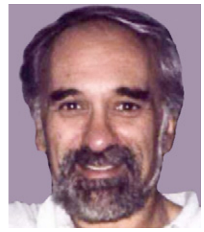

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