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State observability in recurrent neural networks*

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Abstract: We obtain a characterization of observability for a class of nonlinear systems which appear in neural networks research.

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1. Introduction

Systems consisting of a large number of interconnected 'neurons' evolving according to difference (in discrete time) or differential (in continuous time) equations have attracted considerable attention lately; see for instance the material on 'recurrent nets' in [4]. The basic models considered are those in which the dynamics take one of the following two forms, in discrete or continuous time (to simplify notations, we drop time arguments t, and use superscripts ' + ' and '$\cdot$' to indicate time-shift and time-derivative, respectively):

$$x^+ (\text{or } \dot{x}) = \boldsymbol{\sigma}(Ax + Bu), \qquad y = Cx, \tag{1}$$

where A, B, and C are, respectively, real matrices of sizes $n \times n, n \times m$, and $p \times n$, and $\boldsymbol{\sigma}$ indicates the application of a function

$$\sigma : \mathbb{R} \to \mathbb{R}$$

to each coordinate of an n-vector, $\boldsymbol{\sigma}(x_1, \ldots, x_n) = (\sigma(x_1), \ldots, \sigma(x_n))$. See Figure 1 for a block diagram, where Δ indicates either a unit delay or integration. The complete model is specified once that σ and the triple (A, B, C) are given. (In continuous time, one needs to assume also that σ is at least locally Lipschitz, so that existence and local uniqueness of the differential equation holds.)

Many questions, mirroring those for linear systems (for which σ is the identity) can be posed. In the recent work [1], we explored realization questions, and in particular the fact that all the entries of the matrices A, B, and C can be recovered (up to a small number of symmetries) from the zero-initial state input/output behavior, assuming suitable minimality assumptions, and as long as σ is nonlinear enough. This is somewhat surprising, since for the linear case one can only recover the parameters up to basis changes, and it is reminiscent of old work of Rugh and coworkers, as well as Boyd and Chua (see for instance [5, 3]) on uniqueness of interconnections containing nonlinearities. (Reference [2] explains the relation between those more classical facts and the result in [1].)

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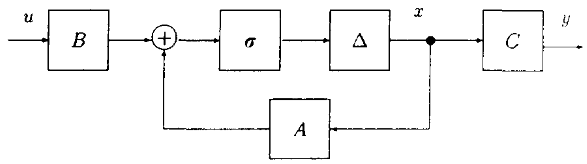

Fig. 1. Block diagram of recurrent net.

In this paper, we look at questions of *observability*, i.e. state distinguishability for a known system, as opposed to determination of the systems parameters with a known initial state. Our main result is that observability can be characterized, if one assumes certain conditions on the nonlinearity and on the system, in a manner very analogous to that of the linear case. Recall that for the latter, observability is equivalent to the requirement that there not be any nontrivial A-invariant subspace included in the kernel of C. We show that the result generalizes in a natural manner, except that one now needs to restrict attention to certain special 'coordinate' spaces.

The paper is organized as follows. We first state precise definitions and results, then prove the results, and in the last sections we compare with linear systems and give some further remarks.

2. Statement of main result

The function σ will be assumed to satisfy the following *independence property* ('IP' from now on): Given any positive integer l, any nonzero real numbers $b_1, \ldots, b_l$, and any real numbers $\beta_1, \ldots, \beta_l$ such that

$$(b_i, \beta_i) \neq \pm (b_j, \beta_j) \quad \forall i \neq j,$$

the functions

$$1, \sigma(b_1 u + \beta_1), \ldots, \sigma(b_l u + \beta_l)$$

are linearly independent, i.e.,

$$c_0 + \sum_{i=1}^{l} c_i \sigma(b_i u + \beta_i) = 0 \quad \forall u \in \mathbb{R} \;\Rightarrow\; c_0 = c_1 = \cdots = c_l = 0.$$

The following result [2] provides sufficient conditions for a given function σ to satisfy property IP; these conditions are weak enough to allow inclusion of most examples of interest in neural networks.

Fact 1. *Assume that σ is a real-analytic function, and it extends to an analytic function $\sigma: \mathbb{C} \to \mathbb{C}$ defined on a subset $D \subseteq \mathbb{C}$ of the form*

$$D = \{z \mid |\operatorname{Im} z| \leq \lambda\} \setminus \{z_0, \bar{z}_0\}$$

for some $\lambda > 0$, where $\operatorname{Im} z_0 = \lambda$ and z_0 and $\bar{z}_0$ are singularities, that is, there is a sequence $z_n \to z_0$ so that $|\sigma(z_n)| \to \infty$, and similarly for $\bar{z}_0$. Then σ satisfies property IP.

Note that if σ has a meromorphic extension which has a unique pole of minimal positive imaginary part, then it satisfies the above hypotheses. Most rational functions are like this; more interestingly, consider the main example in neural networks research, namely,

$$\sigma(z) = \frac{1}{1 + e^{-z}}. \tag{2}$$

Here the set of poles is the set

$$\{k\pi \mathrm{i}, k \text{ odd}\}$$

and one can take $z_0 = \pi\mathrm{i}$ above. Another example which appears often in the context of neural nets is that of $\arctan(x)$. Here, integrating $1/(1+z^2)$, one can find a branch defined on the complement of $\{\operatorname{Re} z = 0, |\operatorname{Im} z| \geq 1\}$, so one may pick $z_0 = \mathrm{i}$. See [2] for much more on property IP and related matters.

We will provide results under a restriction on the class of systems (1). We state this condition next. For any matrix M, M_i denotes the i-th row of M. Fix a pair of positive integers m, n. Let

$$\mathscr{B}_{n,m} = \left\{ B \in \mathbb{R}^{n\times m} \,\middle|\, \begin{array}{ll} B_i \neq 0 & \text{for all } i = 1, \ldots, n, \\ B_i \neq \pm B_j & \text{for all } i \neq j. \end{array} \right\}. \tag{3}$$

We drop the subscripts n, m when clear from the context. Observe that in the special case in which $m = 1$, a vector b is in $\mathscr{B}$ if and only if all its entries are nonzero and have different absolute values.

We denote by $\mathscr{S}$ the set of all systems (1) for which $B \in \mathscr{B}$ and σ satisfies the property IP.

Let e_i, $i = 1, \ldots, n$ denote the canonical basis elements in $\mathbb{R}^n$. A subspace V of the form $V = 0$ or

$$V = \operatorname{span}\{e_{i_1}, \ldots, e_{i_l}\}, \quad l > 0 \tag{4}$$

will be called a *coordinate subspace*. Coordinate subspaces are exactly those that are invariant under all the projections

$$\pi_i : \mathbb{R}^n \to \mathbb{R}^n, \quad \pi_i e_j = \delta_{ij} e_i.$$

Sums of coordinate subspaces are again of that form. Thus, for each pair of matrices (A, C) with $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, there is a unique largest A-invariant coordinate subspace included in $\ker C$; we denote this by $\mathscr{O}_c(A, C)$. One way to compute $\mathscr{O}_c = \mathscr{O}_c(A, C)$ is by the following recursive procedure:

$$\mathscr{O}_c^0 := \ker C,$$

$$\mathscr{O}_c^{k+1} := \mathscr{O}_c^k \cap A^{-1}\mathscr{O}_c^k \cap \pi_1^{-1}\mathscr{O}_c^k \cap \cdots \cap \pi_n^{-1}\mathscr{O}_c^k, \quad k = 0, \ldots, n-1,$$

$$\mathscr{O}_c := \mathscr{O}_c^n.$$

(This can be implemented by an algorithm which employs a number of elementary algebraic operations which is polynomial on the size of n and m. Alternatively, one could give a graph-theoretic algorithm based on structural observability techniques.)

Recall that a system is observable if for each two distinct initial states there is some control sequence that gives a different output when the system is started at those states. This definition can be formalized in the obvious way both for discrete- and continuous-time systems; see e.g. [7] for details, as well as for references to equivalences between this definition and 'single experiment' definitions. We now state the main result (in fact, two different results, one for discrete and another one for continuous time, but the proofs, given in the next section, will be essentially the same for both cases).

Theorem 1. *Let $\Sigma \in \mathscr{S}$. Then Σ is observable if and only if*

$$\ker A \cap \ker C = \mathscr{O}_c(A, C) = 0.$$

The condition $\mathscr{O}_c(A, C) = 0$ is equivalent to: no nonzero A-invariant coordinate subspace is included in $\ker C$.

Recall that the pair of matrices (A, C) is said to be observable – in the sense of classical linear systems theory; see e.g. [7], Section 5.2 – if the largest A-invariant subspace included in $\ker C$, denoted $O(A, C)$, is zero. Since both $\mathscr{O}_c(A, C)$ and $\ker A \cap \ker C$ are subspaces of $O(A, C)$, the following Corollary holds.

Corollary 2.1. *If $\Sigma \in \mathscr{S}$ and the pair of matrices (A, C) is observable then Σ is observable.*

A usual case is that in which A is invertible. In that case, $\ker A \cap \ker C = 0$. So in particular, we have the following Corollary.

Corollary 2.2. *If $\Sigma \in \mathcal{S}$ and $\det A \neq 0$, then Σ is observable if and only if $\mathcal{O}_c(A, C) = 0$.*

Remark 2.3. It is perhaps remarkable that this latter condition is formally the same as the observability condition (see e.g. [6]) that results for bilinear systems with transition matrices $A, \pi_1, \ldots, \pi_n$ and output matrix C.[1]

As any coordinate subspace has the form $V = \sum_j \pi_{i_j}(\mathbb{R}^n)$, for some finite set of indices $i_1, \ldots, i_l$, for such a space $CV = 0$ implies that $C\pi_{i_j} = 0$ for all j. In other words, if all columns of C are nonzero then $\mathcal{O}_c = 0$. Thus, we have yet another sufficient condition.

Corollary 2.4. *If $\Sigma \in \mathcal{S}$, $\ker A \cap \ker C = 0$, and each column of C is nonzero, then Σ is observable.*

3. Proof of Theorem 1

First we introduce some more notations and prove some intermediate technical results.

For each matrix D, we denote by I_D the following set:

$$I_D = \{i \mid \text{the } i\text{-th column of } D \text{ is zero}\} = \{i \mid D\pi_i = 0\}. \tag{5}$$

Note that, for a coordinate subspace V as in equation (4), $V \subseteq \ker D$ if and only if all $i_j \in I_D$.

Lemma 3.1. *Assume that $D \in \mathbb{R}^{q \times n}$ and $B \in \mathcal{B}_{n,m}$, and that σ satisfies property* IP. *Then the following two properties are equivalent, for each pair of vectors $\xi, \zeta \in \mathbb{R}^n$:*

(1) *$\xi_j = \zeta_j$ for all $j \notin I_D$.*

(2) *$D\sigma(\xi + Bu) = D\sigma(\zeta + Bu)$ for all $u \in \mathbb{R}^m$.*

Proof. Obviously, the first property implies the second one. Assume now that the second equality holds for some pair ξ, ζ, but that for this pair, there exists some $J \notin I_D$ so that $\xi_J \neq \zeta_J$. Pick any row index i so that the entry $D_{iJ} \neq 0$. We will prove that

$$\sum_{j=1}^{n} D_{ij}\sigma(\xi_j + (Bu)_j) = \sum_{j=1}^{n} D_{ij}\sigma(\zeta_j + (Bu)_j) \quad \forall u \in \mathbb{R}^m \Rightarrow D_{ij} = 0 \;\; \forall j = 1, \ldots, n, \tag{6}$$

which contradicts the fact that $D_{iJ} \neq 0$. Since the terms for which $\xi_j = \zeta_j$ can be cancelled out, we may assume without loss of generality that $\xi_j \neq \zeta_j$ for all j. (Not all terms cancel out in this manner because, by assumption, $\xi_J \neq \zeta_J$.)

First note that, since $B \in \mathcal{B}$, there must exist some $\bar{u} \in \mathbb{R}^m$ such that

- $B_i\bar{u} = (B\bar{u})_i \neq 0$ for all $i = 1, \ldots, n$,
- $B_i\bar{u} = (B\bar{u})_i \neq \pm (B\bar{u})_j = \pm B_j\bar{u}$ for all $i \neq j$.

Indeed, each of the equations $B_iu = 0$, $(B_i + B_j)u = 0$, and $(B_i - B_j)u = 0$ defines a hyperplane in $\mathbb{R}^m$, so we only need to avoid their (finite) union.

Now pick elements $u \in \mathbb{R}^m$ of the form $u = \bar{u}v$ and let $b_i := B_i\bar{u}$ in equation (6). We have that

$$\sum_{j=1}^{n} D_{ij}\sigma(b_jv + \xi_j) - \sum_{j=1}^{n} D_{ij}\sigma(b_jv + \zeta_j) = 0$$

[1] We thank Leonid Gurvits for pointing this out to us.

for all $v \in \mathbb{R}$. Consider the functions $\sigma(b_j v + \xi_j)$ and $\sigma(b_j v + \zeta_j)$ for $j = 1, \ldots, n$. As the numbers b_j are all nonzero and have distinct absolute values, and because σ satisfies property IP, the only way in which these functions could be linearly dependent is if $(b_j, \xi_j) = (b_j, \zeta_j)$ for some j, i.e. if $\xi_j = \zeta_j$, contradicting the assumption that these are all distinct. □

Next we establish some intermediate results useful for proving Theorem 1.

Lemma 3.2. *If x is indistinguishable from z then $(Ax)_i = (Az)_i$ for all $i \notin I_C$.*

Proof. We want to prove that, both for discrete time and for continuous time, x indistinguishable from z implies

$$C\sigma(Ax + Bu) = C\sigma(Az + Bu) \quad \forall u \in \mathbb{R}^m. \tag{7}$$

After this is shown, applying Lemma 3.1 with $D = C$, $\xi = Ax$, and $\zeta = Az$ to equation (7) provides the desired result.

In the discrete-time case, equation (7) holds since each side of equation (7) represents the output that one obtains by applying the one-step control u to the states x (left-hand side) and z (right-hand side).

Consider now the continuous-time case. Then, for any control value $u \in \mathbb{R}^m$, let $u(t)$ be the control function constantly equal to u. Denote by $x(t)$ and $z(t)$ the solutions of the differential equation (1) starting at x and z, respectively. Since both of these solutions exist at least on a small enough interval $[0, \varepsilon)$, indistinguishability implies $Cx(t) = Cz(t)$ for all $t \in [0, \varepsilon)$. Taking derivatives in this equality, we conclude that

$$C\dot{x}(t)|_{t=0} = C\dot{z}(t)|_{t=0},$$

which, in turn, says that also in this case equation (7) holds. □

For any two pairs of states $(x, z), (\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n$, we denote

$$(x, z) \rightsquigarrow (\xi, \zeta)$$

if, for the discrete-time case, there exists some input sequence $u_1, \ldots, u_l$, for some $l \geq 0$, such that, if we initialize the system at x (or at z), we reach ξ (or ζ respectively). For the continuous-time case, we require that there exists some (measurable, essentially bounded) control function $u(t): [0, T] \to \mathbb{R}^m$, such that, if we solve the differential equation (1) starting at x (or z), then the solution is defined on the entire interval $[0, T]$, and at time T the state ξ (or ζ) is reached.

Note that, with this notation, two states $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$ are distinguishable with respect to the system (1), in the standard sense of control theory, if and only if there is some pair $(\xi, \zeta) \in \mathbb{R}^n$ such that $(x, z) \rightsquigarrow (\xi, \zeta)$ and $C\xi \neq C\zeta$. Observability means that every pair $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$ with $x \neq z$ is distinguishable.

Proposition 3.3. *Let $\Sigma \in \mathscr{S}$, and pick any pair of states $x, z \in \mathbb{R}^n$. This pair is distinguishable if and only if: either $x - z \notin \ker C$ or there exists a pair of states x', z' so that $(x, z) \rightsquigarrow (x', z')$ and $A_j x' \neq A_j z'$ for some $j \notin I_C$.*

Proof. Pick any pair of (distinct) states x, z. We first prove necessity.

Case 1: Assume we are dealing with the discrete-time case. If x is distinguishable from z, then either $Cx \neq Cz$ or $Cx = Cz$ and there exists a pair (ξ, ζ) such that $(x, z) \rightsquigarrow (\xi, \zeta)$, and $C\xi \neq C\zeta$. If the second condition holds, then let (x', z') and $u \in \mathbb{R}^m$ be such that $(x, z) \rightsquigarrow (x', z')$ and $\sigma(Ax' + Bu) = \xi$ and $\sigma(Az' + Bu) = \zeta$ (note that such a u exists since necessarily $(x, z) \neq (\xi, \zeta)$). Then by Lemma 3.1 – applied with $D = C$, $\xi = Ax'$, and $\zeta = Az'$ – there is some $j \notin I_C$ such that $A_j x' \neq A_j z'$.

Case 2: Assume we are in the continuous-time case, and $Cx = Cz$. Then since the pair is distinguishable there exists, as before, a pair (ξ, ζ) such that $(x, z) \rightsquigarrow (\xi, \zeta)$, and $C\xi \neq C\zeta$. Let $u(\cdot): [0, T] \to \mathbb{R}^m$ be the control function which steers (x, z) to (ξ, ζ). (Note that necessarily $T > 0$.) We now prove by contradiction that there exists some $t \in (0, T]$ such that $A_j x(t) \neq A_j z(t)$ for some $j \notin I_C$ (where $x(t)$ (or $z(t)$) denotes the solution of the differential equation (1) with control function $u(t)$ and initial condition x (or z)). Assume that our conclusion does not hold, i.e. for all $t \in (0, T]$:

$$A_j x(t) = A_j z(t) \quad \forall j \notin I_C.$$

This implies that $(\dot{x}(t))_j = (\dot{z}(t))_j$ for all $j \notin I_C$. Thus, by integrating, we have

$$(x(t))_j = (z(t))_j + (x_j - z_j) \quad \forall j \notin I_C.$$

Since for each $j \in I_C$ the j-th column of C is zero, and since $Cx = Cz$, the previous equation implies:

$$Cx(t) = Cz(t) \quad \forall t \in [0, T],$$

which, in particular, says that $C\xi = Cx(T) = Cz(T) = C\zeta$, giving the desired contradiction.

Conversely, assume that the property holds. If $Cx \neq Cz$, the states are distinguishable. Otherwise, from Lemma 3.2 we get that x' is distinguishable from z', which, in turn, implies that x is distinguishable from z. □

For any subspace V of $\mathbb{R}^n$, and any two $x, z \in \mathbb{R}^n$, we denote

$$x \equiv z \bmod V$$

if $x - z \in V$. Observe that if V is a coordinate subspace, then

$$x \equiv z \bmod V \;\Rightarrow\; \sigma(x) \equiv \sigma(z) \bmod V. \tag{8}$$

The next lemma establishes a useful property of A-invariant coordinate subspaces. Note that the conclusion for discrete time is slightly different from the one for continuous time.

Lemma 3.4. *Let V be an A-invariant coordinate subspace. Assume that $x \equiv z \bmod A^{-1}(V)$. Pick any (ξ, ζ) such that $(x,z) \rightsquigarrow (\xi, \zeta)$*, then

(1) *for the discrete-time setting, we have $\xi \equiv \zeta \bmod V$,*

(2) *for the continuous-time setting, we have $\xi - x \equiv \zeta - z \bmod V$.*

Proof. The discrete-time result is easy to see. If V is A-invariant and a coordinate subspace, then $x \equiv z \bmod A^{-1}(V)$ implies $Ax + Bu \equiv Az + Bu \bmod V$ for all $u \in \mathbb{R}^m$. Thus arguing inductively on the length of controls, and using equation (8), our conclusion follows.

Now we establish the continuous-time result. Without loss of generality, we may assume that there exists $1 \leq k \leq n$ such that

$$V = \operatorname{span}\{e_1, \ldots, e_k\}$$

Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

with $A_1 \in \mathbb{R}^{k \times k}$, $A_2 \in \mathbb{R}^{k \times n-k}$, $A_3 \in \mathbb{R}^{n-k \times k}$, $A_4 \in \mathbb{R}^{n-k \times n-k}$, $B_1 \in \mathbb{R}^{k \times m}$, and $B_2 \in \mathbb{R}^{n-k \times m}$. Since V is A-invariant, we must have $(Ae_i)_l = a_{li} = 0$ for all $l \in \{k+1, \ldots, n\}$ and $i \in \{1, \ldots, k\}$. So $A_3 = 0$.

For each $y \in \mathbb{R}^n$, we denote $y = (y^1, y^2)$, where $y^1 = (y_1, \ldots, y_k)$ and $y^2 = (y_{k+1}, \ldots, y_n)$. With this notation, we have $y \equiv \tilde{y} \bmod V$ if and only if $y^2 = \tilde{y}^2$.

Let $p = Ax$ and $q = Az$. Since $x \equiv z \bmod A^{-1}(V)$, we have $p^2 = q^2$. Let $u(t)\colon [0, T] \to \mathbb{R}^m$ be the control function that steers (x, z) to (ξ, ζ). Denote by $x(t)$, $z(t)$ the corresponding trajectory starting at x and z, respectively, and let $p(t) = Ax(t)$, $q(t) = Az(t)$. Since $A_3 = 0$, we have

$$\dot{p}^2(t) = A_4 \boldsymbol{\sigma}(p^2(t) + B_2 u(t)),$$

$$\dot{q}^2(t) = A_4 \boldsymbol{\sigma}(q^2(t) + B_2 u(t)).$$

Thus, $p^2(t)$ and $q^2(t)$ are both solutions of the same differential equation; since $q^2(0) = p^2(0)$, by uniqueness of solutions we may conclude $p^2(t) = q^2(t)$ for all $t \in [0, T]$. This implies $\dot{x}^2(t) = \dot{z}^2(t)$, for all $t \in [0, T]$. So, we have

$$x^2(t) - x^2(0) = \int_0^t \dot{x}^2(s)\,\mathrm{d}s = \int_0^t \dot{z}^2(s)\,\mathrm{d}s = z^2(t) - z^2(0).$$

Evaluating the previous equation at $t = T$, we get

$$\xi^2 - x^2 = \zeta^2 - z^2,$$

which implies that $\xi - x \equiv \zeta - z \bmod V$, as desired. □

Remark 3.5. Since V is A-invariant, then $V \subseteq A^{-1}(V)$, thus, in particular, the previous lemma applies when $x \equiv z \bmod V$.

Now we are ready to prove Theorem 1. We will in fact establish the following stronger fact:

Two states x, z are indistinguishable if and only if $x \equiv z \bmod (A^{-1}\mathcal{O}_c \cap \ker C)$. $\quad (*)$

We first show the sufficiency of this condition. Assume that $Cx = Cz$ and $Ax \equiv Az \bmod \mathcal{O}_c$. Now we apply Lemma 3.4 to any pair (ξ, ζ) such that $(x, z) \rightsquigarrow (\xi, \zeta)$. For the discrete-time case, we get that $\xi - \zeta \in \mathcal{O}_c \subseteq \ker C$, so $C\xi = C\zeta$. For the continuous-time case, we have that $\xi - \zeta - (x - z) \in \mathcal{O}_c \subseteq \ker C$, thus, also in this case, we conclude $C(\xi - \zeta) = C(x - z) = 0$. So, in both cases, the chosen states cannot be distinguished.

Now we show necessity of the condition. That is, we need to see that if $x - z \notin A^{-1}\mathcal{O}_c \cap \ker C$ then the states are distinguishable. We wish to apply the criterion in Proposition 3.3. We may assume that $x - z \in \ker C$, since otherwise the states are obviously distinguishable. Since $\ker A \cap \ker C \subseteq A^{-1}\mathcal{O}_c \cap \ker C$, $Cx = Cz$ implies that $Ax \neq Az$. So for some j it is the case that $\pi_j Ax \neq \pi_j Az$. Hence the following set is nonempty:

$$J := \{ j \mid \exists (x', z')\ [(x, z) \rightsquigarrow (x', z') \text{ and } x' - z' \notin \ker \pi_j A] \}.$$

Consider the following coordinate subspace:

$$V = \operatorname{span}\{ e_j \mid j \in J \}.$$

Note that by definition (case where $(x, z) = (x', z')$), $A_j x \neq A_j z \Rightarrow j \in J$, i.e., $Ax - Az \in V$, or equivalently, $x - z \in A^{-1}V$.

If we prove that V is A-invariant, then it will follow that either $CV \neq 0$ or, by definition of $\mathcal{O}_c$, V is included in $\mathcal{O}_c$. In this latter case, we would have that $x - z \in A^{-1}V \subseteq A^{-1}\mathcal{O}_c$, contradicting the choice of the pair x, z. Thus it must be the case that $CV \neq 0$, which is the same as saying that J must contain some element not in I_C, and then Proposition 3.3 applies. Thus, we only need to prove invariance.

Pick an index $j \in J$. By definition of J, we can write $(x, z) \rightsquigarrow (x', z')$ and $A_j x' \neq A_j z'$. Writing $\xi := Ax'$ and $\zeta := Az'$, we have that

$$\xi_j \neq \zeta_j. \tag{9}$$

We wish to prove that $Ae_j \in V$, i.e. we need to see that, for each given $l \notin J$, $a_{lj} = \pi_l(Ae_j) = 0$. So take one such l.

Then, for the discrete-time case, since $l \notin J$, it must be the case that

$$\pi_l A \sigma(Ax' + Bu) = \pi_l A \sigma(Az' + Bu) \quad \text{for all } u \in \mathbb{R}^m. \tag{10}$$

Assume that $a_{lj} \neq 0$. Consider the matrix $D = \pi_l A$; since $a_{lj} \neq 0$, $j \notin I_D$. We are then in the situation of Lemma 3.1; this results in a contradiction between equations (10) and (9).

For the continuous-time setting, we argue as follows. Let $u \in \mathbb{R}^m$, and denote by $u(t)$ the control function constantly equal to u. Let $x'(t)$ and $z'(t)$ be the corresponding trajectories starting at x' and z', respectively; note that these trajectories are defined at least on a small interval $[0, \varepsilon)$. Since $l \notin J$, we must have

$$(Ax(t))_l = (Az(t))_l \quad \forall t \in [0, \varepsilon).$$

By taking the derivative with respect to t in the previous equation, and evaluating at $t = 0$, we again get that equation (10) holds. Thus, we now conclude as before.

Now it remains to show that condition (*) implies Theorem 1. By the definition of observability, it is enough to see that the following two conditions are equivalent:

$$A^{-1}\mathcal{O}_c \cap \ker C = 0, \tag{11}$$

$$\ker A \cap \ker C = \mathcal{O}_c = 0. \tag{12}$$

Since $\mathcal{O}_c$ is A-invariant, we have

$$\mathcal{O}_c \subseteq A^{-1}\mathcal{O}_c \cap \ker C.$$

Since also $\ker A \cap \ker C \subseteq A^{-1}\mathcal{O}_c \cap \ker C$, it is clear that (11) implies (12). Moreover, if (12) holds, then $A^{-1}\mathcal{O}_c = \ker A$, thus also the converse holds. $\square$

4. Some examples

It is interesting to note that the observability conditions found in Theorem 1, namely,

$$\ker A \cap \ker C = \mathcal{O}_c(A, C) = 0,$$

are *necessary* for the observability of any system of type (1), even if it does not belong to the class $\mathscr{S}$. However, as soon as a particular system Σ is not in $\mathscr{S}$, the previous conditions are no longer sufficient.

In this section, we provide two examples which show that the assumption that the function σ satisfies Property IP, is essential to conclude Theorem 1. To see that also the assumption $B \in \mathscr{B}_{n,m}$ is needed, see Examples 5.1 and 5.2.

Example 4.1. Let $\sigma(\cdot)$ be any periodic smooth function of period T; clearly such a function does not satisfy property IP. Consider the following system, with $n = 2$ and $p = m = 1$:

$$x^+ (\text{or } \dot{x}) = \sigma(x + bu),$$

$$y = x_1 - x_2,$$

where b is any vector in $\mathscr{B}_{2,1}$.

These systems satisfy all our observability conditions – except for the fact that σ does not have property IP – but they are not observable. Indeed, we consider $\bar{x} = (T, T)$. Then $C\bar{x} = 0$, and, since σ is periodic of period T, it is easy to see that both for the discrete-time, and for the continuous-time cases, $\bar{x}$ is indistinguishable from 0.

Example 4.2. Assume that $\sigma(x) = x^2$, which again does not satisfy property IP. Consider first the discrete-time system, with this function σ, $n = 2$, $m = p = 1$, and matrices A, B, and C as follows:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = (1, 2), \qquad C = (-4, 1).$$

Note that this system also satisfies all our observability conditions except for the property IP, and it is not observable. To show this last fact, we argue as follows. Let α be any real number, and consider the state $x_\alpha = (\alpha, 4\alpha)$. We claim that, with the obvious notation,

$$x^+(u) = (\alpha', 4\alpha') \quad \text{for some } \alpha' \in \mathbb{R}.$$

Note that if this property holds, then the system is not observable since all these states are in $\ker C$. We have

$$4x_1^+(u) = 4(2\alpha + u)^2 = (4\alpha + 2u)^2 = x_2^+(u).$$

Thus, our claim holds with $\alpha' = (2\alpha + u)^2$.

Let us consider now the same model in continuous time. It is possible to prove that this continuous-time system also is not observable. Fix, as before, a state of the type $x_\alpha = (\alpha, 4\alpha)$, which is in $\ker C$. Then it is easy to see that, if $u(t)$ is a control function, and we denote by $x(t) = (x_1(t), x_2(t))$ the corresponding trajectory starting at x_α, then $x_2(t) = 4x_1(t)$ for all t.

5. Some comparisons with linear systems

Corollary 2.1 showed that, if $B \in \mathscr{B}$, observability of the pair (A, C) implies that the system of interest is observable. But the converse is not true. For an example, consider any system in which each column of C is nonzero but A is a nonsingular diagonal matrix, such as $A = I$. In this case, the pair (A, C) is not observable (if $n > 1$ and $\ker C \neq 0$), but, from Corollary 2.4, we have that, if $B \in \mathscr{B}$, the system is observable.

On the other hand, observability of the pair (A, C) is no longer sufficient if one does not assume $B \in \mathscr{B}$. To illustrate this point, consider the following two examples, the first in discrete time and the second in continuous time.

Example 5.1. Pick any two nonzero real values x_1, x_2 in the image of σ such that

$$x_1\sigma^{-1}(x_2) \neq x_2\sigma^{-1}(x_1). \tag{13}$$

Such values always exist for nonlinear σ. Consider the discrete-time system with $n = 2$, $p = 1$, and $B = 0$:

$$A = \begin{pmatrix} \sigma^{-1}(x_1)/x_1 & 0 \\ 0 & \sigma^{-1}(x_2)/x_2 \end{pmatrix}, \qquad C = (x_2, \; -x_1).$$

Given (13) it is easy to see that the pair (A, C) is observable; however, the nonlinear system is not. In fact, the state $x = (x_1, x_2)$ is an equilibrium state and $Cx = 0$, so it is indistinguishable from zero.

Example 5.2. Assume that σ is a smooth Lipschitz function, which satisfies property IP and such that $\sigma(0) = \sigma(\bar{x}) = 0$. Pick any two nonzero distinct real values x_1, x_2, and consider the continuous-time system with $n = 2$, $p = 1$, and $B = 0$:

$$A = \begin{pmatrix} \bar{x}/x_1 & 0 \\ 0 & \bar{x}/x_2 \end{pmatrix}, \qquad C = (x_2, \; -x_1).$$

Since $x_1 \neq x_2$, it is easy to see that the pair (A, C) is observable; however, the nonlinear system is not. In fact, the state $x = (x_1, x_2)$ is an equilibrium state and $Cx = 0$, so it is indistinguishable from zero.

Of course, if one uses instead a B in $\mathscr{B}$ in these previous examples, then the systems become observable. This shows that observability may depend on the matrix B, which is a characteristic of nonlinear systems.

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