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Conditions for Abstract Nonlinear Regulation*

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This paper studies the general problem of the existence of output (dynamic) stabilizers for a control system. The controller is not assumed to have any special structure. Necessary and sufficient conditions are given in terms of new notions which generalize the usual ones of stabilizability and detectability. In the particular cases of analytic and bilinear systems, these conditions are considerably simplified.

1. INTRODUCTION

A typical regulation problem is the following one. Let S be a control system, and let "0" be an equilibrium state of S. One wishes to obtain a (feedback) controller which drives every state of S asymptotically to 0 while it applies inputs to S which themselves approach 0 (internal stability). In general, this controller has access only to (partial) measurements of the state of S. Further, one assumes that arbitrary (finite support) disturbances may affect states and measurements, but that these disturbances cannot be directly observed by the controller. For linear (time invariant, finite dimensional) systems, such a controller exists if and only if S is stabilizable and detectable (see, e.g., Wonham (1974)). The first of these properties, which we shall call "asymptotic controllability," or just "asycontrollability," means that each state can be driven (open loop) asymptotically to the origin. The second, which we shall call "0-detectability," says that the subsystem defined by the set of "unobservable states," i.e., the set of those states which are indistinguishable from 0, is asymptotically stable.

In this paper, we present a result which can be interpreted as a generalization, to nonlinear systems, of the above result. There has been, of course, extensive research on the topic of constructing regulators for many kinds of nonlinear systems, but no characterization in the spirit of those in the linear theory has been suggested. Much past research dealt with controllers having a "smooth" or even algebraic structure—for instance, bilinear controllers for bilinear systems. While such special controllers are of course to be desired if they exist, it appears to be impossible to derive a general theory under such artificial constraints. There are in fact many

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examples of simple control problems for which no "nice" synthesis is possible. One way around this difficulty is to relax somewhat the structure and to introduce piecewise linear controllers, as done for instance in the well-developed case of sliding-mode systems (see, e.g., Utkin (1977) and Young (1978)). Even when using such an approach, however, there seems to be no way to derive complete necessary and sufficient characterizations. (A somewhat related approach, via sampled *PL* controllers, is described in Sontag (1981), and provides sufficient conditions for regulability which are very close to being necessary.)

In view of the above, we take the most general approach that seems natural. Regulators will be just abstract control systems: a set with well-defined transition and output maps. In order to obtain a mathematically significant theory, we shall restrict attention to the case where the original system to be controlled has a certain amount of structure, at least topological, and progress to systems defined by analytic differential equations and eventually to bilinear systems. For each such class the characterizations become progressively simpler, until in the last case one recovers, for bilinear systems, a characterization equal to that for linear systems. The constructions in the proofs are rather abstract, but they all can be in principle implemented digitally; our objective in this paper, however, is to understand the underlying properties, not to provide actual control algorithms. (Close inspection of the proofs suggests that, with the appropriate definitions, the resulting regulators can be defined via "piecewise-analytic," or at least "piecewise-continuous," equations, depending on the structure of the original system.)

In Section 2 we present the definitions and state the main results. The central theorem is that, for systems S whose state spaces admit a metric for which the system maps are continuous, regulability of $(S, 0)$ is equivalent to preregulability plus indy-detectability. The latter has a rather technical definition: it means, intuitively, that indistinguishability ("indy," for short) classes can be estimated (asymptotically). Preregulability means that S is 0-detectable (definition as in the linear case) and indy-asycontrollable (or, "indy classes are asycontrollable"), meaning essentially that for each indy class there is a control sending all states in the class (uniformly) to 0. The necessary part of this equivalence is easy to prove from the definitions. The intuitive idea of the sufficiency proof is also easy to understand: alternate estimation of indy classes with appropriate control actions; technically, the proof turns out to be rather delicate, and it will occupy most of Section 3. It turns out that indy-detectability is satisfied automatically for analytic systems, so regulability is in that case equivalent to just preregulability. In the bilinear (or, more generally, state-affine case) one can prove, further, that preregulability is in fact equivalent to just asycontrollability and 0-detectability.

The results in this paper were presented in the conference papers of Sontag (1981b, 1982b). Throughout the paper we include suggestions for further research.

2. Definitions and Statement of Results

We shall need a large number of definitions and notational conventions. The latter are unavoidable if the proofs are to be kept at a reasonable length.

Systems and Signal Spaces

A *time-function* will be any function defined on the nonnegative reals $\mathbb{R}$ $(\geqslant 0)$. In any statement involving time functions, "for all t" will mean "for all $t \geqslant 0$" unless otherwise stated. Often, values will belong to a set having a distinguished element, to be called always "$\mathbf{0}$," and/or a set endowed with a metric $\mathbf{d}$; some of the following notations assume this. Let w be a time function. Then $t(w) := \sup\{t \mid w(t) \neq \mathbf{0}\}$. The *concatenation at* $b \geqslant 0$ of v and w is denoted by $v\,|b|\,w$ and is equal to $v(t)$ for $t < b$ and to $w(t-b)$ otherwise. The time function 0 is the one having the constant value $\mathbf{0}$. The *b-initial segment* of v-restriction to $[0, b)$ followed by 0— is $v\,|b|\,0$. The right shift by b of v is just $0\,|b|\,v$; this suggests extending the above definition to deal with negative b, so that the *left* shift (with truncation) is $0\,|b|\,v$ with $b < 0$. Note the equality $u\,|a|\,(v\,|b|\,w) = (u\,|a|\,v)\,|(a+b)|\,w$. We adopt the convention that any function which we define on a subinterval I will be alternatively thought of as a time function, extended by $\mathbf{0}$ outside I.

For any metric space $(X, \mathbf{d})$ and subset A of X, we denote by $B(A, r)$ [resp., $B^*(A, r)$] the open [resp., closed] ball $\{x \mid \mathbf{d}(x, A) < r\}$ [resp., $\leqslant$]. When an element $\mathbf{0}$ has been distinguished in X, $\#(x) := \mathbf{d}(x, \mathbf{0})$, and A is omitted in the above for $A = \{\mathbf{0}\}$. If v is a time function, $\#(v)$ will denote the sup-"norm", i.e., $\sup\{\#(v(t)), t \geqslant 0\}$, assuming values are on a metric space and there is a distinguished value. Consistently with this, $B(r)$ $[B^*(r)]$ will be also used for time functions, i.e., for the set $\{v \mid \#(v) < r\}$ $[\leqslant r]$. For any T, the set $B(r; T)$ is the set of v with $\#(v) < r$ and with $t(v) \leqslant T$, and similarly for B^*. No confusion should arise from the fact that the same notations—$\#(\cdot)$, $\mathbf{0}$, etc.—will be used in any given discussion to refer to objects associated to different sets; the meanings will be clear from the context.

One rather obvious terminology convention: most definitions and proofs will involve estimates depending on various (distinct) parameters—say T depending on k and e, b depending only on e, etc. A phrase like "for all $k, e > 0$ there exist positive $T(k, e)$ and $b(e)$ such that..." will be used sometimes instead of saying, more precisely: "there exist functions

T: $\mathbb{R}(>0) \times \mathbb{R}(>0) \to \mathbb{R}(>0)$ and b: $\mathbb{R}(>0) \to \mathbb{R}(>0)$ such that, for all $k > 0$ and $e > 0$ the quantities $T(k, e)$ and $b(e)$ satisfy...."

Let $\mathbf{V}$ be a metric space with an element $\mathbf{0}$ and for which the balls $B^*(r)$ are all compact. A *signal space* V is a set of time functions with values in the (underlying signal-value set) $\mathbf{V}$ which (1) is closed under concatenations, (2) contains 0, (3) consists of locally bounded time functions, i.e., $\#(v\,|t|\,0)$ is finite for all v in V and all t, and (4) is extended, i.e., a time function v with all truncations $v\,|t|\,0$ in V is necessarily itself in V.

The cartesian product $U \times V$ of two signal spaces is itself a signal space with underlying set $\mathbf{U} \times \mathbf{V}$ and metric $\mathbf{d}((u, v), (u', v')) = \mathbf{d}(u, v) + \mathbf{d}(u', v')$.

Let V, W be two fixed signal spaces (of "input" and "output" signals, respectively). With respect to these, a *system* $S = (X, \phi, h)$ is given by a set X, a map ϕ: $\mathbb{R}\,(\geqslant 0) \times X \times V \to X$, and a map h: $X \times \mathbf{V} \to \mathbf{W}$, such that the following axioms hold for all x in X, all u, v in V, and all t, s:

(2.1) $\phi(0; x, v) = x$,

(2.2) $\phi(s; \phi(t; x, u), v) = \phi(t + s; x, u\,|t|\,v)$,

(2.3) if $u\,|t|\,0 = v\,|t|\,0$, then $\phi(t; x, u) = \phi(t; x, v)$,

(2.4) given K, $T > 0$, there exists $K' > 0$ so that, for all v in $B(K; T)$, $w(\cdot) = h(\phi(\cdot\,; x, v), v(\cdot))$ is in W, and $w\,|T|\,0$ is in $B(K'; T)$.

Note that of course K' depends on x. An *initialized* system $(S, 0)$, or just S, is given by a system S and a state 0 in S which satisfies:

(2.5) $\phi(t; 0, 0) = 0$ for all t, and

(2.6) $h(0, \mathbf{0}) = \mathbf{0}$.

Note the use of "0" both to denote a state and elements of (input and output) signal spaces; there is a mild inconsistency in not denoting this state by "**0**." The above definition is rather standard, except perhaps for (2.4), which must be added in the abstract setup but is automatically satisfied in the usual ("finite dimensional") cases. The causality axiom (2.3) follows from the consistency (2.1) and semigroup (2.2) axioms, but we include it for emphasis.

Some particular classes of systems will be of interest. A *strictly causal* system is one for which h is independent of v; more generally, if $V = C \times D$ and $W = E \times F$, then one says, for example, that E is strictly causal on the C-coordinate if the first coordinate of $h(x, \mathbf{c}, \mathbf{d})$ does not depend on $\mathbf{c}$. A *discrete time* (or a "sampled data" system) is one for which transitions occur at integer times only and depend only on samples of the input: $\phi(t; x, v) = x$ for $t < 1$, $\phi(1; x, v)$ depends only on $v(0)$. This models the case of difference equations in the strictly causal case (with a "one-second clock" added in each interval); for outputs depending on inputs one would modify the state space to store samples of past inputs. Note that the present definition allows for interconnections of discrete and other systems. A *continuous-time* system is one arising (in the obvious way) from equations

$$\dot{x}(t) = f(x(t), v(t)), \qquad w(t) = h(x(t), v(t)), \tag{2.7}$$

where the state space X is a differentiable manifold, the map $f: X \times \mathbf{V} \to T(X)$ is continuous, each $f(\cdot, v)$ is a complete vector field on X, and h is continuous. The space of input signals V consists of all piecewise continuous time functions to $\mathbf{V}$ (one could take locally bounded measurable inputs without changing any of the results), and it is assumed that solutions $x(t)$ exist and are unique for all $x(0)$ and all t. (Note that we are implicitely making the usual—and rather restrictive, in our view—assumption that there are no finite escape times. This assumption simplifies considerably the exposition, but it would be interesting to have the general case treated in the future.) An *analytic* (continuous time) system is one for which X and $\mathbf{W}$ are real analytic manifolds, $\mathbf{V}$ is a subset of an Euclidean space with connected interior and no isolated points, and both f and h are real analytic (see Sussmann (1979)). Finally, a (continuous time) *state-affine* system is one for which X, $\mathbf{V}$, $\mathbf{W}$ are Euclidean, f is affine in x, and h is a constant linear function of x, i.e., one has equations

$$\dot{x}(t) = F(u(t))\, x(t) + G(u(t)), \qquad y(t) = Hx(t), \tag{2.8}$$

with $F(\cdot)$ and $G(\cdot)$ continuous. As an initialized system, 0 is the origin in X. (*Bilinear* systems have F, G linear on u.)

The most important class in what follows is that of *metric* systems. These are systems S for which X is a metric space with all $B^*(r)$ compact, and for which h and ϕ are jointly continuous in all their arguments (for the compact-open topology on V). The continuous systems defined before are all metric.

We need to introduce a few notions for a general system S. The *output* $\text{out}[x/v]$ is the signal in W defined by $w(\cdot) := h(\phi(\cdot\,; x, v), v(\cdot))$, for any x in X and v in V. Two states x, x' are *v-indistinguishable* iff $\text{out}[x/v] = \text{out}[x'/v]$. They are *indistinguishable* if this happens for all v in V. An *observable* system is one for which no two states are indistinguishable. For a state x, input v, and subset A of X containing x, the *indy class* (resp., *v-indy class*) of x *rel* A is the set $[x//A]$ (resp., $[x/v/A]$) consisting of all states in A which are indistinguishable (resp. v-indistinguishable) from x. When $A = X$, we write just $[x]$ and $[x/v]$, respectively. This notation is consistent with the one for outputs since the latter depend only on indy classes. Note that, for example, $[0/0]$ is for initialized systems the set of all states x giving identically zero outputs when the input signal 0 is fed into the system.

Let I be the equivalence relation defined by: xIx' iff $[x] = [x']$, and consider the quotient set X/I. When X is a topological space, X/I will be endowed with the (usual) finest topology for which the projection $[\cdot]$ is continuous. The ($A-$) *saturation* $[B]$ (resp., $[B//A]$) of a subset B (resp., rel A) is the union of the sets $[x]$ for x in B (resp., $[x//A]$ for x in $A \cap B$). The set B is *saturated* (resp., rel A) iff $B = [B]$ (resp., $=[B//A]$). By continuity of the maps out$[\cdot/v]$, X/I is a Hausdorff space when X is a metric system.

Regulation Concepts

We shall say that a map $f: V \to W$ between signal spaces is *stable* iff the following two properties hold:

(2.9) for any $k, T, e > 0$ there is a T' so that $\#(f(v)(t)) < e$ whenever v is in $B(k: T)$ and $t \geqslant T'$;

(2.10) for any $e > 0$ there exist $d, T > 0$ so that $\#(f(v)) < e$ whenever v is in $B(d: T)$.

An initialized system will be called *stable* iff its *i/o* map $f(v) := \text{out}[0/v]$ is stable.

In other words, outputs must converge to zero under any finite support inputs (to be thought of as "disturbances" or "perturbations"), and this convergence is uniform on the "magnitude" of the disturbance; further, small disturbances should give rise to small outputs. This is just one of many possible definitions, and we use it because it is simple, mathematically convenient, and intuitively reasonable. The results to be given can be extended to cover stability under non-finite support but "sufficiently rapid decay" disturbances; the proofs are basically the same, but there seems to be no simple (elegant) way to make the corresponding statements precise. We leave as a suggestion for further research the search for similar results under other definitions, e.g., via extended spaces. From a purely mathematical standpoint, it would be highly desirable to have a definition of stability which is closed under cascades.

A (deterministic, general) *regulation problem* is specified by (i) an initialized system $(P, 0)$ (the *plant*) whose input and output spaces split as $U \times V$ and $W \times Y$, respectively, with the Y-coordinate strictly causal on the U-coordinate, and (ii) a class *OBJ* of maps from V into $U \times W \times Y$. The signal spaces V, U, W, Y will be called the spaces of disturbances, controls, output-objectives, and measurements, respectively. A *solution* to such a problem is provided by an initialized system $(Q, 0)$ (the *controller*) which satisfies the following properties:

(2.11) the input (resp., output) signal space of Q is Y (resp., U),

(2.12) the interconnection $P * Q$ (see below) is well-posed, and

(2.13) the *i/o* map $f(v) := \text{out}[(0,0)/v]$ of $P * Q$ is in *OBJ*.

In general, by the *interconnection* $P * Q$ of two systems $P = (X, \phi, h)$ and $Q = (X', \phi', h')$ with compatible signal spaces as above, we mean a system $(X \times X', \phi^*, h^*)$ such that (i) the input signal space of $P * Q$ is V, the output space $U \times W \times Y$, and (ii) for any v in V, and any x in X, x' in X', let $(x(t), x'(t)) = \phi^*(t; (x, x'), v)$, $y(t) :=$ Y-coordinate of $h(x(t), v(t))$, $u(t) := h'(x'(t), y(t))$; then the following must hold:

(2.14) $\text{out}^*[(x, x')/v] = (\text{out}'[x'/y], \text{out}[x/(u, v)]) = (u, w, y)$, and

(2.15) $\phi^*(t; (x, x'), v) = (\phi(t; x, (v, u)), \phi'(t; x', y))$ for all t.

If both P and Q are initialized systems, one defines $P * Q$ to be initialized at $(0, 0)$. We shall say that the interconnection $P * Q$ is *well-posed* if there is a unique such $P * Q$. (This can be equivalently expressed in terms of uniqueness of the signals u and y such that the above properties hold.) The $x(\cdot)$, $x'(\cdot)$, u, y will be refered to as the "closed-loop" state trajectories and control and measurement signals.

We shall be interested here only in the state stabilization problem, but we feel that the above definition should be appropriate to the modeling of many other interesting regulation problems (decoupling when V, W are further split and OBJ consists of diagonal maps, etc.). One possible variation is to require the *i/o* maps associated to *every* initial state to be in OBJ, but this can be made equivalent to the above if one includes enough "disturbances" to set initial states. A rather interesting fact is that even some system theoretic problems not commonly thought of as "regulation" problems fit neatly in the above; for instance, if OBJ consists of a single map and P is the trivial system with $y := v$ and $w := u$ then a "controller" is just a *realization* of f; an inversion problem, on the other hand, can be modeled by letting $y := f(v)$, $w := u$, for given f, with $OBJ = \{\text{delays}\}$ or $\{\text{integrators}\}$. Here we restrict ourselves to:

(2.16) DEFINITION. The plant P is *regulable* iff the regulator problem has a solution when $OBJ = \{\text{stable maps}\}$.

As explained in the Introduction, we are going to treat only a particular case of this problem, namely, that of state stabilization under any finite support perturbations. Specifically, *we make the following assumptions on P for the rest of the paper:*

(2.17) full state as output-objective: W-coordinate of $h(x, u, v) = x$;

(2.18) *independent* state and output disturbances: V is a product $A \times B$, in such a way that ϕ is independent of B and the second coordinate of h is independent of A;

(2.19) the disturbances are *full* (see (2.21)–(2.22) below); and

(2.20) the system P is metric.

The argument A (resp., B) will be deleted from h (resp., ϕ).

The notion of full disturbance corresponds to requiring that arbitrary effects can be achieved by the perturbations. The typical example, and the standard case in the regulation literature, is that of additive disturbances, e.g., $\dot{x} = f(x, u) + \mathbf{a}$, $y = h(x) + \mathbf{b}$, for continuous time systems on an Euclidean space. The axioms are as follows:

(2.21) For each T, u the map $\phi[T, u](\mathbf{a}) := \phi(T; 0, u, \mathbf{a})$ (i) is open at $\mathbf{a} = 0$ with respect to the compact-open topology on A, and (ii) for each k, u there is a k' such that the image of $B(k'; T)$ under $\phi[T, U]$ contains $B(k)$.

(2.22) Let $T > 0$. Then, (i) for each $e > 0$ there is a $d > 0$ such that, for each admissible trajectory $x(\cdot)$ with $\#(x(t)) < d$ for $t < T$ and for each y in $B(d; T)$, there is a $\mathbf{b}$ in B with $\#(b \mid T \mid 0) < e$ and $y(t) = Y$-coordinate of $h(x(t), \mathbf{b}(t))$ for $t < T$, and (ii) for each $k > 0$ there is a $k' > 0$ such that if $x(\cdot)$ is a trajectory with $x(t)$ in $B(k)$ for all $t < T$, and if y is in $B(k; T)$, then there is a $\mathbf{b}$ with $\#(\mathbf{b} \mid T \mid 0) < k'$ and with $y(t) = h(x(t), \mathbf{b}(t))$ for $t < T$.

We introduce also the *underlying* system $S := P/0$ of the plant P; this is the system (X, ϕ, h) obtained when the maps ϕ, h are restricted to $v = 0$ and the first coordinate of h is ignored. This system, with input and output spaces U and Y, respectively, will be very important in what follows since most properties will depend only on S. (Suggestion for further research: if (2.19) does not hold, one may derive the results using for S the set of states reached from $x = 0$, and if (2.17) does not hold the results can be derived with some variation relating to controllability to the set of states giving $w = 0$.)

We now introduce the notions which will be used to characterize regulability. Let P be a fixed plant, and take $S = P/0$. All notions of indistinguishability will be with respect to the underlying system S, not to P. The same notations ϕ, h will be used for the system maps of P and S; each has one less argument ($\mathbf{a}$, or $\mathbf{b}$, respectively) in the case of S. Definitions 2.23 and 2.24 depend only on S, and Definition 2.26 only involves P in the well-posedness of the feedback system for arbitrary disturbances. Recall that X has a metric $\mathbf{d}$.

(2.23) DEFINITION. The system S is *0-detectable* iff $[0/0]$ is asymptotically stable, i.e., (i) $\phi(t; x, 0)$ converges to $x = 0$ (as $t \to \infty$) for any x in $[0/0]$, and (ii) for each $e > 0$ there is a $d > 0$ such that $\#(\phi(\cdot; x, 0)) < e$ for $\#(x) < d$.

Note that $[0/0]$ is positively invariant under $u = 0$, so the above could have been also defined as stability (in the sense of this paper) for the "subsystem" $[0/0]$ of S, assuming one introduces appropriate "disturbances" to set the initial states in $[0/0]$. Note also that a standard stability argument can be used to prove that the convergence is uniform on compacts, i.e., that for each $k > 0$ and $r > 0$ there is a T such that $\phi[T, 0]$ maps $B(k)$ into $B(r)$.

The next definition requires, intuitively, that each state x be open-loop asymptotically controllable to the origin, given only the knowledge of the indy class $[x]$; the technical conditions ask for for uniform convergence on compacts, and for small excursions and control values for small states:

(2.24) DEFINITION. The system S is *indy-asycontrollable* iff there exists a function $d\colon \mathbb{R}(>0) \to \mathbb{R}(>0)$ and for each indy class L there is a control signal u such that:

(i) if $\#(x) < d(e)$ for some x in L and some e, then both $\#(u) < e$ and $\#(\phi(\cdot; x, u)) < e$; and

(ii) for each k, r there is a T such that, whenever there is an x in L with $\#(x) < k$, then $\#(\phi(T; x, u)) < r$.

(2.25) DEFINITION. The system S is *asycontrollable* iff S is indy-asycontrollable with respect to the *identity* measurement function.

The most important of Definitions 2.24 and 2.25 will be the former. Note that in particular this condition implies that $L = [0]$ is asymptotically stable when $u = 0$, but this is in general weaker than Definition 2.23 because $[0]$ may be a proper subset of $[0/0]$. Note that Definitions 2.24 and 2.25 coincide for observable systems. In the "classical" linear case, Definition 2.24 is equivalent to Definition 2.23 plus Definition 2.25. No definition like 2.24 appears to have been given before. We could have defined asycontrollability not requiring that controls converge to zero. Since we are studying here "regulation with internal stability," the present definition is the appropriate one. The reference Sontag (1982) deals with a Lyapunov ("direct method") characterization of asycontrollability for continuous time systems; it would be interesting to obtain analogous conditions for indy-asycontrollability and for 0-detectability. Note that we are suggesting here "internal" characterizations, *not* the search for conditions under which a given regulator configuration gives a solution to a regulator problem (which is an interesting but different problem). The work of Vidyasagar (1980) may be very useful in this regard.

(2.26) DEFINITION. The system S is *preregulable* iff it is 0-detectable and indy-asycontrollable.

While the intuitive content of the next definition is very natural –indy classes can be estimated on compacts by a suitable "detector",– we would like to suggest for further study the search for a simpler definition compatible with the results to be obtained. Note the interpretation of the $Q(k, e)$ below as "detectors" for states in the large compact $B^*(k)$ which do not "disturb" the states in a small ngbd $B(e)$.

(2.27) DEFINITION. The plant P is *indy-detectable* iff (i) for each $e > 0$

there is a $d(e) > 0$, (ii) for each $k, k', e > 0$ there is an $m(k, e, k') > 0$, (iii) for each $k, r, e > 0$ there is a $T(k, e, r) > 0$, (iv) for each $k, e > 0$ there is a system $Q(k, e)$ with $P * Q(k, e)$ well-posed, a state $q(k, e)$ in each such system, and a $b(k, e) > 0$, and (v) there is given for each such k, e a function $\mathbf{i}: X' \to X/I$ (primes indicate objects associated to $Q(k, e)$), such that the following properties hold for arbitrary k, e, r. Consider first the closed loop system $P * Q(k, e)$ and, for each x in X the trajectory with $x'(0) = q(k, e)$, $x(0) = 0$, $v = 0$, and let $T := T(k, e, r)$; then:

(a) if $\#(x) < d(e')$, then $\#(x(t)) < e'$ and $\#(u(t)) < e'$ for all $t \leqslant T$ and all e' in the interval $[e, 1]$;

(b) if $\#(x) < k$, then $\#(x(t)) < b(k, e)$ for all $t \leqslant T$, and $x(T)$ is in $B(\mathbf{i}(x'(T)), r)$;

(c) if $\#(x) < k'$ for some k', then $\#(x(t)) < m$ and $\#(u(t)) < m$ for all $t \leqslant T$ and $m = m(k, e, k')$.

Consider now the system P by itself, with input signals $u = v = 0$.

It is then also required that:

(d) if $\#(x) < d(e)$, and if $y|t'|0 = 0$ for some t', then either $|\phi(t': x, 0)/0| = |0/0|$ or $\#(x(t)) < e$ for $0 \leqslant t \leqslant t'$, and

(e) if $\#(x) < k'$ and $y|t'|0 = 0$ for some k', t', then either $|\phi(t': x, 0)/0| = |0/0|$ or $\#(x(t)) < m(1, 1, k')$ for $t \leqslant t'$.

The definition could be generallized to allow for the estimation function to depend on the present output $y(t)$; this would give an equivalent concept since one may always enlarge the state space of Q to allow for the memory of y. The 1 in (e) is chosen arbitrarily; it is only used in order to simplify notations by not adding yet another estimate depending only on k'. We shall see later that:

(2.28) THEOREM. *Every plant P with S analytic is indy-detectable.*

But many –even smooth– systems are not indy-detectable. For instance, consider the system –plant– with $X = A = Y = U = \mathbb{R}$ (W is irrelevant and B can be taken trivial) and equations $x = u + \mathbf{a}$, $y = h(x)$, where h is any function which is bijective on the interval $[-2, -1]$ and is zero outside $[-3, -1]$. The underlying system S ($\mathbf{a} = 0$) is in fact observable, but the above definition cannot hold for, say, $r = 0.1$, $e = 0.5$. Indeed, compare two trajectories remaining in $B(0.5)$. Both result in $y = 0$, independently of $x(0)$, so the controller state trajectory $x'(\cdot)$ is independent of the initial state of the plant, say $x(0) = x$ or $= z$. It follows that the control u is also the same, and thus the distance $\mathbf{d}(x(t), z(t))$ is constant in t (where $z(\cdot)$ is the trajectory starting at z). But (b) requires that for some $t = T$ this distance be less than

0.2, because the set (indy class) $\mathbf{i}(x'(T))$ consists of a single state (observability). Taking $x = 0$, $z = 0.4$ gives a contradiction.

These are the main results to be proved:

(2.29) THEOREM. *Regulability = indy-detectability + preregulability.*

(2.30) COROLLARY. *For S analytic, regulability = preregulability.*

(2.31) THEOREM. *For S state-affine, preregulability = 0-detectability + asycontrollability.*

3. PROOFS

Necessary Part

Let P be a regulable plant, $S = P/0$, and Q a regulator for P. Recall that (2.17)–(2.20) are supposed to hold, and that OBJ = all stable maps. We wish to show that P is preregulable and indy-detectable. A state x' in X' is *reachable* (from 0) iff there is some input signal y for Q and some t such that $x' = \phi'(t; 0, y)$. The following lemma gives consequences of regulability for S.

(3.1) LEMMA. *Consider $P * Q$, and assume that $v = 0$. Then, for each $k, e > 0$, and for each x' in X', there exist positive numbers $d(e)$, $m(k, x')$, and $T(k, e, x')$ such that:* (a) *if $\#(x) < d(e)$ and $x' = 0$, then the closed loop trajectories have $\#(x(\cdot)) < e$ and $\#(u) < e$, and* (b) *if $\#(x) < k$ and x' is reachable, then* (i) *$\#(x(t)) < e$ and $\#(u(t)) < e$ for all $t \geqslant T = T(k, e, x')$, and* (ii) *$x(\cdot)$, y, u are in $B(m(k, x'))$.*

Proof. Let $e > 0$ *be given.* By (2.10), there are $e' > 0$ and $T > 0$ such that $\#(\text{out}[(0,0)/v]) < e$ whenever v is in $B(e'; T)$. Since P is metric, and using the local compactness of X and $\mathbf{V}$, the map sending the triple $(\mathbf{a}, u, x)$ into the trajectory $\phi(\cdot; x, (u, \mathbf{a}))$ is continuous, assuming the compact open topology is used for state trajectories. Thus, $\phi(t; 0, 0, 0) = 0$ implies that there exists for the above T an e' small enough so that $\#(x(t)) < e$ for $t \leqslant T$ if v is as above and $u = 0$. We assume that e' satisfies also this property. By 2.22(i), for this e' there is a d such that, whenever $x(\cdot)$ is a trajectory which is in $B(d)$ for $t < T$, there is some $\mathbf{b}$ in $B(e'/2; T)$ such that $h(x(t), \mathbf{b}(t)) = 0$ for $t < T$. By 2.21(i), this d can be chosen small enough that for each x in $B(d)$ there is an $\mathbf{a}$ in $B(e'/2; T)$ such that $\phi[T, 0]$ sends $\mathbf{a}$ to x. Further, d can be taken less than e. We let $d(e) :=$ this last d. Assume now that x is in $B(d(e))$, and pick $\mathbf{a}$ as above. The continuity argument on ϕ gives that $\#(x(t)) < e$ for all $t \leqslant T$. Thus, there is a $\mathbf{b}$ as above such that $y(t) =$

$h(x(t), \mathbf{b}(t)) = 0$ for $t < T$. Let $v := (\mathbf{a}, \mathbf{b})$. Then, if $x(0) = 0$, $x'(0) = 0$, all future $x(t)$ $(= w(t))$ and $u(t)$ are in $B(e)$, and $y\,|T|\,0 = 0$, $x(T) = x$. It follows that $x'(T) = 0$. Thus, $\phi^*(t\colon (x, 0), 0) = \phi^*(t; \phi^*(T; (0, 0), v), 0) = \phi^*(t + T, (0, 0), v\,|T|\,0)$, and it follows that (a) holds.

We now prove (b). We first establish that for any k and any reachable x' there exists a $c = c(k, x')$ and a T such that, whenever $\#(x) < k$, there is a v in $B(c)$ such that $\phi^*(T; (0, 0), v) = (x, x')$. Since x' is reachable in Q, there is a y in its input signal space and a T such that $\phi'(T; 0, y) = x'$. Let $u(t) := h'(x'(t), y(t))$ for $t \leqslant T$. By 2.21(ii), there is for this k a $c > 0$ such that, for each x in $B(k)$ there is an $\mathbf{a}$ in $B(c/2)$ so that $\phi(T; 0, u, \mathbf{a}) = x$. By Definition 2.4 applied to $P/0$, for each such a c there is a k' such that all intermediate $x(t)$ are in $B(k')$, when the input to P has $\mathbf{a}$ in $B(c/2)$ and u fixed as above. By 2.22(ii), then, c can be chosen large enough that, for any trajectory $x(\cdot)$ starting in $B(k')$, there is a $\mathbf{b}$ in $B(c/2)$ with $h(x(t), \mathbf{b}(t)) = y(t)$ for $t < T$. This $c = c(k, x')$ satisfies the desired property.

Assume now that e, k, x' are given. Choose a $T = T(x')$ so that $T + T'$ satisfies the required properties. Let $N = N(c, T, T')$ be such that all of $x(t)$, $u(t)$, $y(t)$, are in $B(N)$ whenever $t \leqslant T + T'$, v is in $B(c; T)$, and initial state $= 0$ (definition of "system" applied to $P * Q$). To obtain m, let T' be obtained for (say) $e = 1$, and let $m(k, x')$ be larger than $N(c, T, T')$ and 1. This completes the proof of Lemma 3.1.

(3.2) PROPOSITION. *S is 0-detectable.*

Proof. Let $[x/0] = [0/0]$. Consider in $P * Q$ the initial state $(x, 0)$, and assume that $v = 0$. Consider the signals $u = 0$ in U and $y = 0$ in Y. Let $x(t) = \phi(t; 0, 0)$, $x'(t) = 0$ for all t. By well-posedness, it follows that $\phi^*(t; (x, 0), 0) = (x(t), 0)$. The conclusion follows from Lemma 3.1.

(3.3) PROPOSITION. *S is indy-asycontrollable.*

Proof. Let $d(\cdot)$ be as Lemma 3.1. Let L be an indy class. Pick any x in L, and consider $\phi^*(t; (x, 0), 0)$. We claim that the ensuing u and y are independent of the particular x chosen: this follows from well-posedness, since $\text{out}[x/u] = \text{out}[z/u]$ whenever $[x] = [z] = L$. Again the result follows from Lemma 3.1.

(3.4) PROPOSITION. *P is indy-detectable.*

Proof. Again the proof uses Lemma 3.1. We let $d(\cdot)$ be as there, and define $Q(k, e) := Q$, and $q(k, e) := 0$, for all k, e, T' via $T'(k, e, r) :=$ the $T(k, r, 0)$ from Lemma 3.1, $b(k', e) = m(k, e, k') := m(k', 0)$, and the functions $\mathbf{i}$ all constant and equal to $[0]$. Given e, k, r, the axioms (a), (b), and (c) in 2.27 then follow from Lemma 3.1. In particular, note that (a) holds for *any* e' (not just those in $[e, 1]$), because Q is independent of e, and for (b) note that $\mathbf{d}(x(T), \mathbf{i}(x'(T))) = \mathbf{d}(x(t), [0]) \leqslant \mathbf{d}(x(t), 0)$. To prove (d), assume that $x(0) = x$ is in $B(d(e))$, $x'(0) = 0$, $u = v = 0$, and $y\,|t'|\,0 = 0$ for some t'. Let $z := x(t')$. Consider the evolution of $P * Q$ with $v := 0$ and starting at $(z, 0)$. Let u', y', $z(\cdot)$, $z'(\cdot)$ be the trajectories obtained. The signals $0\,|t'|\,u'$ and $0\,|t'|\,y'$ are consistent with the trajectory equal to $x(t)$ for $t \leqslant t'$ and to $z(t' - t)$ otherwise, and similarly for x'. By well-posedness, these are the trajectories corresponding to the initial state $(x, 0)$ and $v = 0$. Thus, $\#(x(t)) < e$ for all t, and in particular for $t \leqslant t'$, as wanted. The proof of (e) is analogous.

Sufficiency

Throughout this section, P is an indy-detectable, preregulable plant. We shall need a couple of easy technical lemmas.

(3.5) LEMMA. *Given a compact subset Z of X, and an open subset N, there exists an open set N' such that* (i) *N' is saturated* rel Z, *and* (ii) *for any indy class L such that N contains $L \cap Z$, N' satisfies $L \cap Z \subseteq N' \cap Z \subseteq N \cap Z$.*

Proof. Consider the restriction θ to Z of the projection $[\cdot]$. Since Z is compact and X/I is Hausdorff, θ is a *closed* (continuous) map. Let B be the complement of $N \cap Z$ in Z. This set is closed in the relative topology of Z. Thus, $[B//Z]$ (=preimage under θ of $\theta(B)$) is also closed in Z, and hence in X. Then $N' :=$ complement (in X) of $[B//Z]$ satisfies the required properties.

(3.6) LEMMA. *Each of the functions $d(\cdot)$ appearing in Definitions* 2.23, 2.24, *and* 2.27 *may be assumed to be continuous and strictly increasing.*

Proof. Note that any function d' with $d'(e) < d(e)$ for all $e > 0$ also satisfies the conditions in each case. Note also that, when $e' > e$, d can be redefined if desired at e' by $d'(e') := d(e)$, again without affecting the validity of the conditions. Given d, then, define first d' by: $d'(e) := d(1/n)$ in the interval $[1/n, 1/(n-1))$, for each $n > 1$, and $d'(e) := d(1)$ otherwise. Now replace d' by any continuous strictly increasing function majorized by the piecewise constant d'.

Only part of the following lemma will be needed later, but the full statement is of interest in itself. Note the difference with Definition 2.24: T is now independent of L, and $x(t)$, $u(t)$ are in $B(r)$ *for all* $t \geqslant T$; on the other hand, u depends now on k and r. Only indy-asycontrollability is needed in Lemmas 3.7–3.8.

(3.7) LEMMA. *For each k', r', $e > 0$ with $r' < k'$, $e < k'$ there are positive $d'(e)$, $c(k')$, and $T(k', r') > 1$ such that, for each indy class L', there* *is an input u' with:* (i) *if x is in $L' \cap B^*(d'(e))$, then both $\phi(t: x, u')$ and $u'(t)$ are in $B(e)$ for all t, and* (ii) *if x is in $L' \cap B^*(k')$, then $x(t) = \phi(t; x, u')$ is in $B(r')$ for all $t \geqslant T'(k', r')$, and $\#(x(t)) < c(k')$, $\#(u) < c(k')$.*

Proof. Define a function $j(e) := d(d(e)/2)$, where d is as in Definition 2.24, assumed continuous and strictly increasing via Lemma 3.6. Let also $d'(e) := j(e)/2$. For each class $L \neq [0]$, let $e(L)$ be the inf of the e such that $L \cap B^*(j(e))$ is nonempty. Since L is closed, $e(L) \neq 0$. Pick k', r', e, and a class L.

Assume first that $L \neq [0]$. Apply Definition 2.24 to L, to obtain an input u as there. Let $r := d(r')$, $k := k' + 1$, and pick T as in Definition 2.24(ii). Pick a number c larger than k', larger than the (finite) number $\#(u \mid T \mid 0)$, and larger than all possible values $\#(\phi(t; x, u')$ for all $t \leqslant T$, x in $B^*(k')$ and $\#(u') \leqslant \max\{\#(u \mid T \mid 0), k'\}$. Since $\phi[T, u]$ is continuous, there is a d'' such that any x, z at distance $<d''$ get mapped to x', z' at distance $<d(e(L))/2$, for x, z in $B^*(k)$. Let m be a number less than both d'' and $j(e(L))/2$. Note that $\phi|T, u|$ maps $E := L \cap B^*(k')$ into $B(d(r'))$. Let N be an open set which contains the former and which is also mapped into $B(d(r'))$. We may assume that N is contained in the m-ngbd of E. By Lemma 3.5, we may further assume that N is saturated rel $B^*(k')$. Consider any indy class L' intersecting N at a nonempty set F. Let G be the image of F under $\phi[T, u]$. Since $[x] = [z]$ always implies that $[\phi(t; x, u)] = [\phi(t; z, u)]$, there is a (unique) indy class H containing F. Let w be the input associated to H via Definition 2.24. In particular, for any x in G, $\phi(t; x, w)$ remains in $B(r')$ for all t. Further, $\#(w) < r' < c$. Suppose that L' intersects some $B^*(d'(e))$, for some e, say at x. Since N is saturated rel $B^*(k)$, x is in N (because $d'(e) < e < k$). Thus, there is some z in L with $\mathbf{d}(x, z) < m < j(e(L))/2$. Since $\#(x) < d'(e) = j(e)/2$, it follows that $j(e(L)) \leqslant \#(z) < j(e(L))/2 + j(e)/2$. Thus, $e(L) < e$. It follows that L intersetcs $B(j(e)) = B(d(d(e)/2))$, and z is in the intersection. Also, this implies that u is in $B(j(e))$, so also in $B(e)$. By Definition 2.24, $\phi(T; z, u)$ is in $B(d(e)/2)$. Since $\mathbf{d}(x, z) < m < d$, also $x' := \phi(T; x, u)$ is in $B(d(e))$. Thus, $\#(\phi(\cdot; x', w)) < e$ and $\#(w) < e$. Concatenating u and w, one has for each L' intersecting N that the conclusions hold, except of course that T and c depend on the chosen L $(\neq [0])$.

We claim that the same local result holds for $L = [0]$. Again apply Definition 2.24, to obtain T, with $r = d(r')$, $k = k' + 1$. Note that now $u = 0$. Since $E := [0] \cap B^*(k')$ is mapped by $u = 0$ into $B(d(r'))$, there is by continuity on u and x an $a > 0$ such that $\phi(T; x, w)$ is in $B(d(r'))$ whenever $\#(w) < a$ and x is in an a-ngbd of E. Let N be an open ngbd of E which is saturated rel $B^*(k)$ and which is contained in the $B(d(a))$-ngbd of E. Consider any indy class L' intersecting N at a nonempty set F. Choose for

L' an input u as in Definition 2.24. If L' intersects $B^*(d'(e))$, note that $d'(e) < d(e)$, so L' also intersects $B(d(e))$, and hence by Definition 2.24 future $x(t)$ and $u(t)$ are in $B(e)$. If x is in F, it is also in $B(d(a))$, so also $\#(u) < a$, and hence also $\phi(T; x, u)$ is in $B(d(r'))$, as wanted. With any c larger that a and larger than the $x(t)$ which may appear when starting in the closure of N, one has the conclusion for ngbds of $L = [0]$.

Cover the set $B^*(k')$ by the open sets $N(L)$ constructed for each L and take a finite subcover. It is enough now to take c (resp., T) as the largest of the $c(L)$ (resp., all $T(L)$, and 1) associated to this subcover.

(3.8) LEMMA. *Let $T(\cdot, \cdot)$ and $d(\cdot)$ be the functions in the statement of Lemma 3.7. For each $k, r, e > 0$ with $r < k$ and $e \leqslant 1 \leqslant k$ there exists a $c(k)$, and for each indy class L intersecting $B^*(k)$ there exist an input u and a open set N, such that* (i) *$L \cap B^*(k) \subseteq N$,* (ii) *$N$ is saturated rel $B^*(k)$,* (iii) *for any x in $N \cap B^*(k)$, $\#(\phi(T; x, u)) < r$ and $\#(x(t)) < c(k)$ for all $t \leqslant T$,* (iv) *if $e \leqslant e' \leqslant 1$ and x is in $N \cap B^*(d(e'))$, then $\#(x(t)) < e'$ for $t \leqslant T$, and* (v) *$\#(u) < c(k)$.*

Proof. Let k, r, e, L be given. Pick u as in Lemma 3.7. Let M be the set of all x in X such that for each e' in $[e, 1]$ the following property holds: $\#(x) \leqslant d(e')$ implies $\#(\phi(t; x, u)) < e'$ for all $t \leqslant T$. An easy compactness argument (plus continuity of $d(\cdot)$) gives that M is open. By Lemma 3.7, L is contained in M. Since $L \cap B^*(k)$ is mapped by $\phi[T, u]$ into $B(r)$, it has an open ngbd M' which is also sent to $B(r)$. Let N' be the intersection of M and M' and pick N such that it satisfies (i) and (ii) and such that $N \cap B^*(k)$ is included in $N' \cap B^*(k)$. Assume that x is in the first of these sets. Then x is in M', so it is mapped into $B(r)$. Define $c(k)$ as larger than $c'(k) :=$ the "$c(k)$" of Lemma 3.7, and so that it bounds all values $\#(\phi(t; x, u'))$ for $\#(u') < c'(k)$, $t \leqslant T$, and $\#(x) \leqslant k$. Thus, (iii) holds. Since (v) holds by construction, it only remains to prove (iv). But this is immediate from the definition of M.

(3.9) LEMMA. *For each k', k'', r, e, $e' > 0$, $e \leqslant 1$, there are $T = T'(k', r, e) > 1$, positive $c' = c'(k')$, $m' = m'(k', e, k'')$, $g = g(e')$, $b' = b'(k', e)$, and systems $Q = Q(k', r, e)$ with* states *$x' = x'(k', r, e)$ such that the following properties hold:* (i) *the interconnection $P * Q$ is well posed, and for $v = 0$, $x(0) = x$, and $x'(0) = x'$ the following hold for the ensuing closed loop trajectories;* (ii) *if $\#(x) < g(e')$ for an e' in the interval $[e, 1]$, then $\#(x(t)) < e'$ and $\#(u(t)) < e'$ for $t \leqslant T$,* (iii) *if $\#(x) < k'$, then $\#(x(T)) < r$, and $\#(x(t)) < c'$ and $\#(u(t)) < b'$ for $t \leqslant T$; and* (iv) *if $\#(x) < k''$, then $\#(u(t)) < m'$ for $t \leqslant T$.*

Proof. Let d be as in Lemma 3.8, and let d' be the function called d in Definition 2.27. Define $g(e) := d'(d(e))$. Consider the detectors in Definition

2.27, and let R be the $Q(k', d(e))$ there, $q = q(k', d(e))$. Let $X(R)$ be the state space of R. Using b from Definition 2.27, let k be any number larger than $b(k', e)$, r, and 1. Let $c'(k') := c(k)$, and let $b'(k', e)$ be any number larger than $b(k', e)$ and $c(k)$. Let $m'(k', e, k'')$ be any number greater than $c(k)$ and $m(k', e, k'')$. We shall define $T'(k', r, e)$ to be the sum of the $T(k, r)$ in Lemma 3.8 and of $T = T(k', d(e), \theta')$ (right-hand term is the estimate in Definition 2.27), where the number θ' will be constructed below.

We claim that there exists an open covering $\{N_i, i = 1,..., n\}$ of $B^*(k)$ with all N_i saturated rel $B^*(k)$, inputs $\{u_i\}$ as in Lemma 3.8, and a map $\mathbf{1}: X' \to \{1,..., n\}$ such that: (1) $N_i \cap B^*(k)$ is mapped into $B(r)$ by u_i at time $T(k, r)$, with all the properties in Lemma 3.8 holding, and (2) when R is started at q and $x(0)$ is in $B(k')$, then $x(T)$ is in N_i for $i = \mathbf{1}(x'(T))$. We now prove this claim. Pick any indy class L, and apply Lemma 3.8 to get a corresponding N, u. Let $\theta(L)$ be such that the $\theta(L)$-ngbd of $L \cap B^*(k)$ is contained in N. Pick $\theta'(L) := \theta(L)/2$. Consider $B(L, \theta'(L))$ and apply Lemma 3.5 to this set, with $Z = B^*(k)$. Let $N'(L)$ be the saturated open thus obtained. Cover $B^*(k)$ by these $N'(L)$, and take a finite subcover $\{N'_i = N'_i(L_i), i = 1,..., n\}$. By N_i, u_i we denote the N, u associated to the corresponding L_i. Let θ' be a number which is less than 1 and less than all the θ'_i. This is the θ' used above to obtain the time T that the detector R will be operated. Given any indy class L intersecting $B^*(k)$, one of the N'_i contains this intersection. Let $\mathbf{j}: X/I \to \{1,..., n\}$ be a choice map giving an index i for which this happens. Let $\mathbf{1}(z) := \mathbf{j}(\mathbf{i}(z))$. If R is now operated, with $x(0) = x$ in $B(k')$, one has $x(T)$ in $B(\mathbf{i}(x'(T)), \theta')$ by definition of indy-detectability. Thus, $d(x(T), z) < \theta'$ for some z in $\mathbf{i}(x'(T))$. But $x(t)$ is always in $B(b(k', e))$, so $\#(z) \leqslant k$. By the definition of $\mathbf{j}$, $\mathbf{i}(x'(T)) \cap B^*(k)$ is contained in $B(L_i, \theta_i)$, where $i = \mathbf{1}(x'(T))$. Thus, $\mathbf{d}(x(T), L_i) < \theta_i$, and it follows that $x(T)$ is in N_i, as claimed.

We now define $Q = Q(k', r, e)$. The state space X' will be the set $(X(R) \cup \{1,..., n\}) \times \mathbb{R}$ minus the set of all (p, s) with p in $X(R)$ and $s \geqslant T$. The state x' will be $(q, 0)$. States of the type (p, t) in $X(R) \times \mathbb{R}$ will be said to be "of type 1", the (i, t) in the rest of type 2. We let ϕ'', h'' denote the system functions of R. Then, for states of types 1 and 2 respectively:

$$\phi'(t; (p, s), y) := (\phi''(t; p, y), t + s) \qquad \text{if} \quad t + s < T$$

$$:= (\mathbf{1}(\phi''(T - s; p, y)), t + s) \qquad \text{otherwise,}$$

$$\phi'(t: (i, s), y) := (i, t + s), \tag{3.10}$$

$$h'((p, t), \mathbf{y}) := h''(p, \mathbf{y}); \qquad h'((i, t), \mathbf{y}) := u_i(t - T). \tag{3.11}$$

Then (3.10)–(3.11) define indeed a system, and the interconnection P^*Q is well posed: one needs only to check all possible combinations of types of states and times $<T$ or $>T$. For instance, the signals u, y needed for the closed loop operation are constructed separately for $t < T$ (well posedness of

P^*R) and for $t \geqslant T$ (basically open-loop control in that interval). All the desired properties hold by construction.

(3.12) LEMMA. *Consider the system $S = P/0$. For each $k, e > 0$ there exist positive* $\mathbf{j}(e)$, $\mathbf{n}(k)$ *such that, for any t, x for which* $\text{out}[x/0]\,|t|\,0 = 0$, *the following properties hold:* (a) *if $\#(x) < \mathbf{j}(e)$ for some e, then $\#(\phi(t'; x, 0)) < e$ for all $t' < t + 1$, and* (b) *if $\#(x) < k$ for some k, then $\#(\phi(t'; x, 0)) < \mathbf{n}(k)$ for all $t' < t + 1$.*

Proof. By continuity of $\phi(t; \cdot, 0)$ there is a $g(\cdot)$ such that $B(g(e))$ is mapped into $B(e)$ for all $t \leqslant 1$. By 0-detectability, there is a $d'(\cdot)$ such that $\#(\phi(\cdot; x, 0)) < e$ whenever $[x/0] = [0/0]$ and $\#(x) < d'(e)$. Finally, let d be as in Definition 2.27. Define $\mathbf{j}(\cdot) := \min\{d'(\cdot), d(g(\cdot))\}$. Pick an x in $B(\mathbf{j}(e))$ and t arbitrary. Assume that $y\,|t|\,0 = 0$. By indy-detectability, either (i) $x(t')$ is in $B(g(e))$ for $t' \leqslant t$ or (ii) $x(t)$ is in $[0/0]$. In case (i), the choice of g gives the desired conclusion. If $[x(t)/0] = [0/0]$, then, since also $y(t') = 0$ for $t' < t$, it follows that $\text{out}[x/0] = 0$, so x is itself in $[0/0]$. By 0-detectability, (a) holds. To prove (b), note that the estimate $\mathbf{n}(\mathrm{k})$ exists for x in $[0/0]$, and if x is not in this set, the estimate also exists by Definition 2.27 plus the boundedness of $\#(\phi(t; x; 0))$ for x bounded and $t \leqslant 1$.

The proof that a given system description involving different kinds of states indeed defines a system Q, and that an interconnection $P * Q$ is well-posed, is in general very tedious since it involves careful checking of all axioms for each kind of state. The following technical lemma can be used, at least in the case that we shall need to consider, to establish directly these properties.

(3.13) LEMMA. *Assume given a family of systems $Q(i)$, $i \geqslant 0$, for which the interconnections $P * Q(i)$ are well-posed for each i. States $q(i)$ and a subset $Z(i)$ are specified for each state set $X(i)$. Further, a function $f_i: Z(i) \to \mathbf{N}$ (= nonnegative integers) is given for each i. Let $\mathbf{s}(x, y) := \inf\{t \mid \phi(t; x, y)$ is in $Z(i)\}$, for each input y and each x in $X(i)$. It is assumed that there is an $\mathbf{s}(x)$ (possibly infinite) such that $\mathbf{s}(x, y) \geqslant \mathbf{s}(x) > 0$ for all y, for each x not in $Z(i)$, and that $\mathbf{s}(q(i)) = T(i) \geqslant 1$ for all i. The claim is that there is a unique system Q such that $P * Q$ is well-posed, and such that for each x in $X(i)$:*

$$\begin{aligned} \phi(t; x, y) &= \phi_i(t; x, y) && \textit{if} \quad t < \mathbf{s}(x, y), \\ &= \phi_i(t - \mathbf{s}(x, y); q(j), 0\,|(\mathbf{s}(x, y) - t)|\,y) && \textit{otherwise}, \end{aligned} \tag{3.14}$$

where $j := f_i(\phi_i(\mathbf{s}(x, y); x, u))$, and where the state space of Q is the disjoint union of the $X(i)$ modulo the identifications $x = q(f_i(x))$ for each x in $Z(i)$. The output map is assumed to be $h(x, \mathbf{y}) := h_i(x, \mathbf{y})$ for those x that are in $X(i)$ but not in $Z(i)$.

Proof (sketch). The definition of ϕ is by induction on n, defining $\phi(t; x, y)$ for each (t, x, y) such that, either $\mathbf{s}(x) > 1$ and $t \leqslant n$, or $\mathbf{s}(x) \leqslant 1$ and $t \leqslant n - 1$. Formulas 3.14 then define ϕ for all values. The axioms for Q being a system follow from those for the subsystems $Q(i)$. To prove well-posedness, consider a state x of P and a state x' of Q, say in $X(i)$. By the above identification, one may assume that x' is not in $Z(i)$. Thus, a unique pair of closed-loop signals (u, y) exists, by well-posedness of $P * Q(i)$, for $t < \mathbf{s}(x)$. Using this y one obtains the next j as in formula 3.14, and the result follows by induction. Note the use of the assumption that $T(i) > 1$ in the induction step, ensuring that there are finitely many transitions, from one type of state to another type of state, in any finite time interval.

We now start constructing the controller P. The definition will use the construction in Lemma 3.13. Define, using Lemma 3.9, $e(1) := 1$ and inductively $e(i + 1) := g(e(i))$. For each $a > 0$, define $f(a)$ to be any integer j such that $e(j) < a$.

A system $Q(0)$ is introduced as follows. Its state set is $X(0) :=$ set of initial segments of observations, i.e., the set of pairs (y, t), y in Y, with $y | t | 0 = 0$. We denote by 0 the pair $(0, 0)$, and define $\phi(t; 0, y) := 0$ if $T(y) := \inf\{t \mid y(t) \neq 0\}$ is greater than t, $\phi(t; 0, y) := (y \mid (t - T(y)) \mid 0, t - T(y))$ otherwise. For a state (y, t) with $t > 0$, define $\phi(t'; (y, y') := (y | t | y', t + t')$. The output h is constantly $=0$. Let $Z(0) := \{(y, t) \mid t = 1\}$, and $f_0(y, 1) := f(\#(y | 0.5 | 0))$. The hypothesis of 3.13 hold then for $Q(0)$, when $q(0) := 0$.

Define now functions $K(i, j)$ and $m(i, j)$ as follows, by induction on i. Let $K(1, j) := j$ for all $j \geqslant 1$. Assume that $K(i, j)$ has been defined for all j. Let $k'(i) := K(i, i)$. Consider using Lemma 3.9 the system $Q'(i) := Q(k'(i), e(i + 2), e(i + 1))$, and obtain $T(i) := T(k'(i), e(i + 2), e(i + 1))$. Let $m(i, j) := m'(k'(i), e(i + 1), K(i, j))$, for all j. The induction is then completed by letting $K(i + 1, j)$ be any number bounding

$$\{\#(\phi(t; x, u)) \mid t \leqslant T(i), \#(x) \leqslant K(i, j), \#(u) \leqslant m(i, j)\}. \tag{3.15}$$

Note that $K(i + 1, j) > K(i, j) > j$ for each i. Now introduce the systems $Q(i)$ as follows. The state space $X(i)$ of $Q(i)$ is $X'(i) \times \mathbb{R}(\geqslant 0)$, where $X'(i)$ is the state space of $Q'(i)$, and let $q(i) := (x'(i), 0)$, using the states $x'(i)$ given in Lemma 3.9. The set $Z(i)$ consists of all the (x, t) with $t = T(i)$. If ϕ'_i is the transition map of $Q'(i)$, then let for $Q(i)$:

$$\phi_i(t'; (x, t), y) := (\phi'_i(t'; x, y), t' + t), \tag{3.16}$$

and output $h_i((x, t), \mathbf{y}) := h'_i(x, \mathbf{y})$. Let f_i be constantly equal to $i + 1$. The hypothesis in Lemma 3.13 hold by well posedness of the original $P * Q'(i)$.

Note that $s(x, t) := T(i) - t \leqslant T(i)$ for all x. And $s(q(i)) = T(i)$, so the notations $T(\cdot)$ in Lemmas 3.13 and 3.9 are consistent.

Let Q be constructed from these $Q(i)$ via Lemma 3.13. It must be proved that $P * Q$ is stable. We shall use primes (′) to indicate objects associated to Q. For simplicity, conditions (2.9)–(2.10) will be only proved explicitly for u and x: by the continuity of the output measurement y on x, the full result for $\text{out}[(0, 0)]/v] = (u, x, y)$ can be obtained trivially from this. Note the following two properties that hold by construction. Let $v = 0$, and consider the states $x(0) = 0$ in P and $x'(0) = q(i)$ in X'. Then, for each $i \geqslant 1$:

(3.17) assume $\#(x) < e(i + 1 - i')$, $i' \geqslant 0$, and take any $j \geqslant i$; let $T := 0$ if $j = i$, $= T(i + 1) + \cdots + T(j)$ otherwise; then $\#(x(t))$ and $\#(u(t))$ are necessarily $< e(j - i')$ for $t \geqslant T$;

(3.18) assume $\#(x) < K(i, j), j \geqslant i$, and let $T := T(i) + \cdots + T(j)$; then $\#(x(t))$ and $\#(u(t))$ are $< e(j)$ for $t \geqslant T$.

Let k, T be given, and consider a disturbance v in $B(k; T)$. Let $j > 0$ be chosen such that $T \leqslant T(0) + \cdots + T(j)$. Then $x'(T)$ must be in one of the sets $X(i)$ with $0 \leqslant i \leqslant j$. Assume first that $i > 0$. Let $t' := s(x'(T))$. Then $x'(T + t') = q(i + 1)$. Let K be a number large enough so that $x(T + t')$ is in $B(K)$. Note that K can be chosen independently of the particular v since $x(T)$ is bounded for v in, say, $B(k; T + 1)$, and since the input disturbance is identically zero in the interval $(T, T + T')$. Let $j \geqslant i$ be such that $K < K(i, j)$—recall that always $K(i, j) > j$. Then property (2.9) in the definition of stability follows from (3.18). Assume now that $i = 0$, and suppose that $x(T)$ is in $[0/0]$ and $x'(T) = 0$. Then (2.9) follows by 0-detectability. If $x = x(T)$ is not in $[0/0]$, consider $t' := \inf$ of those t for which the output y of S has $y(t) \neq 0$, assuming $x(0) = x$ and control $u = 0$. Then $x(T + t')$ is again bounded as a function only of k and T, by the second part of Lemma 3.12. Since $x'(T + t') = q(1)$, one may again apply (3.18). If $x'(T) \neq 0$, then there is a $t' < 1$ as above, so we again apply (3.18). Thus, (2.9) is true in every case.

To prove (2.10), let e be given. Define $T := 0.5$. Pick a j' such that $e(j') < e$. Let d be such that, with $u = 0$ and $\#(v) < d$, necessarily $\#(x(t)) <$ j$(e(j' + 1))$—cf. Lemma 3.12—and $\#(y(t)) < e(j')$ for $t \leqslant 0.6$ (any number larger than 0.5 is suitable here). Let $t'' := \inf\{t \mid y(t) \neq 0$ if $u = 0$ is applied to $P\}$. If $y = 0$, then in the closed loop operation always $x'(t) = 0$. Thus, $u = 0$ is indeed applied (note the delay which prevents this reasoning from being circular). In that case, Lemma 3.12 gives the desired result. The other possibility is that t'' is finite. Then $x'(t'' + 1) =$ some $q(i)$, where $i = f(a)$, $a =$ magnitude of the output y restricted to $[t'', t'' + 0.5)$. By the choice of $d, a < e(j')$. By definition of f, then, $i > j'$. Again by Lemma 3.12, one knows that $\#(x(t'' + 1)) < e(j' + 1)$. Let $i' := i - j'$. By (3.17), then,

$\#(x(t)) < e(i-i') < e(j') < e$ for all t, and the same happens for the control values $\#(u(t))$.

This completes the proof of regulability. Close inspection of the above proof suggests an area for further research: develop all the theory in the category of "piecewise continuous" systems, where one requires all maps appearing to be continuous on each of the elements of a covering; the sets in this covering could be, e.g., intersections of open and closed sets, and in order for the setup to be of applied interest one should require this covering to be locally finite around each nonzero state. Similarly for piecewise analytic, etc. Note that these constructions result a priori in very large state spaces; as remarked in the Introduction we are here interested only in abstract regulability. But various simplifications are immediate. In most situations one is only interested in appropiate behavior for disturbances which are bounded in magnitude and/or in controlling to a sufficiently close tolerance around the origin, not necessarily asycontrollability. All that is required is then to only consider those $Q(i)$ controlling states of a given bounded magnitude, to a ngbd $B(j)$ small enough.

Some Particular Classes of Systems

We wish to prove Theorem 2.28. The following lemma is useful.

(3.19) LEMMA. *Let P be a plant, and assume that the following properties hold for $S = P/0$, for some $T > 0$:* (i) *for each e there is a u with $\#(u) < e$ and such that u determines final states, i.e., for any x, z in X, if* $\text{out}[x/u]\,|T|\,0 = \text{out}[z/u]\,|T|\,0$, *then* $[\phi(T; x, u)] = [\phi(T; z, u)]$, *and* (ii) *for any x in X, either* $\text{out}[x/0]\,|T|\,0 \neq 0$ *or* $[\phi(T; x, 0)] = [0/0]$. *Then P is indy-detectable.*

Proof. Given $e > 0$, pick $d(e)$ such that $\phi(t; x, u)$ is in $B(e)$ whenever $t \leqslant T$, $\#(x) < d(e)$, and $\#(u) < d(e)$. Such a d always exists by continuity because 0 is an equilibrium state. Let $T(k, e, r) :=$ constantly T, and let $m = m(k, e, k')$ be any number bounding $\#(\phi(t; x, u))$ for $t \leqslant T$, $\#(x) \leqslant k'$, and $\#(u) \leqslant 1$. Let $b(k, e) := m(k, e, k)$. The system $Q(k, e)$ will depend only on e; in fact, both the state space X and the transition map ϕ are even independent of e. Pick X as the set of initial segments of inputs (y, t) with $y\,|t|\,0 = y$. Let $\phi(t'; (y, t), y') := (y\,|t|\,(y'\,|t'|\,0), t + t')$. For each e, take an u as in the statement, such that $\#(u) < \min\{1, d(e)\}$. Define for $Q(k, e)$ the (strictly causal) output map $h(y, t) := u(t)$. Finally, choose $q(k, e) := (0, 0)$, and let $\mathbf{i}(y, t) :=$ the indy class $[\phi(T; x, u)]$ characterized by $y\,|T|\,0$, if any such class exists, and arbitrary otherwise. (In the case to be considered, analytic systems, one may replace Y by a finite dimensional space.) The systems $Q(k, e)$ are all well defined, and the interconnection $P * Q(k, e)$ is well posed—trivially, since there is no feedback involved. The axioms (a), (b), (c), for indy-detectability are satisfied by construction. We now prove (d). Let t' be arbitrary and let $\#(x) < d(e)$. Assume that $\text{out}[x/0]\,|t'|\,0 = 0$. Let first $t' \geqslant T$. Then $[\phi(T; x, 0)/0] = [0/0]$, so also $[\phi(t'; x, 0)/0] = [0/0]$, as wanted. If instead $t' < T$, then $\#(\phi(t; x, 0)) < e$ for all $t \leqslant t'$, by definition of $d(\cdot)$. To prove (e) the argument is analogous: just note that $\#(\phi(t; x, 0))$ is bounded when x and t are both bounded.

For analytic systems, the hypothesis of the lemma hold; this is an immediate consequence of the main result in Sussmann (1979); part (ii) is just an exercise in analyticity. This establishes Theorem 2.28. We remark also that one has the same result for (at least) polynomial discrete time systems; see Sontag (1979).

Turning now to preregulability, we ask when is this property equal to just asycontrollability plus 0-detectability. Call a system *weakly* preregulable if these two latter properties hold. (This is unrelated to "weak regulation" in the sense of Sontag (1981).) The following analytic system is weakly preregulable but not preregulable (see Sontag (1982b) for details). Let the state space of S be the plane, $U = \mathbb{R}$, and equations:

$$\dot{x} = 4x \arctan(xz - 1)(u + 1) + 2x(1 - z^2)(u^2 + 1),$$
$$\dot{z} = -z(u + 1), \tag{3.20}$$

and output $y = z$. It is interesting that this system is pretty well behaved near $(0, 0)$ since its linearization there is asymptotically stable. Study of this example suggests the definitions below.

Call the system S *locally* indy-asycontrollable if there exists a ngbd N of 0 and a function d such that the properties in (2.4) hold for all indy classes rel N. Call S *strongly* locally indy–a.c. if there is some saturated such ngbd. In the "hyperbolic" case when the linearization of S at zero makes sense and is regulable, one concludes only local indy–a.c., not the strong notion.

(3.21) LEMMA. *A weakly regulable and strongly locally indy–a.c. system S is necessarily preregulable.*

Proof. We must prove indy-asycontrollability. First modify d such that for no $e > 0$ is $\#(x) < d(e)$ for an x not in N. Let $a > 0$ be such that $B(a)$ is included in N. Let $k > 0$, and pick any x in $B^*(k)$. By asycontrollability, there are a u and a T such that $\#(\phi(t; x, u)) < a$ for all $t \geqslant T$. Using an argument as in Lemma 3.7, one can conclude that there as a whole ngbd of x that satisfies the same property, for a fixed $T(x)$, and for inputs u bounded by a constant dependent only on x. By compactness, one can find a $b(k)$ and a $T(k)$ such that all states in $B^*(k)$ can be controlled into $B(a)$ in time $T(k)$ using inputs in $B(d(k))$. We may assume that $b(\cdot)$ and $T(\cdot)$ are increasing functions. Let $c(k)$ be a bound on the values $\#(\phi(t; x, u))$, for $\#(u) = b(k)$,

$\#(x) \leqslant k$, and $t \leqslant T(k)$. Consider $T(\cdot, \cdot)$ from the definition of strong local indy-asycontrollability. Define now a new $T'(k, r) := T(k) + T(c(k), r)$.

Let now $L \neq 0$ be an indy class, and assume that L does not intersect N (otherwise, what follows is true by hypothesis). Let k be smallest possible such that L intersects $B^*(k)$. Let x be any state in this intersection. Find u so that $\phi(T(k); x, u)$ is in $B(a)$. Then all of L is mapped into a indy class L' having a representative in $B(a)$. Thus (saturation) L' is included in N. There is then an input u' driving L' to the origin, and $u \,|\, T(k) |\, u'$ drives L to the origin. Condition (i) of Definition 2.24 is irrelevant here because L does not intersect N. Assume however that L intersects some $B(k')$, say at a state z. Necessarily, $k' > k$. So $b(k') > b(k)$, and thus $\#(\phi(T(k); z, u) < c(k')$. It follows that, for the chosen input for L', z is mapped into $B(r)$ in time at most $T(k', r)$. This completes the proof.

Particular cases of interest of the above are that of local indy-a.c. with $[0]$ compact, or the case where there is (exact) local controllability. We use this lemma to give a proof of Theorem 2.31.

Let S be a weakly preregulable state affine system. Consider $[0]$, which is a subset of $[0/0]$. This set is a subspace for which we may decompose the equations for S as

$$\dot{x} = F(u)\, x + G(u), \tag{3.22}$$

$$\dot{z} = A(u)\, z + E(u)\, x + B(u), \tag{3.23}$$

with $y = Hx$, and where $[0]$ is the space of vectors of the form $\{(0, z)\}$. Since $[0/0]$ is asymptotically stable, $[0]$ also is (under input $= 0$). Thus, $A(0)$ is a Hurwitz matrix. By Bellman (1969, Theorem 2.2.2), and by continuity of A on u, there is an $e > 0$ such that $\dot{x}(t) = A(u(t))\, x(t)$ is asystable for any u in $B(e)$. In the definition of asycontrollability, let d be such that, for any (x, z) in $B(d)$, there is a u in $D(e)$ controlling (x, z) asymptotically to zero. Consider now $N :=$ saturated of $B(d) = \text{space}\{(x, z) \,|\, \#(x) < d\}$. Pick any indy class $L(x)$ which contains an $(x, 0)$ in $B(d)$. Let u control $(x, 0)$ as above. In general,

$$z(t) = \mathbf{A}(t)\, z + g(u, x)(t), \tag{3.24}$$

where $\mathbf{A}$ is the fundamental matrix of $\dot{z} = A(u(t))\, z$. For $z = 0$, then $g(u, x)(t)$ converges to zero. Thus, for any other z this is still true, for the same fixed u, because the first term in the right converges by the choice of e. Thus, $L(x)$ is controlled, and the estimates $d(\cdot)$, $T(\cdot, \cdot)$ follow from those for asycontrollability applied to $(x, 0)$ plus the linear theory. Thus, S is strongly locally indy–a.c., and the result follows from the previous lemma.

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