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REAL ADDITION AND THE POLYNOMIAL HIERARCHY

Eduardo D. SONTAG *

Department of Mathematics, Rutgers University, Hill Center, Busch Campus, New Brunswick, NJ 08903, U.S.A.

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The kth alternation level of the theory of real numbers under addition and order is log-complete for the kth level of the polynomial hierarchy.

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1. Introduction

There has been some interest in the literature on decision methods for $\mathrm{Th}(\mathbb{R}, +, <)$, the theory of real numbers with addition (and order). See, for instance, [6], [3], and [2], as well as [4] and [1] in the more general context of reals with multiplication. This theory is of interest also from the point of view of piecewise linear algebra and its potential applications in control system theory [9,10]. If n is the length of a formula in the language L associated to this theory, it is known that the decision problem has deterministic time complexity at least exponential and at most doubly exponential. Here we remark that the kth level of this problem (i.e., deciding the truth of a sentence given in prenex form with $k-1$ quantifier alternations) is complete for Σ_k^p. In particular, the satisfiability problem ($k=1$) is NP-complete, and for each *fixed* level k we have a deterministic *single* exponential time bound.

The proof proceeds roughly as follows. In the next section we establish some facts on polyhedra, in particular, that a nonempty polyhedron defined by equations with 'small' coefficients must contain 'small' points, and, more importantly, that every projection of such a polyhedron can also be defined by equations with small coefficients. This gives rise to a theoretical procedure for quantifier elimination in the last section. The procedure results in exponentially large formulas, but the lengths of the numbers appearing in these formulas remain small, so an alternating algorithm can be deduced which only checks rational points of size polynomial in the size of the input data

2. Some polyhedral geometry

We need a few facts regarding the geometry of polyhedra. We use [8] and [7] as our main references, but much of the terminology is standard. A *flat* in $\mathbb{R}^d$ is an affine submanifold (= translate of a subspace) of $\mathbb{R}^d$. A *polyhedron* $P \subseteq \mathbb{R}^d$ is by definition an intersection of finitely many hyperplanes and closed half spaces,

$$a_1x_1 + \cdots + a_dx_d \leqslant a_0. \tag{2.1}$$

(It is of course redundant to include hyperplanes explicitly.) The *affine hull* of P is aff(P) = smallest flat containing P. The *relative interior* ri(P) is the interior of P relative to aff(P); its *dimension* is the dimension of aff(P). A *line-free* polyhedron is one that does not contain any lines (= one-dimensional flats). If P is the intersection of the hyperplanes

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$H_1, \ldots, H_r$ and the half spaces $K_1, \ldots, K_s$, and if $H_{r+1}, \ldots, H_{r+s}$ are the bounding hyperplanes of the K_i (obtained by replacing $\leqslant$ by $=$ in (2.1)), we call $\Gamma := \{H_1, \ldots, H_{r+s}\}$ a *supporting family* (of hyperplanes) *for* P. Let $\{K_1, \ldots, K_u\}$, $u \leqslant s$, be the set of those half spaces among the above for which P is in fact included in the corresponding H_{r+i}. If we replace K_i by the corresponding H_{r+i}, for $i = 1, \ldots, u$, we obtain a new representation as an intersection, giving rise to the *same supporting family*, but with the property that ri(P) and aff(P) can be represented as follows:

$$\text{aff}(P) = \text{intersection of the } \{H_i, i = 1, \ldots, r+u\}, \tag{2.2}$$

$$\text{ri}(P) = \text{aff}(P) \text{ intersected with the open halfspaces } \text{int}(K_i), i = u+1, \ldots, s. \tag{2.3}$$

A *vertex* (or extreme point) of the polyhedron P is a point not in the relative interior of any segment contained in P; equivalently [7, result 2.6.3], a zero-dimensional face of P. If Γ as above is a supporting family for P and v is a vertex of P, then v is the intersection of those H_i containing it; thus, by a trivial dimensionality argument, v is the intersection of precisely d such hyperplanes. The Minkovski–Weil theorem [8, Theorem 2.12] says that P is a polyhedron iff it is generated by finitely many points and rays. More precisely, there exist two sets of points $V = \{v_1, \ldots, v_r\}$ and $D = \{y_1, \ldots, y_s\}$ such that

$$P = \text{conv}(V; D) := \left\{ \sum \lambda_i v_i + \sum \mu_j y_j, \sum \lambda_i = 1, \text{ all } \lambda_i \geqslant 0, \mu_j \geqslant 0 \right\}. \tag{2.4}$$

We call $V \cup D$ an *extreme family* for P. With such a representation, ri(P) is the set of points as in (2.4) but such that all λ_i and μ_j are strictly positive. Appropriate sets V and D can be obtained as follows. First write P as the union of finitely many line-free polyhedra P_i; this can be done, for instance, through intersection with the various orthants of $\mathbb{R}^d$. Each such $P_i = \text{conv}(V_i; D_i)$, where V_i is the set of vertices of P_i and D_i is a set of representatives for the directions of recession of P_i. A *direction of recession* (or vertex at infinity) is a ray $\mathbb{R}_+ y$, where y is a vector in $\mathbb{R}^d$ such that, for some $v_0 \in P$, the line $v_0 + \mathbb{R}_+ y = \{v_0 + \mu y, \mu \geqslant 0\}$ is a one-dimensional face of P. If y is of this form and Γ_i is a supporting family for P_i, then there are $d-1$ hyperplanes in Γ_i, having equations $\sum_j q_{ij} x_j = b_i$, $i = 1, \ldots, d-1$, which intersect precisely at a line containing $v_0 + \mathbb{R}_+ y$. Thus $\sum_j q_{ij} y_j = 0$ for each i. These $d-1$ equations determine $\mathbb{R}_+ y$ uniquely except for sign. The sets V, D can be obtained now as the union of the V_i and D_i respectively. Note that one can always obtain rational y_i, and that the x_i are necessarily rational, if P admits a supporting family with rational-coefficient equations. Rationality will be implicitly assumed in all that follows.

Denote by $\#(q)$ the *length* of the rational q: if $q = a/b$ with a, b relatively prime, then $\#(q)$ is the sum of the number of bits in the binary representations of a and b (disregarding signs). If $v \in \mathbb{R}^d$, then $\#(v)$ denotes the sum of the lengths of the entries of v. If H is a hyperplane (definable by rational equations), consider the possible equations

$$q_1 x_1 + \cdots + q_d x_d = q_0 \tag{2.5}$$

for H. For any such equation, we sum the lengths of the rationals q_i. We denote the smallest such possible sum, over all representations of H, by $\#(H)$. If Γ is a finite family $\{H_i, i = 1, \ldots, k\}$ of hyperplanes, we let

$$\#(\Gamma) := \max\{\#H_i, i = 1, \ldots, k\}.$$

If $V \cup D$ is an extreme family for P, we write

$$\#(V \cup D) := \max\{\#(v_i), \#(y_j), i = 1, \ldots, r, j = 1, \ldots, s\}.$$

The main point of the following series of lemmas is that these dual kinds of representations (extreme family, supporting family) are equivalent in a polynomial-space sense.

Lemma 2.1. *There is a polynomial* p_1 *such that for every polyhedron* $P \subseteq \mathbb{R}^d$ *and each supporting family* Γ *for* P, *there is an extreme family* $V \cup D$ *for* P *such that*

$$\#(V \cup D) \leqslant p_1(d) \#(\Gamma).$$

Proof. Without loss (see above construction) we may assume that P is line-free; notice that $\#(\Gamma)$ does not increase when adding constraints for intersections with orthants ($x_1 \geqslant 0$, $x_2 \leqslant 0, \ldots$). Let v be any vertex of P. As remarked above, v is the intersection of d elements of Γ, say $H_1, \ldots, H_d$. Let A and b be the following matrix and vector respectively:

$$A := \begin{pmatrix} q_{11} & \cdots & q_{1d} \\ \vdots & & \vdots \\ q_{d1} & \cdots & q_{dd} \end{pmatrix}, \qquad b := \begin{pmatrix} q_{10} \\ \vdots \\ q_{d0} \end{pmatrix} \tag{2.6}$$

obtained from a representation of the hyperplanes in Γ that achieves $\#(\Gamma)$, with H_i having equations

$$q_{i1}x_1 + \cdots + q_{id}x_d = q_{i0}.$$

Thus v is the solution of the system $Ax = b$. In general, let $\#(C)$ be the sum of the $\#(c_{ij})$, for any matrix C; here, the composite matrix (A, b) has $\#(A, b) \leqslant (d^2 + d)\#(\Gamma)$. For any $d \times d$ matrix C, $\#(\det(C))$ is bounded by a polynomial in d and $\#(C)$: this follows by Hadamard's inequality; in fact, the following very nice bound is given in [5]:

$$\#(\det(C)) \leqslant (d^2+2)\#(C) + \tfrac{1}{2}(d \log d). \tag{2.7}$$

By Cramer's rule, the coordinates v_i are quotients of determinants of submatrices of (A, b). So $\#(v) \leqslant 5d^4\#(\Gamma)$. To obtain directions of recession the argument is similar: we must find small vectors y in any one-dimensional intersection L of $d-1$ hyperplanes $\Sigma_j q_{ij}x_j = 0$, $i = 1, \ldots, d-1$ (see discussion preceding the statement of the lemma). But this can be reduced to finding the unique solution of a nonsingular system of d equations, as follows. Introduce A, b as in (2.6) (but with $d-1$ rows). Since rank $A = d-1$, some row vector of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ can be adjoined to A in order to obtain a matrix A′ of full rank. Adjoining the entry $q_{d0} = 0$ to b, there results a nonsingular system $A'x = b'$. Let z be the unique solution of this system. Then, z has a small size (same argument as above), and either $y = z$ or $-z$ has the property that $\mathbb{R}_+ y = L$ and can hence be used to represent the direction of recession L. □

Lemma 2.2. *There is a polynomial* p_2 *such that for every nonempty polyhedron* $P \subseteq \mathbb{R}^d$ *and each extreme family* $V \cup D$ *for* P *there is a point* z *in* ri(P) *such that* $\#(z) \leqslant p_2(d)\#(V \cup D)$.

Proof. Assume that P has dimension $e \leqslant d$. Thus there are $e+1$ affinely independent elements in $V \cup D$, say $v_1, \ldots, v_r$, $y_{r+1}, \ldots, y_{e+1}$. Let σ be the (generalized) simplex

$$\sigma := \operatorname{conv}(\{v_1, \ldots, v_r\}; \{y_{r+1}, \ldots, y_{e+1}\}).$$

Thus $\dim \sigma = e$, so $\operatorname{ri}(\sigma) \subseteq \operatorname{ri}(P)$. Pick now the point

$$z := (1/r)(v_1 + \cdots + v_r) + y_{r+1} + \cdots + y_{e+1}.$$

Then $z \in \operatorname{ri}(\sigma)$, and $\#(z) \leqslant 6d^3\#(V \cup D)$. □

Corollary 2.3. *There is a polynomial* p_3 *such that for every nonempty polyhedron* $P \subseteq \mathbb{R}^d$ *and each supporting family* Γ *for* P *there is a point* z *in* ri(P) *such that*

$$\#(z) \leqslant p_3(d)\#(\Gamma).$$

Proof. The proof clearly follows from the two previous lemmas, taking $p_3 := p_1p_2$. □

Lemma 2.4. *There is a polynomial* p_4 *such that for every polyhedron* $P \subseteq \mathbb{R}^d$ *and each extreme family* $V \cup D$ *for* P *there is a supporting family* Γ *for* P *such that*

$$\#(\Gamma) \leqslant p_4(d)\#(V \cup D).$$

Proof. We claim first that it is sufficient to consider the case in which P contains the origin 0. Indeed, assume that case is established, and pick an arbitrary P. If P is empty, the problem is trivial, so assume P is nonempty. Thus, by Lemma 2.2 there is a z with small size in P. Translating coordinates $x \to x - z$, P is transformed into a polyhedron P_0 containing 0, and $P_0 = \operatorname{conv}(V_0; D)$, with V_0 corresponding to translates by z. So $V_0 \cup D$ increases at most polynomially from $V \cup D$. If the family Γ_0 obtained for P_0 is small, a new translation of coordinates $x \to x + z$ results in a family Γ for P with $\#(\Gamma)$ again small. Thus we may assume that P contains 0.

Let $P^0 \subseteq \mathbb{R}^d$ be the *polar* of P. This is the polyhedron consisting of all the vectors q =

$(q_1, \ldots, q_d)'$ such that

$q'v_i \leqslant 1$ and $q'y_j \leqslant 0$

for all v_i in V and all y_j in D (prime indicates transpose). Thus P^0 admits the supporting family Γ^0 consisting of all hyperplanes $v_i'x = 1$ and $y_j'x = 0$; this family has $\#(\Gamma^0) \leqslant \#(V \cup D) + 1$. By Lemma 2.1 applied to P^0 there is an extreme family $V^0 \cup D^0$ for P^0 with $\#(V^0 \cup D^0)$ small. Since P is a closed convex set and contains the origin, it is reflexive in the sense that $(P^0)^0 = P$ [8]. Again applying the above argument, the extreme family $V^0 \cup D^0$ gives rise to a small length support family $\Gamma = (\Gamma^0)^0$, as desired. □

Corollary 2.5. *There is a polynomial* p_5 *such that for every polyhedron* $P \subseteq \mathbb{R}^d$, *each projection* $\pi(x_1, \ldots, x_d) = (x_1, \ldots, x_e)$, $e < d$, *and each supporting family* Γ *for* P, $P_1 = \pi(P)$ *is a polyhedron admitting a supporting family* Γ_1 *such that*

$$\#(\Gamma_1) \leqslant p_5(d) \#(\Gamma).$$

Proof. By Lemma 2.1 there exists an extreme family $V \cup D$ with $\#(V \cup D) \leqslant p_1(d)\#(\Gamma)$. Let $V_1 = \pi(V)$, $D_1 = \pi(D)$. Thus $P_1 = \text{conv}(V_1; D_1)$ and $\#(V_1 \cup D_1) \leqslant \#(V \cup D)$. By Lemma 2.4 applied to P_1 with extreme family $S_1 \cup D_1$ there is a supporting family Γ_1 as desired, with $p_5 = p_1 p_4$. □

3. Complexity results

We consider prenex-form formulas

$$\Phi = Q_{i_1} x_{i_1} \ldots Q_{i_s} x_{i_s} F(x_1, \ldots, x_d) \tag{3.1}$$

formed by propositional combinations of terms τ of the form

$$q_1 x_1 + \cdots + q_r x_r \; \rho \; q_0, \tag{3.2}$$

where ρ is one of the relational symbols $<$, $\leqslant$, $=$, and $\neq$; the q_i are rational constants, represented as quotients of integers written in binary, and the variables x_i are those in the list $x_1, x_{01}, \ldots$ with binary subscripts. Let L be the set of closed such formulas for which there exist real values for the variables in Φ such that Φ is a true sentence, when the rational and relational symbols, and $+$, are interpreted over the reals.

Let $x_1, \ldots, x_d$ be all the variables appearing in F. Without loss of generality we assume that these are the first d variables in the language. We shall assume that all terms τ in F are as in (3.2) but with $r = d$, adding zero coefficients q_i if necessary; padding equations of F with zeroes at most squares its length.

The k*th level* L_k of L is formed from all formulas Φ in L beginning with an existential quantifier and having $k - 1$ alternations (thus, L_0 corresponds to the quantifier-free formulas); more precisely, a formula Φ of L is in L_k if there is grouping of its variables $x_1, \ldots, x_d$ into subsets

$$z_1 = \{x_1, \ldots, x_{i_1}\},\ z_2 = \{x_{i_1+1}, \ldots, x_{i_2}\},$$
$$\ldots, z_k = \{x_{i_{k-1}-1}, \ldots, x_d\}$$

such that Φ has the form

$$\exists z_1 \forall z_2 \ldots Q_k z_k F, \tag{3.3}$$

where $\exists z_1$ stands for the group of quantifications $\exists x_1 \ldots \exists x_{i_1}$, etc., and Q_k is existential if k is odd and universal if k is even. Let L_k^π denote the corresponding π-hierarchy, obtained from formulas that start with universal quantification; we phrase results in terms of L_k, but the analogous ones apply to L_k^π.

We denote quantification over finite ranges by $\exists^a z$ and $\forall^a z$, meaning that, for each variable in the group $z = \{x_1, \ldots, x_j\}$, only (rational) values with $\#(x_i) < a$ are considered in checking truth over $\mathbb{R}$. Let Σ_k^p be the k*th* level set of the polynomial hierarchy [11]. Thus, for any fixed k, L_k is in Σ_k^p if and only if there exists a polynomial p(n) such that a formula Φ as in (3.3) is in L_k iff

$$\exists^{p(|\Phi|)} z_1 \ldots Q_k^{p(|\Phi|)} z_k F$$

is true. Here, $|\Phi|$ denotes the length of Φ. The main result is as follows.

Theorem 3.1. L_k *is in* Σ_k^p.

Before proving the theorem, we establish a couple of technical results. To each term (3.2) in formula Φ (with $r = d$ = number of variables) we associate the hyperplane H in $\mathbb{R}^d$ obtained when ρ is replaced by equality, if this is a nontrivial hyperplane. Let $\Gamma = \Gamma(\Phi)$ be the set of *hyperplanes of* Φ. We denote $\#(\Phi) \coloneqq \#(\Gamma)$. Of course, $\#(\Phi) < |\Phi|$. If $\Phi = \Phi(z)$ is a formula as above, with free variables $z = \{x_1, \ldots, x_\ell\}$, we let

$$S(\Phi) \coloneqq \{z \in \mathbb{R}^\ell \text{ s.t. } \Phi(z) \text{ is true}\}.$$

We first establish the case $k = 0$ of the theorem, i.e., that L_1 is in NP.

Lemma 3.2. *There exists a polynomial* q_1 *such that, if* F *is quantifier-free, with variables* $z = \{x_1, \ldots, x_d\}$, *then* F *is satisfiable, i.e.,* $\Phi = \exists z F(z)$ *holds, if and only if* $\exists^{q_1(d)\#(F)} z F(z)$ *holds.*

Proof. Let q_1 be the polynomial p_3 obtained in Corollary 2.3. We consider the *extended* form F^e of F obtained by the following procedure. First bring F to disjunctive normal form. Then replace each negation by the corresponding positive term, e.g.,

$$\neg(s < c) \to (-s \leqslant -c),$$
$$\neg(s \neq c) \to (s = c).$$

Next replace the $\leqslant$ and $\neq$ signs:

$$(s \leqslant c) \to (s < c) \vee (s = c),$$
$$(s \neq c) \to (s < c) \vee (-s < -c),$$

and bring the expression again into DNF; this is F^e. Thus, F^e is a disjunction of formulas G of type

$$(\alpha_1 = a_1) \wedge \cdots \wedge (\alpha_s = a_s) \wedge (\beta_1 < b_1) \wedge \cdots \wedge (\beta_t < b_t). \tag{3.4}$$

The above construction is such that $\mathscr{S}(F) = \mathscr{S}(F^e)$. Further, this set is the union of the sets $\mathscr{S}(G)$ corresponding to the formulas G appearing in (3.4) above. If F is satisfiable, then at least one such G is satisfiable. Assume then that $Q \coloneqq \mathscr{S}(G)$ is nonempty, for some G. Consider the polyhedron $P \coloneqq \mathscr{S}(G')$ corresponding to the non-strict formula G':

$$(\alpha_1 = a_1) \wedge \cdots \wedge (\alpha_s = a_s) \wedge (\beta_1 \leqslant b_1) \wedge \cdots \wedge (\beta_t \leqslant b_t). \tag{3.5}$$

Since Q is nonempty, $Q = \mathrm{ri}(P)$. Further, and this is the critical observation, the family Γ consisting of all the hyperplanes $(\alpha_i = a_i)$ and $(\beta_j = b_j)$ is a supporting family for P and

$$\#(\Gamma) \leqslant \#(\Phi). \tag{3.6}$$

This last inequality is due to the fact that all these hyperplanes arise from terms in the original formula F, i.e., $\Gamma \subseteq \Gamma(\Phi)$. By Corollary 2.3, Q—and hence $\mathscr{S}(F)$—contains a point z as desired. □

Lemma 3.3. *There exists a polynomial* q_2 *such that, for any* $\Phi = \exists u F(z, u)$ *with free variables* $z = \{x_1, \ldots, x_e\}$ *and with bound variables* $u = \{x_{e+1}, \ldots, x_d\}$, *there is a quantifier-free formula* $\Psi(z)$ *such that* $\mathscr{S}(\Phi) = \mathscr{S}(\Psi)$ *and* $\#(\Psi) \leqslant q_2(d)\#(\Phi)$.

Proof. Let q_2 be the polynomial p_5 from Corollary 2.5. Consider again the expanded form F^e of F. Note that, with the notations of Corollary 2.5, $\mathscr{S}(\Phi) = \pi(\mathscr{S}(F)) = \pi(\mathscr{S}(F^e))$. Since $\mathscr{S}(F^e)$ is the union of the S(G) in the proof of Lemma 3.2, it is enough to consider the projections of such sets: if quantifier-free formulas Ψ_G are found with $\mathscr{S}(\Psi_G) = \mathscr{S}(G)$ for each of these such that $\#(\Psi_G) < q_2(d)\#(G) \leqslant q_2(d)\#(\Phi)$, the disjunction Ψ of such Ψ_G will be as desired. So consider any G. Without loss of generality, $Q \coloneqq \mathscr{S}(G)$ is nonempty. Again with the notations of the previous proof, consider G' and $P \coloneqq \mathscr{S}(G')$, so that $Q = \mathrm{ri}(P)$. Let P_1 be the projection $\pi(P)$ and $Q_1 \coloneqq \pi(Q)$; thus $Q_1 = \mathrm{ri}(P_1)$. Let $\Gamma = \Gamma(G')$ be the set of hyperplanes of formula G'. By Corollary 2.5, applied to P and Γ, P_1 can be written as an intersection of halfspaces and hyperplanes giving rise to a supporting family Γ_1 with $\#(\Gamma_1) \leqslant q_2(d)\#(\Gamma)$. Thus, $P_1 = \mathscr{S}(\Psi_1)$, where Ψ_1 is some formula as in (3.5) whose hyperplanes are in Γ'. We need then a formula Ψ such that $\mathscr{S}(\Psi) = \mathrm{ri}(P_1)$. By (2.3), Ψ can be obtained from Ψ_1 through changing some of the inequalities $\leqslant$ to equalities and others to strict inequalities. This preserves the hyperplanes, and hence $\#(\Psi) = \#(\Gamma_1)$. □

Proof of Theorem 3.1. We may now prove the theorem, by induction on the number k of alternations. Specifically, we shall prove by induction that all formulas Φ in L_k (and L_k^T) can be decided by checking points whose coordinates have length less than $p_k(d)\#(\Phi)$, where

$$p_k(d) := \left(8q_1(d)q_2(d)^{k-1}\right)^k.$$

Note that the case $k=1$ follows from Lemma 3.2. Since $\#(\Phi) < n := |\Phi|$, it will then follow that $p(n) := np_k(n)$ satisfies the requirements.

Let then Φ be as in (3.3) (the case of π-formulas is analogous). Using Lemma 3.3 repeatedly, we eliminate $k-1$ quantifiers, starting with Q_k. Thus, there exists a quantifier-free formula $\Psi(z_1)$ such that

$$\mathscr{S}(\forall z_2 \ldots Q_k z_k F(z_1, \ldots, z_k)) = \mathscr{S}(\Psi),$$

and $\#(\Psi) \leqslant q_2(d)^{k-1}\#(\Phi)$. It follows from Lemma 3.2 that Φ is true iff $\exists^\lambda z_1 \Psi(z_1)$ holds, where $\lambda = q_1(d)q_2(d)^{k-1}\#(\Phi)$. For any fixed ξ with $\#(\xi) \leqslant \lambda$, consider the formula

$$\Phi_\xi := \forall z_2 \ldots Q_k z_k F(\xi, z_2, \ldots, z_k).$$

Since only a substitution is performed, $\#(\Phi_\xi) \leqslant 8q_1(d)q_2(d)^{k-1}\#(\Phi)$. We conclude that Φ is true if and only if one of the formulas Φ_ξ holds, for some ξ as above. By the induction hypothesis, each such Φ_ξ can be decided by checking at points $(\xi, z_2, \ldots, z_k)$ such that the components of the z_i have lengths less than $p_{k-1}(d-1)\#(\Phi_\xi)$. Thus these lengths are less than $p_k(d)\#(\Phi)$. Since the same bound holds for ξ, the induction step is completed. □

Remark. As a decision method for (all of) L, we obtain an alternating space complexity of the order of

$$2^{cn^2 \log n},$$

which is considerably worse than that in [6], namely 2^{cn}. This strongly suggests that the actual constructions given in this paper could be improved considerably.

Let B_k be, as in [11], the k*th* level quantified Boolean satisfiability problem. Given a formula in B_k, with variables $x_1, \ldots, x_d$, replace each occurrence of x_i by the term $(x_i = 1)$ and each occurrence of its negation $(\neg x_i)$ by the term $(x_i = 0)$, for each $i = 1, \ldots, d$. Adding the conjuncts $(x_i = 0) \vee (x_i = 1)$ for all i, such a formula is log-space reduced to a formula in L_k, preserving truth. From this, the above theorem, and the completeness of $B_k \cap$ 3DNF or $B_k \cap$ 3CNF (see, e.g., [11]) we have the following.

Corollary 3.4. L_k *is log-complete in* Σ_k^p.

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References

[1] M. Ben-Or, D. Kozen and J. Reif, The complexity of elementary algebra and geometry, in: Proc. 16th ACM Symp. on Theory of Computing, 1984.
[2] L. Berman, The complexity of logical theories, Theoret. Comput. Sci. 11 (1980) 71–77.
[3] A.R. Bruss and A.R. Meyer, On time-space classes and their relation to the theory of real addition, Theoret. Comput. Sci. 11 (1980) 59–69.
[4] G. Collins, Quantifier elimination for real-closed fields by cylindrical algebraic decomposition, in: Lecture Notes in Computer Science 35 (Springer, Berlin, 1975) 134–183.
[5] M. Dowd, Gaussian elimination runs in polynomial time, Report LCSR-TM-3, Lab. for Computer Science Research, Rutgers Univ., 1982.
[6] J. Ferrante and C. Rackoff, A decision procedure for the first order theory of real addition with order, SIAM J. Comput. 4 (1975) 69–76.
[7] B. Grumbaum, Convex Polytopes (Wiley, New York, 1967).
[8] V. Klee, Some characterizations of convex polyhedra, Ann. Math. 132 (1959) 79–107.
[9] E.D. Sontag, Remarks on piecewise-linear algebra, Pacific J. Math. 98 (1982) 183–201.
[10] E.D. Sontag, Nonlinear regulation: The piecewise linear approach, IEEE Trans. Autom. Control AC-26 (1981) 346–358.
[11] L.J. Stockmeyer, The polynomial-time hierarchy, Theoret. Comput. Sci. 3 (1977) 1–22.