POLE SHIFTING FOR FAMILIES OF LINEAR SYSTEMS DEPENDING ON AT MOST THREE PARAMETERS*

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ABSTRACT

We prove that for any family of n-dimensional controllable linear systems, continuously parameterized by up to three parameters, and for any continuous selection of n eigenvalues (in complex conjugate pairs), there is some dynamic controller of dimension $3n$ which is itself continuously parameterized and for which the closed-loop eigenvalues are these same eigenvalues, each counted 4 times. An analogous result holds also for smooth parameterizations.

0. Introduction

This paper deals with the control of *parameterized families of linear control systems*. The study of such families is motivated by problems of adaptive control as well as what is commonly referred to as "gain scheduling" for nonlinear systems. In this introduction we shall first provide a quick overview of these motivations, starting with very elementary material. Then we will describe past work, and we shall state our main result, which asserts that for families depending on at most *three* parameters, one can solve pole-assignment problems using controllers whose complexity grows linearly with the dimension of the system. The proof of this result will involve a certain amount of topological and algebraic machinery.

It is worth recalling some of the basic principles of linear control theory. Consider for instance a spring-mass system with no damping,

$$\ddot{y} + ky = u$$

where u is an external forcing term (the "control" or "input") and k is the stiffness (Hooke's) constant. Here y is the displacement from the equilibrium position. One objective (imprecisely stated), is as follows: given any initial $y(0), \dot{y}(0)$, find a control function $u(t), t \geq 0$ such that

$$y(t) \to 0 \;\text{ and }\; \dot{y}(t) \to 0$$

fast and with no oscillations. To achieve this goal, one applies *proportional-derivative* feedback (with $\alpha, \beta > 0$)

$$u(t) = -\alpha y(t) - \beta \dot{y}(t)$$

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corresponding to the intuitive idea that we wish to push in the direction opposite to the displacement from equilibrium (the "α" term), but at the same time "applying the brakes" if moving too fast (the damping term "β"). The *closed loop system* that results from substituting this control law in the dynamical equation is

$$\ddot{y}(t) + \beta\dot{y}(t) + (k + \alpha)y(t) = 0 \ .$$

Note that for each pair of negative real numbers λ and μ, there exist *gains* α, β such that all solutions of the closed loop system have the form

$$ae^{\lambda t} + be^{\mu t}$$

(or $ate^{\lambda t}$ if $\lambda = \mu$), which decay exponentially. Indeed, we may solve for α, β: using the characteristic equation

$$z^2 + \beta z + (\alpha + k) = (z - \lambda)(z - \mu)$$

one gets

$$\alpha = \lambda\mu - k, \ \beta = -\lambda - \mu$$

as the needed gains. It follows that any desired decay rate can be achieved. In fact, any roots λ and μ can be obtained for the characteristic equation, subject only to the requirement that if either is not real then the other must be its conjugate (so that α and β are real).

Assume now that the stiffness coefficient k is a parameter which has not been measured at the time when we want to design the control law. Of course, the control law will depend on k. But we can certainly precompute the *form* of u using the formulas $\alpha = \lambda\mu - k, \beta = -\lambda - \mu$. (Assuming that λ, μ have been decided upon, based on the desired performance characteristics for the closed loop system.) Note that the unknown parameter k appears linearly, in particular polynomially, in the form of u. One could think of building a controller device with a dial marked "k" which, when set to the appropriate value of this parameter, will simply evaluate the two linear functions given above and use this as a control law. Together with the choice of an estimation procedure for k, this gives rise to an "indirect adaptive control" algorithm.

In that context, it is of interest to know in general when the construction of a controller can be carried out continuously, or smoothly, or even algebraically in a suitable sense, just like it is possible with this example. This gives the rise to the study of *control of parameterized families of systems*; reference [So1] gives an introduction to the topic. More generally, this is a subarea of the theory of *systems over rings*. See for instance the text [BBV] for systems over rings, and [Os] for more on adaptive control and families of systems.

As we shall discuss below, it is in general impossible to carry out these constructions even continuously, and one needs to employ instead *dynamic*, –also called *integral*– feedback. In this type of controller, one does not just feed back into the state a linear combination $u(t) = -\alpha y(t) - \beta\dot{y}(t)$ of state variables, but in addition one uses integrated versions of y. In order to make this more precise, we switch first to the *state space* formulation of control problems. For the above example, this is the formulation as a system of two first order equations: Let $q \in \mathbb{R}^2$ be the vector with components $q_1 := y$ and $q_2 = \dot{y}$, so the equation can be written as

$$\dot{q} = Aq + Bu, \quad A = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In general a *linear system* is a pair (A, B) of two real matrices, where A is $n \times n$ and B is $n \times m$; n is called the *dimension* of the system, and m is the *number of independent controls* (or "inputs"). For instance, a model of a robotic arm with $n = 2m$ links can for small displacements be modeled by such a system, if there is an actuator (e.g. an electric motor) at each joint.

If in the above example we let

$$F := (\alpha, \beta)$$

then the closed loop behavior is defined by the equation (substitute $u := -Fq$)

$$\dot{q} = (A + BF)q.$$

Thus the main problem becomes that of modifying the spectrum of $A+BF$, for fixed (A, B) and varying F. The *Pole Shifting Theorem* says that for any *controllable* linear system (A, B), and for each monic real polynomial p of degree n, there is some $F \in \mathbb{R}^{n\times n}$ so that the characteristic polynomial of $A+BF$ is the desired p. Controllability, or "reachability," is a generic condition on systems, the condition that

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$$

and it corresponds to the property that one can steer any state to any other state by applying suitable controls.

We now may define a *continuous* (resp., *smooth,*) *family of systems parameterized by* X as a pair (A, B) of matrices (A is $n \times n$, B is $n \times m$), the entries of which are $C^k, k = 0$ (respectively, $k = \infty$) functions $X \to \mathbb{R}$. In the continuous case we assume that X is a topological space, in the second that it is a smooth (paracompact) manifold. Equivalently of course, we may define a family as a pair $A \in R^{n\times n}$ and $B \in R^{n\times m}$, where $R = C^k(X, \mathbb{R})$, seen as a ring with pointwise operations. The integer n is again called the dimension of the family.

The main question is now: *if a pair is pointwise controllable* (i.e., it is controllable for each parameter value), *and if a set of eigenvalues* $\lambda_1(x), \ldots \lambda_n(x)$ *is desired for each* $x \in X$, *does there exist an* $F \in R^{m\times n}$ *such that the eigenvalues of* $A(x) + B(x)F(x)$ *are precisely these?* For the problem to make sense, we must assume of course that the desired eigenvalues appear in complex conjugate pairs, and that they depend continuously (or smoothly) on x. We introduce then the following definition, for either $R = C^0(X, \mathbb{R})$ or $R = C^\infty(X, \mathbb{R})$: the (parameterized) polynomial $p \in R[\lambda]$ is a (continuously or smoothly, respectively) *splitting polynomial* if there exist functions

$$\lambda_i : X \to \mathbb{C}$$

(continuous or smooth, respectively,) such that

$$p(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$$

(equality holds at each point in X). For instance, every constant real polynomial is smoothly splitting. Note that the pole-shifting theorem guarantees the existence of a family F_x so that $A(x) + B(x)F_x$ has the desired eigenvalues, but this family of feedback laws is not necessarily continuous.

For the spring-mass system in the introduction, seen as smoothly parameterized by k, any polynomial is achievable. This is also true in general provided that the number of controls m be 1, but it is false otherwise, as illustrated next. The example to follow serves also to introduce another motivation for the study of control for families of systems, that of continuous gain scheduling (see e.g. [BR] for more on related topics).

Consider the two-dimensional nonlinear system with $m = 2$ controls,

$$\begin{aligned} \dot{y} &= z + 2vw + w \\ \dot{z} &= -y + w^2 + v - v^2 \,, \end{aligned}$$

where we are denoting $q = (y, z)$ and $u = (v, w)$ for the states and inputs respectively. For each fixed value $\overline{u} = (\overline{v}, \overline{w})$ of these controls, there is a corresponding equilibrium state $\overline{q} = (\overline{y}, \overline{z})$, namely $\overline{y} = \overline{w}^2 + \overline{v} - \overline{v}^2, \overline{z} = -2\overline{vw} - \overline{w}$. Suppose that we desire a *continuous gain schedule* $K(v, w)$, that is a continuous 2×2 matrix of functions such that, for each fixed $(\overline{v}, \overline{w})$, the linear control law

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} + K(\overline{v}, \overline{w}) \begin{pmatrix} y - \overline{y} \\ z - \overline{z} \end{pmatrix}$$

solves a desired control objective up to first order, around the equilibrium point $(\overline{y}, \overline{z})$.

For instance, assume that we wish to place the spectrum of the linearization of the system at the locations $-1, -2$. Taking Jacobians, and dropping bars for notational simplicity, what we want is a continuous matrix F such that $A + BF$ has its poles at $-1, -2$ for all parameter values, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2w & 1 + 2v \\ 1 - 2v & 2w \end{pmatrix}.$$

It is not hard to see that these systems are completely controllable, for every value of the parameter. Thus we are in the situation of the problem stated earlier.

We claim that *it is impossible to obtain a continuous F assigning the eigenvalues $-1, -2$.* This is because in that case the continuous matrix function

$$D := A + BF + I$$

has constant rank 1 (its kernel has constant dimension 1, being the eigenspace corresponding to the eigenvalue -1). Thus its kernel defines in a natural way a line bundle over $\mathbb{R}^2$, which must be trivial together with a complement. In matrix-theoretic terms, what this means is that there must exist continuous and everywhere invertible matrices $P(v,w)$ and $Q(v,w)$ such that

$$PDQ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

from which it follows that

$$BFQe_2 = -(A+I)Qe_2 \ .$$

Since $A+I$ is invertible, this means that Bu, where $u = FQe_2$, is a continuous everywhere nonzero linear combination of the columns of B. Restricting to the circle of radius $1/2$, and writing $2v = \cos\theta, 2w = \sin\theta$, the columns of B generate the "Möbius band" bundle. A linear nonzero combination would give rise to a nontrivial section, a contradiction. More explicitly, we need to show that there cannot be any continuous 2π-periodic function $u(\theta) = (u_1, u_2)'$ such that $B(\theta)u(\theta)$ is always nonzero. Since

$$B(\theta) = 2\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}$$

it follows that

$$a(\theta) = u_1(\theta)\cos\frac{\theta}{2} + u_2(\theta)\sin\frac{\theta}{2}$$

is always nonzero, which gives the desired contradiction since

$$a(2\pi) = -u_1(2\pi) = -u_1(0) = -a(0)$$

and a is continuous.

Note that if instead a complex-valued u is allowed, it *is* possible to obtain a constant rank product Bu; indeed, the (constant) choice

$$u = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

gives that $B(v,w)u$ is a nonzero (complex) vector for all real parameters v and w. This gives rise, via standard constructions in systems over rings, to a *complex-valued* parameterized feedback $F(v,w)$ such that the eigenvalues of the complex matrix $A+BF$ are the desired $-1, -2$. As observed in the last section of [HS1], such a feedback can be interpreted as dynamic feedback over the reals. Specifically, if we write

$$F(v,w) = F_1(v,w) + iF_2(v,w)$$

then the equation $\dot{q} = (A+BF)q$ can be decomposed into its real and complex parts:

$$\dot{q}_1 = (A+BF_1)q_1 - BF_2q_2$$

and

$$\dot{q}_2 = (A+BF_1)q_2 + BF_2q_1 \ .$$

This can be understood as the interconnection of the original system (the first equation) and a controller as follows. The quantity $F_1q_1 - F_2q_2$ is fed as an input to the original system and the variable q_2 is the state of a dynamic controller which evolves according to a differential equation driven by the observations of the state of the original system. The eigenvalues of the composite system (of dimension 4) are again $-1, -2$, but each is now counted twice.

From an engineering design point of view, dynamic controllers are natural to implement. Thus one may ask about the possibility of continuously (or smoothly) parameterized dynamic feedback controllers. Algebraically, the question becomes that of whether there exists an extension of the original system, of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}$$

where I is an identity matrix of some size k (the *dimension of the controller*) and such that for this extension every polynomial, or at least a sufficiently large class of polynomials, is achievable. We call this the *k-extension* of the original family. Purely mathematically, considering k-extensions is analogous to looking for "stable" versions of problems in the sense of K-theory.

There is then a general result, due to P. Khargonekar, that asserts that such controllers can always be built so that eigenvalues of the composite system are basically arbitrary (see [BBV] and [So1]). Unfortunately, the dimension of the necessary controller (the integer k) must be taken for this general result to be n^2 for a system of dimension n. This motivated the problem, stated in [So2], of trying to obtain dynamic controllers whose dimension grows only *linearly* with the dimension of the system to be controlled. In this paper, we provide a partial answer to this question. Our main theorem is as follows. For background on CW-complexes, see for instance [Ma, §7.3]; in particular, the theorem applies for any open subset of $\mathbb{R}^3$.

Main Theorem. *Assume that (A, B) is a continuously (respectively, smoothly) parameterized pointwise controllable family over X, and that p is any continuously (respectively, smoothly) splitting polynomial (of the same degree as the dimension of the family). Assume that X is a CW-complex (respectively, smooth manifold) of dimension 3. Then, the polynomial p^4 is achievable for the $3n$-extension of (A, B).*

The idea of the proof is as follows. We first view the family (A, B) as a family with *quaternionic* values, and then we prove a theorem that says that every polynomial is achievable for families over the quaternions. A quaternionic controller is then interpreted as a dynamic controller over the reals, just as done above over the complexes.

The proof of the theorem on feedback over quaternions is based on a generalization to noncommutative rings of a construction due (for principal-ideal domains) to Eising ([Ei]) and summarized and generalized in [BBV] and [So2]. It relies in turn on a lemma that says that if B is a matrix that is everywhere nonzero then there must exist a matrix L with the property that BL has constant rank one. This lemma can be interpreted also in terms of singular distributions, and is mentioned in [Fl], page 77 as the critical step in this line of work. It was known to be true for real matrices parameterized by one-dimensional X ([HS2]), and for complex matrices and $\dim X = 2$ [We]; here we extend it to quaternionic matrices and 3-dimensional X. This extension relies very heavily on the ideas of [We], and to a lesser extent, [HS2]. Of course, there is a technical problem in even defining "characteristic polynomial" (and so achievable polynomial) over the quaternions, so we must use instead a notion (arbitrary triangularizability) introduced in [So2].

We conjecture that the result remains true for the case when X is four-dimensional, but as of yet have been unable to provide a proof in that case.

1. Quaternions

In this section we shall recall basic facts about, and set terminology, regarding quaternions, matrices over quaternions and families of such matrices.

We denote by $\mathbb{H}$ the ring of quaternions. This is the set of all expressions

$$x = a + bi + cj + dk \quad a, b, c, d \in \mathbb{R}$$

seen as a division ring (noncommutative field) under the product induced by

$$ij = -ji = k, \;\; jk = -kj = i, \;\; ki = -ik = j.$$

The set

$$\mathbb{H}^m = \left\{ \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} : \quad p_i \in \mathbb{H}, \quad i = 1, 2, \ldots, m \right\}$$

is a vector space over $\mathbb{H}$ with the usual "+" and $\mathbb{H}$ *acting on the right.* Most of the essential properties of vector spaces over fields hold for $\mathbb{H}^m$; see for instance [Mc] §5.31.

For each $x = a + bi + cj + dk \in \mathbb{H}$, one defines the *conjugate* of x,

$$\bar{x} = a - bi - cj - dk$$

and its *real part*

$$\mathrm{Re}\, x = a.$$

Note that if $x, y \in \mathbb{H}$, then $\overline{xy} = \bar{y}\bar{x}$ and $\bar{x}x = x\bar{x} = a^2 + b^2 + c^2 + d^2 = \|x\|^2$. For each pair

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

of elements of $\mathbb{H}^m$, we define

$$< u, \; v >= \sum_{i=1}^{m} \bar{u}_i v_i$$

and

$$\|u\| =< u, \; u >^{1/2} .$$

Notice that then $< u, \; v >= \overline{< v, \; u >}$ and for any $p, q \in \mathbb{H}$ we have

$$< up, \; v >= \bar{p} < u, \; v >, \; < u, \; vq >=< u, \; v > q.$$

We now define two natural identifications $\phi, \psi : \mathbb{H}^n \to \mathbb{R}^{4n}$. Write any $v \in \mathbb{H}^n$ as follows:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1^1 + v_1^2 i + v_1^3 j + v_1^4 k \\ v_2^1 + v_2^2 i + v_2^3 j + v_2^4 k \\ \vdots \\ v_n^1 + v_n^2 i + v_n^3 j + v_n^4 k \end{pmatrix} = v^1 + v^2 i + v^3 j + v^4 k.$$

Then we let

$$\phi v = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \\ v_1^4 \\ v_2^1 \\ \vdots \\ v_n^4 \end{pmatrix}, \quad \psi v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix}.$$

Any $\mathbb{H}$-linear mapping $\alpha : \mathbb{H}^n \to \mathbb{H}^n$ corresponds to *left multiplication*

$$\alpha v = Tv$$

by a (uniquely determined) $T \in \mathbb{H}^{n \times n}$. To any such α we associate the transformations of $\mathbb{R}^{4n}$ induced by the identifications ϕ and ψ:

$$\alpha_R = \phi \alpha \phi^{-1}, \quad \alpha^R = \psi \alpha \psi^{-1}.$$

Suppose that $T = (t_{ij})$ with $t_{ij} = t_{ij}^1 + t_{ij}^2 i + t_{ij}^3 j + t_{ij}^4 k$. Let

$$T_1 = (t_{ij}^1), \quad T_2 = (t_{ij}^2), \quad T_3 = (t_{ij}^3), \quad T_4 = (t_{ij}^4).$$

Then

$$\alpha^R \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix} = T^R \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix},$$

where

$$T^R = \begin{pmatrix} T_1 & -T_2 & -T_3 & -T_4 \\ T_2 & T_1 & -T_4 & T_3 \\ T_3 & T_4 & T_1 & -T_2 \\ T_4 & -T_3 & T_2 & T_1 \end{pmatrix}. \tag{1.1}$$

Also,

$$\alpha_R \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \\ v_1^4 \\ v_2^1 \\ \vdots \\ v_n^4 \end{pmatrix} = T_R \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \\ v_1^4 \\ v_2^1 \\ \vdots \\ v_n^4 \end{pmatrix},$$

where T_R is the block matrix $T_R = ((t_{ij})_R)$ whose (i,j)-th block is

$$(t_{ij})_R = \begin{pmatrix} t_{ij}^1 & -t_{ij}^2 & -t_{ij}^3 & -t_{ij}^4 \\ t_{ij}^2 & t_{ij}^1 & -t_{ij}^4 & t_{ij}^3 \\ t_{ij}^3 & t_{ij}^4 & t_{ij}^1 & -t_{ij}^2 \\ t_{ij}^4 & -t_{ij}^3 & t_{ij}^2 & t_{ij}^1 \end{pmatrix}. \tag{1.2}$$

Since $\alpha_R = \phi\psi^{-1}\alpha^R\psi\phi^{-1}$, T_R is similar to T^R as a linear transformation over $\mathbb{R}^{4n}$.

Let $\alpha_1,\ \alpha_2$ be two linear mappings from $\mathbb{H}^n$ to $\mathbb{H}^n$. Then $(\alpha_1\alpha_2)_R = \phi\alpha_1\alpha_2\phi^{-1} = (\phi\alpha_1\phi^{-1})(\phi\alpha_2\phi^{-1}) = (\alpha_1)_R(\alpha_2)_R$. So if $S,\, T \in \mathbb{H}^{n\times n}$, then $(ST)_R = S_RT_R$. Similarly, $(ST)^R = S^RT^R$. Since $I_R = I^R = I$, this implies that T_R and T^R are invertible if T is invertible, and $(T^{-1})_R = (T_R)^{-1},\ (T^{-1})^R = (T^R)^{-1}$.

Now take $T \in \mathbb{H}^{m\times n}$. The column rank of T, denoted by $\operatorname{rank}_c T$, is the dimension of the subspace of $\mathbb{H}^m$ spanned by the columns of T. Let

$$\mathbb{H}_n = \{(\, p_1 \quad p_2 \quad \cdots \quad p_n\,) :\ p_i \in \mathbb{H},\ i = 1,2,\ldots,n\}.$$

Then $\mathbb{H}_n$ is a vector space over $\mathbb{H}$ with the usual "+" and $\mathbb{H}$ *acting on the left*. The row rank of a matrix T over $\mathbb{H}$, denoted by $\operatorname{rank}_r T$, is the dimension of the subspace of $\mathbb{H}_n$ spanned by the rows of T (with $\mathbb{H}$ *acting on the left*). By Proposition 10 in [Bu] II-§10.12,

$$\operatorname{rank}_c T = \operatorname{rank}_r T\ .$$

So we can use $\operatorname{rank} T$ to denote either $\operatorname{rank}_c T$ or $\operatorname{rank}_r T$.

See Proposition 13 in [Bu] II-§10.13 for a proof of the following result:

Lemma 1.1: Let $A \in \mathbb{H}^{m\times n}$ with $\operatorname{rank} A = r$. Then there exists invertible matrices S and T over $\mathbb{H}$ such that:

$$SAT = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.3}$$

In particular, if $m = n$, then A has rank n if and only if it is invertible. ♢

Equation (1.3) can be written as:

$$S_RA_RT_R = \begin{pmatrix} (I_{r\times r})_R & 0 \\ 0 & 0 \end{pmatrix}.$$

From here we can see that $\operatorname{rank} A_R = 4 \operatorname{rank} A$ and similarly, $\operatorname{rank} A^R = 4 \operatorname{rank} A$. Therefore we have the following results:

Corollary 1.2: If A is a matrix over $\mathbb{H}$, then the following statements are equivalent:

(1) A is invertible,

(2) A_R is invertible,

(3) A^R is invertible. ♢

Corollary 1.3: If A, B and C are matrices over $\mathbb{H}$ and $C = AB$, then $\operatorname{rank} C \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$. $\diamondsuit$

Corollary 1.4: If $A \in \mathbb{H}^{m\times n}$ with $n \leq m$, then $\operatorname{rank} A = n$ if and only if the equation $Ax = 0$ has the unique solution $x = 0$. $\diamondsuit$

For $A = (a_{ij}) \in \mathbb{H}^{n\times m}$, define $A^* = (\bar{a}_{ji})$. From $\bar{x}\bar{y} = \overline{yx}$ for $x, y \in \mathbb{H}$, it follows that $S^*T^* = (TS)^*$ for $S \in \mathbb{H}^{n\times m}, T \in \mathbb{H}^{m\times n}$, and $(A^*)_R = (A_R)^*, (A^*)^R = (A^R)^*$ for any matrix A over $\mathbb{H}$.

We now turn to parameterized matrices. Let X be a topological space. We think of $\mathbb{H}$ as a normed space with $\|a\| = (a\bar{a})^{1/2}$ for $a \in \mathbb{H}$, and we let $R = C(X, \mathbb{H})$ be the ring of continuous functions from X to $\mathbb{H}$. For each matrix M over R and each $x \in X$ we let $M(x)$ be the matrix obtained by evaluating M at x. Then

$$\operatorname{rank} M(x) = \frac{\operatorname{rank}(M(x))_R}{4}.$$

Thus $\operatorname{rank} M(x)$ is a lower semi-continuous function of x. M is said to have *constant rank* if $\operatorname{rank} M(x)$ is constant as a function of $x \in X$.

Take $M \in R^{n\times n}$. Suppose that there is some $x_0 \in X$ such that $M(x_0)$ has full rank. Then there exists a neighborhood U of x_0 such that $M(x)$ is full rank for $x \in U$. Thus $M(x)^{-1}$ exists for $x \in U$. Since $(M(\cdot))_R^{-1}$ is continuous, so is $M(\cdot)^{-1}$. Hence if M has full rank constantly, then M has an inverse $M^{-1} \in R^{n\times n}$.

2. Basic Modules

Let $M \in R^{n\times m}$ where $R = C(X, \mathbb{H})$ and X is a topological space. We say that M is *basic* if $M(x) \neq 0$ for each $x \in X$. In this section we will prove that if $M \in R^{n\times m}$ is basic and if X is a CW-complex of dimension at most 3, then there exists a vector $l \in R^m$ such that Ml is everywhere nonzero.

If $f : X \to \mathbb{H}^m$, we denote by $\|f\|$ its sup norm, $\|f\| = \sup_{x\in X} \|f(x)\| \leq \infty$.

Lemma 2.1: Let

$$X = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\} \subset \mathbb{R}^2.$$

Let $a, b \in C(X, \mathbb{H}^m)$ be such that $a(x) \neq 0$ and $b(x) \neq 0$, for all $x \in X$. Then for every $\varepsilon > 0$, there exists an $r \in C(X, \mathbb{H})$ with $\|r\| < \varepsilon$ such that $< b,\ a + br >$ is never zero on X.

Proof: Note that $< b,\ a + br >=< b,\ a > +\|b\|^2 r$. Since $b\,(x, y) \neq 0$ for $(x, y) \in X$, $\|b\|^2$ is bounded away from 0, i.e, there exists $\sigma > 0$ such that $\|b(x)\|^2 \geq \sigma$, for all $x \in X$. Thus it will be enough to show that:

For any $\varepsilon > 0$, there exists an $r \in R$ with $\|r\| < \varepsilon$ such that $< b, a > +r \neq 0$.

Let $f =< b, a >= f_1 + f_2 i + f_3 j + f_4 k$. If we write $r = r_1 + r_2 i + r_3 j + r_4 k$, then $f\,(x, y) + r\,(x, y) \neq 0$ is equivalent to:

$$\begin{pmatrix} f_1\,(x,y) \\ f_2\,(x,y) \\ f_3\,(x,y) \\ f_4\,(x,y) \end{pmatrix} + \begin{pmatrix} r_1\,(x,y) \\ r_2\,(x,y) \\ r_3\,(x,y) \\ r_4\,(x,y) \end{pmatrix} \neq 0.$$

Now fix $\bar{x} \in [0, 1]$. Let

$$g(y) = \begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix} = \begin{pmatrix} f_1(\bar{x}, y) \\ f_2(\bar{x}, y) \end{pmatrix}.$$

For each given $\varepsilon > 0$, let $\delta = \varepsilon/4$ and consider the open set $\Omega = \{y : \|g(y)\| < \delta\}$. We write this as a disjoint union of countably many intervals $\Omega = \cup_i(\alpha_i, \beta_i)$. Now define

$$s(y) = \begin{pmatrix} s_1(y) \\ s_2(y) \end{pmatrix} = \frac{\delta}{\|g(y)\|} \begin{pmatrix} -g_2(y) \\ g_1(y) \end{pmatrix}$$

for $y \notin \Omega$. Then $s(y) + g(y) \neq 0$ and $\|s(y)\| = \delta \leq \varepsilon/4$ for $y \notin \Omega$.

In $[\alpha_i, \beta_i]$, let $s(y)$ be such that

$$s(\alpha_i) = \begin{pmatrix} -g_2(\alpha_i) \\ g_1(\alpha_i) \end{pmatrix},\ s(\beta_i) = \begin{pmatrix} -g_2(\beta_i) \\ g_1(\beta_i) \end{pmatrix}$$

and its coordinates s_1 and s_2 do not vanish at the same time for any $y \in [\alpha_i, \beta_i]$. Thus $\|s(y)\| > 0$ on Ω. Replace $s(y)$ by $\delta s(y)/\|s(y)\|$ in Ω. (Such an "interpolating" s can be found because any two nonzero vectors in $\mathbb{R}^2$ can be joined by a curve which does not cross the origin. This construction would not be possible if one would have considered instead scalar-valued functions.) Then also $\|s(y)\| = \delta$ for $y \in \Omega$ and $s(y) + g(y) \neq 0$ there, since $\|g(y)\| < \|s(y)\| = \delta$ for $y \in \Omega$.

Let $r_j(\bar{x}, y) = s_j(y)$ for $j = 1, 2$. Then

$$\| \begin{pmatrix} r_1(\bar{x}, y) \\ r_2(\bar{x}, y) \end{pmatrix} \| \leq \varepsilon/4$$

and

$$\begin{pmatrix} f_1(\bar{x}, y) \\ f_2(\bar{x}, y) \end{pmatrix} + \begin{pmatrix} r_1(\bar{x}, y) \\ r_2(\bar{x}, y) \end{pmatrix} \neq 0$$

for $y \in [0, 1]$. By the continuity of $f_j(x, y)$ and $r_j(\bar{x}, y)\,(j = 1, 2)$, there exists $\delta_{\bar{x}} > 0$ such that

$$\begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} + \begin{pmatrix} r_1(\bar{x}, y) \\ r_2(\bar{x}, y) \end{pmatrix} \neq 0,$$

whenever $|x - \bar{x}| < \delta_{\bar{x}}$ and $y \in [0, 1]$.

By compactness of $[0, 1]$, there exist $x_0 < x_1 < \cdots < x_k$ and $\delta_0, \delta_1, \ldots, \delta_k > 0$ such that

$$[0, 1] \subseteq \bigcup_{i=0}^{k} (x_i - \delta_i,\, x_i + \delta_i)$$

and functions $r_j(x_i,\ \cdot)$ so that

$$\begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} + \begin{pmatrix} r_1(x_i, y) \\ r_2(x_i, y) \end{pmatrix} \neq 0$$

for $x \in (x_i - \delta_i, x_i + \delta_i),\ y \in [0, 1]$. We may also assume that the intervals do not overlap, in the sense that

$$x_i < x_{i+1} - \delta_{i+1} \ \text{ and } \ x_i + \delta_i < x_{i+1}$$

for each $i = 0, \ldots, k - 1$. Now take for each $i = 0, \ldots, k - 1$ any fixed

$$\bar{x}_i \in (x_i, x_i + \delta_i) \cap (x_{i+1} - \delta_{i+1}, x_{i+1}).$$

Arguing as above, there exist $r_3(\bar{x}_i, y), r_4(\bar{x}_i, y)$ with $\| \begin{pmatrix} r_3 \\ r_4 \end{pmatrix} \| < \varepsilon/4$ such that

$$\begin{pmatrix} f_3(\bar{x}_i, y) \\ f_4(\bar{x}_i, y) \end{pmatrix} + \begin{pmatrix} r_3(\bar{x}_i, y) \\ r_4(\bar{x}_i, y) \end{pmatrix} \neq 0$$

for $y \in [0, 1]$. Again, there exists also some $\tau_i > 0$, $i = 0, \ldots, k - 1$ such that

$$\begin{pmatrix} f_3(x, y) \\ f_4(x, y) \end{pmatrix} + \begin{pmatrix} r_3(\bar{x}_i, y) \\ r_4(\bar{x}_i, y) \end{pmatrix} \neq 0$$

for $|x - \bar{x}_i| < \tau_i$ and $y \in [0,1]$.

Pick for each i some τ_i' with $0 < \tau_i' < \tau_i''$, $\tau_i'' = \min\{\tau_i, \bar{x}_i - x_i, x_{i+1} - \bar{x}_i\}$ and let $t_i(x)$ be a continuous function with values in $[0,\ 1]$ such that

$$t_i(x) = 1 \quad \text{for } x \leq \bar{x}_i - \tau_i' \quad \text{and} \quad t_i(x) = 0 \quad \text{for } x \geq \bar{x}_i + \tau_i'.$$

Now we extend r_1 and r_2 to all of $[0,\ 1] \times [0,\ 1]$ using on each square $[x_i,\ x_{i+1}] \times [0,\ 1]$ the formula

$$r_j(x,y) = r_j(x_i,y)t_i(x) + r_j(x_{i+1},y)(1 - t_i)(x)$$

for $j = 1,2$.

Finally, let $q(x)$ have value in $[0,\ 1]$ and be such that

$$q(x) = 1 \quad \text{for } x \in [\bar{x}_i - \tau_i', \bar{x}_i + \tau_i']$$

for each i and $q(x) = 0$ outside each $[\bar{x}_i - \tau_i'',\ \bar{x}_i + \tau_i'']$, and define $r_j,\ j = 3,4$ so that

$$r_j(x,y) = r_j(\bar{x}_i,y)q(x)$$

for $(x,\ y) \in [\bar{x}_i - \tau_i'',\ \bar{x}_i + \tau_i''] \times [0,\ 1]$. With these choices $\|r\| < \varepsilon$ and $< b,\ a > +r \neq 0$ for all $(x,\ y) \in X$, as desired. ■

Remark: If the X in lemma 2.1 is changed to $\{(x,y) : 1 \leq (x^2 + y^2)^{1/2} \leq 2\} \subset \mathbb{R}^2$, then the conclusion of lemma 2.1 is still true. This is proved by noticing that the above proof works equally well on $\mathbf{S}^1 \times [0,\ 1]$, provided that one picks $x_k = x_1$ in the above proof. ♢

We let $[Y, S^3]$ denote the set of all homotopy classes of mappings from the topological space Y into the 3-sphere S^3.

Lemma 2.2: Let Y be any topological space so that $[Y, S^3]$ is trivial. Then, any $g \in C(Y,\ \mathbb{H})$ which satisfies $g(y) \neq 0$ for all y is homotopic to the constant mapping $g(y) = 1$ using a nonvanishing homotopy.

Proof: This is an immediate consequence of the fact that S^3 is a strong deformation retract of $\mathbb{H}$. ■

Lemma 2.3: Let

$$X = \{(x,y,z): \quad -2 \leq x \leq 2,\ \ 0 \leq y \leq 1,\ \ 0 \leq z \leq 1\}$$

and $R = C(X, \mathbb{H})$. Let $a, b \in R^m$ such that $a(x,y,z) \neq 0$ for $-2 \leq x \leq 0$ and $b(x,y,z) \neq 0$ for $0 \leq x \leq 2$. Then there exist $f,\ g \in R$ such that

(1) $f(x,y,z) = 1,\ g(x,y,z) = 0$ for $-2 \leq x \leq -1$;

(2) $g(x,y,z) = 1,\ f(x,y,z) = 0$ for $1 \leq x \leq 2$;

(3) $c = af + bg$ never vanishes on X.

Proof: Let

$$X_0 = \{(x,y,z) \in X :\ x = 0\} \subset X.$$

Using lemma 2.1 on X_0 find a small function $r(y,z)$ such that

$$< b,\ a + br >\neq 0$$

on X_0 and $\|r\| < \alpha/\beta$, where $\alpha = \inf\{a(x,y,z) :\ -2 \leq x \leq 0\}$ and $\beta = \sup\{b(x,y,z) : 0 \leq x \leq 2\}$. Let ϕ be a [0, 1]-valued continuous function which is 1 on X_0, and zero for $|x| \geq 1$. Take

$$a_1 = a + br\phi.$$

Then $a_1(x,y,z) \neq 0$ for $-2 \leq x \leq 0$ and $< b,\ a_1 >\neq 0$ on X_0.

We now use lemma 2.2 to obtain a $\lambda :\ X \to \mathbb{H} - \{0\}$ which is continuous and satisfies

$$\lambda(0,y,z) = (< b,\ a_1 >)^{-1}(0,y,z) \quad \text{and} \quad \lambda(x,y,z) = 1, \quad \text{if } |x| \geq 1\ .$$

(The lemma applies with Y being a unit square, which is contractible.) Let $a_2 = a_1\lambda$. Then $< b,\ a_2 >= 1$ on X_0.

Pick $0 < \varepsilon < 1$ such that

$$\text{Re } < b,\ a_2 > > 0, \quad \text{for } |x| < \varepsilon.$$

Pick a [0,1]-valued continuous function $t(x)$ such that

$$t(x) = 1, \quad \text{for } x < -\varepsilon,$$

$$t(x) = 0, \quad \text{for } x > \varepsilon.$$

Let

$$c = a_2 t + b(1-t) = a\lambda t + b(1 - t + r\phi\lambda t).$$

Then

$$c = a_2 \neq 0, \quad \text{for } x \leq -\varepsilon,$$

$$c = b \neq 0, \quad \text{for } x \geq \varepsilon,$$

$$< c,\ c >= t^2\|a_2\|^2 + (1-t)^2\|b\|^2 + 2t(1-t)\text{Re } < a_2,\ b >\neq 0$$

for $|x| \leq \varepsilon$. Thus, c never vanishes on X. ■

Lemma 2.4: Let X be the unit cube in $\mathbb{R}^3$, $R = C(X, \mathbb{H})$. If $B \in R^{n\times m}$ is a basic matrix over R, then there exists an $l \in R^m$ such that Bl never vanishes on X.

Proof: First subdivide X into small cubes so that there exists b_i, some column of B, which never vanishes on each small cube. Now we prove the lemma by using the induction on the number n of these small cubes.

It is trivial when $n = 1$. Suppose the lemma is true for $n \leq l$. Now assume $n = l + 1$. Let the endpoints of the small cubes be $(x_i,\ y_j,\ z_k)$ with $x_1 \leq x_2 \leq \ldots \leq x_{n_1}$, $y_1 \leq y_2 \leq \ldots \leq y_{n_2}$, $z_1 \leq z_2 \leq \ldots \leq z_{n_3}$. Without loss of generality, assume $n_3 > 1$. Let

$$e(i,j,k) = \{(x,\ y,\ z) : x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j,\ z_{k-1} \leq z \leq z_k\}.$$

By inductive assumption, we know there exists some $l_1 \in R^m$ such that Bl_1 never vanishes on e_1 where

$$e_1 = \{e(i,j,k) : 0 \le i \le n_1,\ 0 \le j \le n_2,\ 0 \le k \le n_3 - 1\}.$$

Also, there exists some $l_2 \in R^m$ such that Bl_2 never vanishes on e_2 where

$$e_2 = \{e(i,j,k) : 0 \le i \le n_1,\ 0 \le j \le n_2,\ k = n_3\}.$$

(Here, we notice that both e_1 and e_2 are homeomorphic to the unit cube). Applying lemma 2.3 to e_1 and e_2 with Bl_1 and Bl_2 as a and b there, we get f and g such that $Bl_1f + Bl_2g$ never vanishes on $e_1 \cup e_2$. Then $l_1f + l_2g$ is the desired l. ■

Lemma 2.5: Let $R = C(X, \mathbb{H})$ where X is a CW-complex and assume *either* that X has dimension ≤ 3 *or* that X is contractible. Let M be a matrix over R with constant rank k. Then there exist invertible matrices P, Q over R such that

$$PMQ = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where I_k is the identity matrix of size k.

Proof: Let $M \in R^{n\times m}$ have $\operatorname{rank} M(x) = k$ for all $x \in X$. We must use here the language of vector bundles. Consider the trivial n-dimensional quaternionic bundle F_n over X, that is, the trivial bundle with symplectic structure group. The span of the columns of M induces a k-dimensional subbundle E of F_n. When X is contractible ([Hu], Chapter 3, Corollary 4.8) or if $\dim X \le 3$ ([Hu], Chapter 8, Theorem 1.2), E must be trivial. (The idea of the proof of the latter fact is as follows: inductively on the dimension of E, it is enough to prove that E contains a trivial line subbundle. But as a *real* bundle over X, the dimension of E is at least $4 > \dim X$, so a real trivial line subbundle exists. Its closure under the action of the quaternions on each fiber is then the desired quaternionic subbundle.) In matrix-theoretic terms, this means that there is some (continuous) matrix $L_1 \in R^{m\times k}$ so that

$$V_1 = ML_1$$

has constant rank k. Let E' be a complement of E in F_n, $E \oplus E' = F_n$. By the same argument as above, E' is trivial, so there exists a matrix $V_2 \in R^{n\times(n-k)}$ so that

$$(V_1\ V_2) \in R^{n\times n}$$

is invertible. Let P be the inverse of this matrix, so that

$$PML_1 = \begin{pmatrix} I_k \\ 0 \end{pmatrix} \quad .$$

The columns of L_1 give rise themselves to a k-dimensional subbundle E_1 of the trivial m-dimensional bundle F_m. Let E_2 be its complement; as before it must be trivial and we conclude that there is some matrix $L_2 \in R^{m\times(m-k)}$ so that the composite matrix

$$Q_1 = (L_1\ L_2) \in R^{m\times m}$$

is invertible. Write

$$PMQ_1 = \begin{pmatrix} I_k & C \\ 0 & D \end{pmatrix} .$$

Since rank $M = k$ everywhere, it follows that D is identically zero. With

$$Q = Q_1 \begin{pmatrix} I_k & -C \\ 0 & I_{m-k} \end{pmatrix}$$

the conclusion follows. ■

Remark. If M is smoothly parameterized, then P and Q can be picked smooth. This follows from the above argument, since the critical step is the existence for a matrix M of constant rank k of a smooth matrix L with k columns so that the rank of

$$V = ML$$

is identically k. But if a continuous L is first obtained, it is only necessary to approximate the entries of L by smooth functions. The argument is given in more detail in the proof of theorem 2.9. ♢

Lemma 2.6: Let X be a 2-dimensional CW-complex. $R = C(X,\ \mathbb{H})$. If $B \in R^{m\times n}$ is a basic matrix, then there exists some $l \in R^m$ such that Bl has rank 1 constantly on X.

Proof: Proposition 1 in [We], which generalizes lemma 21 in [HS2], can be applied to show that there exists an $u \in R^m$, in fact complex-valued, such that Bu never vanishes on the 1-skeleton. Indeed, if

$$B = B_0 + B_1 i + B_2 j + B_3 k,$$

then B can be written as:

$$B = B^1 + kB^2$$

where $B^1 = B_0 + B_1 i,\ \ B^2 = B_4 + B_3 i$. Thus

$$B \neq 0$$

is equivalent to

$$\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \neq 0.$$

So one can apply the result in that reference to this composite complex matrix, to obtain an $u \in (C(X,\ \mathbb{C}))^m$ such that

$$\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} u$$

never vanishes on the 1-skeleton. This u is as needed. We let $b = Bu$.

We shall construct for each 2-cell e an $u_e \in R^m$ whose support is included in e, so that $b + Bu_e$ is always nonzero on e. Then $v = \sum_e u_e$ is well-defined and $l = u + v$ will be as desired. So fix a 2-cell e. Notice that b does not vanish in a neighborhood U of the boundary of e. Inside e, find a closed set V which is homeomorphic to a square and such that $V \cup U = e$. By proposition 2 in [We], there exists some $v \in R^m$ such that $a = Bv$ never vanishes on V. Parameterize e as

$$\{x : \|x\| \leq 1 + r_0\}$$

such that both a and b have no zeros on $\{x : \|x\| \leq 1\}$ and $\{x : \|x\| \geq 1\}$ respectively.

Let $\hat{e} = \{x : \|x\| \leq 1\}$. By lemma 2.1, we can replace a by $a + br$ for some r and reparameterize $\hat{e}$ as the unit disc again such that $< b,\ a >$ is never zero on

$$\{x : 2/3 \leq \|x\| \leq 1\}.$$

Next we wish to apply lemma 2.2 to conclude the existence of a nonvanishing function $s \in R$ such that

$$s = 1 \ \text{ on } \{x : \|x\| \leq 2/3\}$$

and

$$s =< b,\ a >^{-1} \quad \text{on } \{x : 3/4 \leq \|x\| \leq 4/5\}.$$

For this, it is only necessary up to homeomorphism to patch the nonzero quaternionic-valued function $< b,\ a >^{-1}$ on a unit circle S^1 with the constant function 1 on the same circle. But this is guaranteed by the lemma; it is well-known that $[S^1, S^3]$ vanishes (S^3 is simply connected); see for instance [Ma], example 6.3.16. Now replace a by as. Then

$$< b,\ a >= 1 \ \text{ on } \{x : 3/4 \leq \|x\| \leq 4/5\}.$$

Let t be a [0, 1]-valued continuous function such that

$$t = 1 \ \text{ on } \ \{x : \|x\| \leq 3/4\}$$

and

$$t = 0 \ \text{ on } \ \{x : \|x\| \geq 4/5\}.$$

Now let

$$u_e = (v - u)t, \quad \text{and} \quad c_e = Bu_e = (a - b)t,$$

both extended as zero outside e. Then the supports of u_e and c_e lie inside e and

$$b + c_e = a \neq 0 \quad \text{on } \ \{x : \|x\| \leq 3/4\},$$

$$b + c_e = b \neq 0 \quad \text{on } \ \{x : \|x\| \geq 4/5\}$$

and

$$\|b + c_e\|^2 =< b + c_e,\, b + c_e >= t^2\|a\|^2 + (1 - t)^2\|b\|^2 + 2t(1 - t) \neq 0$$

on $\{x : 3/4 \leq \|x\| \leq 4/5\}$. Therefore, $b + c_e$ never vanishes on e, as desired. ∎

We now arrive to the main result of this section:

Theorem 2.7: Let X be a 3-dimensional CW-complex, $R = C(X, \mathbb{H})$. For any basic $B \in R^{n\times m}$, there exists some $l \in R^m$ such that Bl is everywhere nonzero.

Proof: Let Y be the 2-skeleton of X. Using lemma 2.6, first find some $l_1 \in (C(Y, \mathbb{H}))^m$ such that Bl never vanishes on Y. Extend l_1 continuous inside each cell; then by proposition 7.3.4. in [Ma], l_1 can be extended to all of X; we again by use l_1 to denote this extension. Let $b = Bl_1$.

Let U be a neighborhood of Y on which b never vanishes. Let U_1 be another neighborhood of Y such that $U_1 \subset \operatorname{clos} U_1 \subset U$. Let $f \in C(X, \mathbb{R})$ such that $U^c \subset \operatorname{supp} f \subset U_1^c$, where $U^c = \{x \in X : x \notin U\}$. Let

$$V = \{x : f(x) = 0\}.$$

Then V is a closed set and $Y \subset V \subset U$. Let

$$B_1 = (\, b \quad fB\,) = B\,(\, l_1 \quad fI_{m\times m}\,)\,.$$

Then B_1 is basic and B_1 has rank exactly 1 on V.

Inside each 3-cell e, find a closed subset s_e homeomorphic to a cube so that $e \subseteq V \cup s_e$ and $s_e \cap Y = 0$. Using lemma 2.4, find $a_e = B_1 l_e$ which never vanishes on s_e. Let $\chi_e \in C(X,\ \mathbb{R})$ be such that $s_e \subset \operatorname{supp} \chi_e \subset e$, and define

$$a = \sum_e a_e \chi_e = B_1 \sum_e l_e \chi_e.$$

Then a never vanishes in a neighborhood E of the closed subset $S (= \cup_e s_e)$, and $a = Bl_2$ for some $l_2 \in R^m$. Let $g \in C(X,\ \mathbb{R})$ be such that $g(x) = 1$ on E^c and $\operatorname{supp} g \subset S^c$. Let

$$B_2 = (\, a \quad gB_1\,) = B_1\,(\, l_2 \quad gI\,)\,.$$

Then B_2 is basic over X and,

$$\operatorname{rank} B_2(x) \le \begin{cases} \operatorname{rank} a(x) = 1, & \text{if } x \in S; \\ \operatorname{rank} B_1(x) = 1, & \text{if } x \in V. \end{cases}$$

Therefore, B_2 has constant rank 1 and $B_2 = BL$ for some matrix L over R. Lemma 2.5 implies that then $B_2 l$ is identically nonzero, for some l, and so $B(Ll)$ also is. ∎

The above result applies in particular to the case in which X has in addition a smooth manifold structure. But in that case, l can be chosen smooth too, as we now prove.

Theorem 2.8: Let X be a 3-dimensional smooth manifold, $R_\infty = C^\infty(X, \mathbb{H})$. If $B \in R_\infty^{m\times n}$ is basic, then there exists some $l \in R_\infty^m$ such that Bl is everywhere nonzero.

Proof: This is a consequence of theorem 2.7. First, find $l_1 \in R^m$ as there such that Bl_1 never vanishes. Let

$$\varepsilon(x) = \frac{\|Bl_1(x)\|}{2\|B(x)\|} \text{ for } x \in X.$$

Note that this is continuous. Now find $l(x) \in R_\infty^m$ such that

$$\|l(x) - l_1(x)\| < \varepsilon(x)$$

(cf. Theorem 4.8 in [Bo]). Thus $Bl(x) \neq 0$ for all $x \in X$, as wanted. ∎

3. Systems

In this Section, we study families of systems over the quaternions,and we establish a pole-shifting theorem for them.

Let $R = C(X,\ \mathbb{H})$ where X is a CW-complex of dimension at most 3.

Definition 3.1: A (n-dimensional) *system* Σ over R is a pair $(A,\ B)$ where $A \in R^{n\times n}$ and $B \in R^{n\times m}$.

Remark: A more general definition of system is often desirable. This more general case would correspond to state-spaces which are projective modules (rather than free modules) and $A,\ B$ would be linear maps among such modules; see e.g. [HS2], [So2]. Fortunately for our purposes, as indicated in lemma 2.5, when $\dim X \leq 3$, every n-bundle over X is trivial, which means that such projective modules are necessarily free, and the simple definition given above will suffice. $\diamondsuit$

Let $\Sigma_1 = (A_1,\ B_1)$ and $\Sigma_2 = (A_2,\ B_2)$ be two systems. We say that Σ_1 *is equivalent to* Σ_2, denoted as $\Sigma_1 \sim \Sigma_2$, if there exists an invertible matrix $T \in R^{n\times n}$ such that $B_2 = TB_1$ and

$$A_1 - T^{-1}A_2T = B_1L$$

for some $L \in R^{m\times n}$.

It is easy to prove that this is an equivalence relation; it is sometimes called *feedback equivalence*.

A system $(A,\ B)$ is *controllable* if for each $x \in X$, there is some integer $l = l(x) > 0$ such that

$$\text{rank}\,(\,B \quad AB \quad A^2B \quad \cdots \quad A^lB\,)\,(x) = n.$$

If $\Sigma_1 \sim \Sigma_2$, then Σ_1 is controllable if and only if Σ_2 is controllable.

Definition 3.2: The n-dimensional system $(A,\ B)$ is *(arbitrarily) triangularizable* if for each $a_1, a_2, \ldots, a_n \in R$, there exists a system $(F,\ G)$ such that $(A,\ B) \sim (F,\ G)$ and

$$F = \begin{pmatrix} a_1 & * & \ldots & * \\ 0 & a_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix}.$$

Lemma 3.3: Suppose T, Q, X, Y are matrices over R. Suppose that there exist matrices L_1, L_2, L_3 such that $TX = YL_1$, $Q = YL_2$, $Y = TXL_3$. Then there exists a matrix S such that (a) $S = XL$ for some matrix L; (b) $TS = Q$.

Proof: Let $S = XL_3L_2$. Then (a) holds. For part (b), we have $TS = TXL_3L_2 = YL_2 = Q$. ∎

Lemma 3.4: Let $(A,\ B)$ be a system. Assume that we have a partition

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $A_{11} \in R^{n_1 \times n_1}$, $A_{22} \in R^{n_2 \times n_2}$ and

$$\begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} = BL \quad \text{for some} \quad L.$$

Let $D \in R^{n_1 \times n_1}$, $H \in R^{n_2 \times n_2}$ be given. Suppose there exists some matrix $E \in R^{n \times n}$ satisfying:

(1) $E = BL_1$ for some L_1 and $E^2 = 0$;

(2) there exists some K such that

$$(A + AE) \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} - \begin{pmatrix} D \\ 0 \end{pmatrix} = BK. \tag{3.1}$$

Then there exist some matrices G and C such that

$$F = \begin{pmatrix} D & C \\ 0 & H \end{pmatrix}$$

and $(A,\ B) \sim (F,\ G)$.

Proof: Let $J = I + E$. Then $J(I - E) = (I - E)J = I$. Write

$$AJ = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$$

and let

$$F = \begin{pmatrix} D & G_2 \\ 0 & H \end{pmatrix}.$$

Claim: $(A,\ B) \sim (F,\ JB)$.

First notice that

$$A - JFJ^{-1} = BL_2 \ \text{ for some } L_2 \tag{3.2}$$

is equivalent to:

$$(AJ - JF) = BL_3 \ \text{ for some } L_3. \tag{3.3}$$

Since $J = I + E$, then $(JF - F) = EF = BL_1F$. Thus (3.2) is equivalent to

$$(AJ - F) = BL_4 \ \text{ for some } L_4. \tag{3.4}$$

But equation (3.1) implies that

$$(AJ - F) \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} = BK. \tag{3.5}$$

while

$$(AJ - F) \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} = AJ \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} - \begin{pmatrix} G_2 \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ G_4 - H \end{pmatrix} = BL(G_4 - H).$$

Thus, (3.4) holds, as desired. ■

Lemma 3.5: Assume that the n-dimensional system $(A,\ B)$ over R decomposes as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where $A_{11} \in R^{n_1 \times n_1}$, $A_{22} \in R^{n_2 \times n_2}$ and $n_1 + n_2 = n$. Let $C_1 = \begin{pmatrix} A_{12} & B_1 \end{pmatrix}$. Assume further that

$$\begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} = BL, \quad \text{for some } L. \tag{3.6}$$

Then $(A,\ B)$ is triangularizable if (A_{11}, C_1) is.

Proof: Take any $a_1, a_2, \dots, a_n \in R$. By assumption there exist a matrix

$$D = \begin{pmatrix} a_1 & * & \dots & * \\ 0 & a_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n_1} \end{pmatrix},$$

and an invertible matrix T_1 such that

$$(A_{11} - T_1^{-1} D T_1) = C_1 X_1, \quad \text{for some} \quad X_1.$$

Let

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & I_{n_2} \end{pmatrix}.$$

Then $(A,\ B) \sim (TAT^{-1},\ TB)$.

Let

$$H = \begin{pmatrix} a_{n_1+1} & 0 & \dots & 0 \\ 0 & a_{n_1+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}.$$

Then if we can find E satisfying:

(a) $E = TBL_1$ for some L_1,

(b) $E^2 N = 0$ and

(c) $(TAT^{-1} + TAT^{-1}E) \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} - \begin{pmatrix} D \\ 0 \end{pmatrix} = TBL_2$ for some L_2,

then according to lemma 3.4, $(TAT^{-1},\ TB)$ would be triangularizable and therefore also $(A,\ B)$ would be.

By assumption,

$$(T_1 A_{11} T_1^{-1} - D) = T_1 C_1 X_1$$

for some X_1 and

$$T_1 C_1 = T_1 \begin{pmatrix} A_{12} & B_1 \end{pmatrix},$$

while

$$\begin{aligned} \begin{pmatrix} A_{12} & B_1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} A_{12} & B_1 \\ A_{22} & B_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ A_{22} & B_2 \end{pmatrix} \\ &= \begin{pmatrix} A \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & B \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ A_{22} & B_2 \end{pmatrix}. \end{aligned}$$

Equation (3.6) implies that

$$\begin{pmatrix} 0 & 0 \\ A_{22} & B_2 \end{pmatrix} = B \begin{pmatrix} L_3 & L_4 \end{pmatrix}$$

for some L_3 and L_4. Thus,

$$\begin{pmatrix} A_{12} & B_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & B \end{pmatrix} L_5$$

for some L_5. Therefore we have

$$\begin{aligned} \begin{pmatrix} T_1 C_1 \\ 0 \end{pmatrix} = \begin{pmatrix} T_1 A_{12} & T_1 B_1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} TA \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & TB \end{pmatrix} K \\ &= \begin{pmatrix} \hat{A} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & \hat{B} \end{pmatrix} K, \end{aligned}$$

for some K, where $\hat{A} = TAT^{-1}, \quad \hat{B} = TB$. Thus,

$$\begin{aligned} \begin{pmatrix} T_1 A_{11} T_1^{-1} - D \\ 0 \end{pmatrix} = \begin{pmatrix} T_1 C_1 X_1 \\ 0 \end{pmatrix} &= \begin{pmatrix} T_1 C_1 \\ 0 \end{pmatrix} X_1 \\ &= \begin{pmatrix} \hat{A} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & \hat{B} \end{pmatrix} K X_1. \end{aligned}$$

Notice that

$$\begin{pmatrix} -\hat{A} & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & 0 \\ 0 & \hat{B} \end{pmatrix} = \begin{pmatrix} \hat{A} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & \hat{B} \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

and

$$\begin{pmatrix} \hat{A} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & \hat{B} \end{pmatrix} = \begin{pmatrix} -\hat{A} & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & 0 \\ 0 & \hat{B} \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

By lemma 3.3 there exists $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ satisfying:

(a) $\begin{pmatrix} T_1 A_{11} T_1^{-1} - D \\ 0 \end{pmatrix} = (\, -\hat{A} \quad I \,) Y$;

(b) $Y = \begin{pmatrix} \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} & 0 \\ 0 & \hat{B} \end{pmatrix} Z$ for some Z.

Property (a) implies that

$$\begin{pmatrix} T_1 A_{11} T_1^{-1} - D \\ 0 \end{pmatrix} = (-\hat{A} Y_1 + Y_2)$$

while (b) implies

$$Y_1 = \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} Z_1 \quad \text{and} \quad Y_2 = \hat{B} Z_2$$

for some Z_1 and Z_2. Let $E = (\, Y_1 \quad 0 \,)$. Then

$$E = (\, Y_1 \quad 0 \,) = \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} (\, Z_1 \quad 0 \,) = T \begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix} (\, Z_1 \quad 0 \,) = TB \,(\, LZ_1 \quad 0 \,)$$

and it can be seen that $E^2 = 0$. Furthermore,

$$\begin{aligned}(\hat{A} + \hat{A}E) \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} - \begin{pmatrix} D \\ 0 \end{pmatrix} &= \begin{pmatrix} T_1 A_{11} T_1^{-1} - D \\ 0 \end{pmatrix} + \hat{A} Y_1 + \begin{pmatrix} 0 \\ A_{21} T_1^{-1} \end{pmatrix} \\ &= Y_2 + \begin{pmatrix} 0 \\ A_{21} T_1^{-1} \end{pmatrix}.\end{aligned}$$

Notice that

$$\begin{pmatrix} 0 \\ A_{21} T_1^{-1} \end{pmatrix} = T \begin{pmatrix} 0 \\ A_{21} T_1^{-1} \end{pmatrix} = TBL_6$$

for some L_6. Therefore we know that

$$(\hat{A} + \hat{A}E) \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} - \begin{pmatrix} D \\ 0 \end{pmatrix} = TBL_2 \quad \text{for some } \; L_2.$$

■

Lemma 3.6: The system $(A_{11},\ C_1)$ in lemma 3.5 is controllable if the system $(A,\ B)$ is.

Proof: Let D_k be the matrix formed by taking the first n_1 rows of $A^k B$, and E_k be formed from the last n_2 rows of $A^k B$.

Claim:

$$D_k = A_{11}^k B_1 + \sum_{i=0}^{k-1} A_{11}^i A_{12} L_i$$

for some L_i's.

Clearly, $D_0 = B_1$. Suppose

$$D_k = A_{11}B_1 + \sum_{i=0}^{k-1} A_{11}^i A_{12} L_i,$$

then

$$\begin{aligned} D_{k+1} &= A_{11}D_k + A_{12}E_k \\ &= A_{11}^{k+1}B_1 + \sum_{i=0}^{k-1} A_{11}^{i+1}A_{12}L_i + A_{12}E_k \\ &= A_{11}^{k+1}B_1 + \sum_{i=0}^{k} A_{11}^{i}A_{12}L_i, \end{aligned}$$

where $L_0 = E_k$. So

$$\mathrm{rank}\,(\,D_0 \quad D_1 \quad D_2 \quad \ldots \quad D_k\,)\,(x) \le \mathrm{rank}\,(\,C_1 \quad A_{11}C_1 \quad A_{11}^2C_1 \quad \ldots \quad A_{11}^kC_1\,)\,(x)$$

for each $x \in X$. But $(A,\ B)$ is controllable, so for each $x \in X$ there exists an integer n_x such that

$$\mathrm{rank}\,(\,D_0 \quad D_1 \quad D_2 \quad \ldots \quad D_{n_x}\,)\,(x) = n_1.$$

Therefore

$$\mathrm{rank}\,(\,C_1 \quad A_{11}C_1 \quad A_{11}^2C_1 \quad \ldots \quad A_{11}^{n_x}C_1\,)\,(x) = n_1.$$

So the system $(A_{11},\ C_1)$ is controllable. ■

Theorem 3.7: Let $R = C(X, \mathbb{H})$ where X is a 3-dimensional CW-complex. Every controllable system $(A,\ B)$ over R is arbitrarily triangularizable.

Proof: Use induction on the dimension n of the system. It is trivial when $n = 1$ because in this case both A and B are in fact scalars and B can never be zero.

Assume that the conclusion is true if the dimension of the system is less then or equal to n. Take a controllable system $(A,\ B)$ with the dimension being $n+1$. By theorem 2.7, there exists a vector l over R such that Bl never vanishes. Applying lemma 2.5, we can find a invertible matrix P over R and a nowhere vanishing scalar c such that

$$PBlc = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Note that $(A,\ B) \sim (\hat{A},\ \hat{B})$ where $\hat{A} = PAP^{-1},\ \hat{B} = PB$. So if $(\hat{A},\ \hat{B})$ is triangularizable, so is $(A,\ B)$.

Write

$$\hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{a}_{nn} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \end{pmatrix},$$

where $\hat{A}_{11} \in R^{n\times n}$, $\hat{B}_2 \in R^{1\times m}$. Let $C_1 = (\,\hat{A}_{12} \quad \hat{B}_1\,)$. By lemma 3.6, $(\hat{A}_{11},\ C_1)$ is controllable and therefore, by the assumption on the induction, triangularizable. Applying lemma 3.5 to the system $(\hat{A},\ \hat{B})$ with $n_2 = 1$, we get the conclusion that $(\hat{A},\ \hat{B})$ is triangularizable which implies the system $(A,\ B)$ also is. ■

Remark 3.8: If the system Σ is smooth, meaning that X is also a smooth manifold and the functions defining A, B are C^∞, then for all a_i's smooth, the system is arbitrarily triangularizable as a smooth system (all matrices appearing are smooth). This is clear from theorem 2.8 and the proof of theorem 3.10. ♢

4. Pole-Shifting

This section contains the proof of the main result. It is based on the idea of seeing a family of systems (over the reals) as a quaternionic family, then applying the results from the previous section, and finally viewing a quaternionic feedback as a particular case of dynamic feedback over the reals.

Now let $R = C(X, \mathbf{R})$ where X is a 3-dimensional CW-complex and let $(A,\ B)$ be a system over R (that is, a family of linear systems parameterized by X).

Proof of the Main Theorem: We view $(A,\ B)$ as a system over $R_{\mathbb{H}} = C(X,\ \mathbb{H})$. Since (A, B) is controllable over R,

$$\operatorname{rank}_{\mathbb{R}} \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}(x) = n$$

for each $x \in X$. Thus

$$\operatorname{rank}_{\mathbb{H}} \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}(x) = n$$

for each $x \in X$ since all the entries of $A(x), B(x)$ are in $\mathbb{R}$. So $(A,\ B)$ is controllable as a system over $R_{\mathbb{H}}$. By theorem 3.7, there exists an invertible matrix T such that

$$(A - T^{-1}FT) = BK,$$

for some K, where

$$F = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Suppose $K = K_1 + K_2 i + K_3 j + K_4 k$. Take

$$L = \begin{pmatrix} K_1 & -K_2 & -K_3 & -K_4 \\ K_2 & K_1 & -K_4 & K_3 \\ K_3 & K_4 & K_1 & -K_2 \\ K_4 & -K_3 & K_2 & K_1 \end{pmatrix} = K^R.$$

Let

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix} - \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{pmatrix} L.$$

Then

$$D = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & I_{3n} \end{pmatrix} H$$

equals $(A - BK)^R$. Since $A - BK = T^{-1}FT$, we have

$$D = (T^R)^{-1} F^R T^R.$$

Since F^R and F_R are similar, $(A-BK)^R \sim F_R$. Notice that

$$(F)_R = \begin{pmatrix} (\lambda_1)_R & * & \cdots & * \\ 0 & (\lambda_2)_R & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_n)_R \end{pmatrix}.$$

For each fixed x, if $\lambda_k = \lambda_k(x) \in \mathbb{R}$, then $(\lambda_k)_R = \lambda_k I_{4\times 4}$. If instead $\lambda_k(x) = c + di \in \mathbb{C}$, then $\bar{\lambda}_k$ is also a root of $p(\lambda)$ since $p(\lambda)$ is a polynomial with real coefficients, and

$$(\lambda_k)_R = \begin{pmatrix} c & -d & 0 & 0 \\ d & c & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}, \quad (\bar{\lambda}_k)_R = \begin{pmatrix} c & d & 0 & 0 \\ -d & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & -d & c \end{pmatrix}.$$

Thus the characteristic polynomial of $D(x)$ is $p(\lambda)^4$ for all $x \in X$. ■

Remark 4.2: Replacing theorem 3.7 by remark 3.8 in the above proof, we get: If X is a smooth manifold and the entries of A and B, as well as the functions λ_i's are smooth, the H can be chosen smooth. This completes the proof of the main theorem. ♢

5. Conclusions.

We have shown that, for 3-dimensional parameter spaces, a slightly weakened version of the pole shifting problem can be solved with integral feedback using a number of integrators that grows linearly with the dimension of the systems in the family. We conjecture that the same result must be true for 4-dimensional parameter spaces. In that case, however, there are a number of technical problems that must be solved. First and most important, one needs a generalization of Theorem 2.7. A straighforward generalization will be impossible, however, since line bundles are not free in this case, so instead one will need the existence of line bundles included in the distribution spanned by the columns of B. Second, the definition of systems, and the inductive proof of the pole-shifting theorem, will have to be modified to account for the nontriviality of line bundles. This is because in the induction one "peels off" a line bundle at a time, and there is no guarantee that the remaining part be trivial even if the original bundle was. So one needs to define systems in a more general sense, corresponding in the terminology of systems over rings to systems with projective but possibly nonfree state spaces. Finally, possible generalizations to more than 4 parameters will need a totally new idea, since there are no division rings of dimension more than 4 extending the reals.

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