Neural Networks, Vol. 6, pp. 975–990, 1993

0893-6080/93 $6.00 + .00

ORIGINAL CONTRIBUTION

For Neural Networks, Function Determines Form

FRANCESCA ALBERTINI[1,2] AND EDUARDO D. SONTAG[1]

[1]Rutgers University and [2]Università degli Studi di Padova

(*Received* 13 *July* 1992; *revised and accepted* 10 *December* 1992)

Abstract—*This article shows that the weights of continuous-time feedback neural networks are uniquely identifiable from input/output measurements. Under weak genericity assumptions, the following is true: Assume given two nets, whose neurons all have the same nonlinear activation function* σ*; if the two nets have equal behaviors as "black boxes" then necessarily they must have the same number of neurons and—except at most for sign reversals at each node—the same weights. Moreover, even if the activations are not a priori known to coincide, they are shown to be also essentially determined from the external measurements.*

Keywords—Neural networks, Identification from input/output data, Control systems.

1. INTRODUCTION

Many recent articles explored the computational and dynamic properties of systems of interconnected "neurons." For instance, Hopfield (1984), Cowan (1968), and Cohen and Grossberg (1983) all studied devices that can be modeled by sets of nonlinear differential equations such as

$$\dot{x}_i(t) = -x_i(t) + \sigma\left(\sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{m} b_{ij}u_j(t)\right), \quad i = 1, \ldots, n \quad (1)$$

or

$$\dot{x}_i(t) = -x_i(t) + \sigma\left(\sum_{j=1}^{n} a_{ij}x_j(t)\right) + \sum_{j=1}^{m} b_{ij}u_j(t), \quad i = 1, \ldots, n. \quad (2)$$

Here, each x_i, $i = 1, \ldots, n$, is a real-valued variable that represents the internal state of the ith "neuron," and each u_i, $i = 1, \ldots, m$, is an external input signal. The coefficients a_{ij}, b_{ij} denote the weights, intensities, or "synaptic strengths," of the various connections. The function $\sigma : \mathbb{R} \to \mathbb{R}$, which appears in all equations, is called the "activation function" and is often taken to be a sigmoidal-type map. It characterizes how each neuron responds to its aggregate input. Some authors assume that a different external signal can control each neuron, but it seems reasonable, from a systems-theoretic perspective, to expect that the number of such inputs is far smaller than the number of state variables, that is, $m \ll n$. Electrical circuit implementations of these equations, employing resistively connected networks of n identical nonlinear amplifiers, and adjusting the resistor characteristics to obtain the desired weights, have been proposed as models of analog computers, in particular in the context of constraint satisfaction problems and in content-addressable memory applications (see, e.g., Hopfield, 1984).

We also assume given a certain number p of probes, or measurement devices, whose outputs signal to the environment the collective response of the net to the stimuli presented in the channels u_i. Each such device averages the activation values of many neurons. Mathematically, this is modeled by adding a set of functions

$$y_i(t) = \sum_{j=1}^{n} c_{ij}x_j(t), \quad i = 1, \ldots, p. \quad (3)$$

The coefficient c_{ij} represents the effect of the jth neuron on the ith measurement.

In vector terms, and omitting from now on the t arguments, we can write the above equations (1) and (2), respectively, as

$$\dot{x} = -\mathbf{x} + \vec{\sigma}(\mathbf{Ax} + \mathbf{Bu}) \quad (4)$$

Acknowledgements: This research was supported in part by U.S. Air Force Grant AFOSR-91-0343 and in part by an INDAM (Istituto Nazionale di Alta Matematica F. Severi) fellowship.

Requests for reprints should be sent to Eduardo Sontag, Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

and

$$\dot{x} = -\mathbf{x} + \vec{\sigma}(\mathbf{Ax}) + \mathbf{Bu}, \tag{5}$$

where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix and where we use the notation $\vec{\sigma}(\mathbf{x})$ to denote the application of the nonlinear function σ to each coordinate of the vector $\mathbf{x}$. We also write the output eq. (3) as

$$y = \mathbf{Cx}, \tag{6}$$

where $\mathbf{C}$ is a $p \times n$ matrix. Note that once the nonlinearity σ is specified, and the choice of model (4) or (5) is made (i.e., once the *architecture* of the network has been fixed), the triple of matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ uniquely determines the network dynamics. The entries of these matrices are usually referred to collectively as the "weights" of the network.

Closely related to the above models are those for which the neurons evolve according to

$$\dot{x} = \vec{\sigma}(\mathbf{Ax} + \mathbf{Bu}). \tag{7}$$

Again with outputs (6), they are a natural generalization of the linear systems that appear in control theory (namely, those for which $\vec{\sigma}$ is the identity mapping). Such generalized linear systems are of interest for many reasons besides neural network applications. They provide a class of "semilinear" systems, for which one might expect a theory closer to that of linear systems than is the case for general nonlinear smooth systems. Moreover, for suitably sharp nonlinearities σ they are approximate models of discontinuous equations such as $\dot{x} = \text{sign}(\mathbf{Ax} + \mathbf{Bu})$. [See Schwarzschild and Sontag (1991) for related work on systems that mix linear dynamics and sign functions.] In discrete time, systems of the type (7) have been recently shown to be at least as powerful as any possible digital computational device (see Siegelmann & Sontag, 1991, 1992) when all weights are rational numbers and a general model of analog computers when the weights are allowed to be real (Siegelmann & Sontag, 1992a).

1.1. Uniqueness of Weights

Stability properties, memory capacity, and other characteristics of the above types of systems have been thoroughly investigated by many authors; see, for example, Hirsch (1989) and Michel, Farrell, and Porod (1989) and references therein. In this article, we are interested in studying a somewhat different issue, namely, to what extent does the function of the net, this is, the "black box" behavior mapping external inputs to output signals, uniquely determine the coefficients a_{ij}, b_{ij}, c_{ij} defining the network? A precise formulation is as follows. Assume that the network is started at the relaxed state

$$\mathbf{x}(0) = 0$$

and an input signal $\mathbf{u}(\cdot)$ is applied. Under appropriate technical assumptions, a solution $\mathbf{x}(t)$ exists for the differential equation—respectively, (4), (5), or (7)—and an output signal $\mathbf{y}(t) = \mathbf{Cx}(t)$ is thus generated. In this manner, and for any fixed architecture, for each triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ there is an *input–output mapping*

$$\lambda_{(\mathbf{A,B,C})} : \mathbf{u}(\cdot) \mapsto \mathbf{y}(\cdot). \tag{8}$$

Our main question is to decide to what extent are the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ determined by the i/o mapping $\lambda_{(\mathbf{A,B,C})}$.

In the special case when σ is the identity, classical linear realization theory—see, for instance, Sontag (1990, chap. 5)—implies that, generically, the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is determined only up to an invertible change of variables in the state space, that is, except for degenerate situations that arise due to parameter dependencies ("noncontrollability" or "nonobservability"), if two triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ give rise to the same i/o behavior then there is an invertible matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{AT} = \bar{\mathbf{A}}, \quad \mathbf{T}^{-1}\mathbf{B} = \bar{\mathbf{B}}, \quad \mathbf{CT} = \bar{\mathbf{C}}. \tag{9}$$

This is the same as saying that the two systems are equivalent under a linear change of variables $\mathbf{x}(t) = \mathbf{T}\bar{\mathbf{x}}(t)$. Conversely, still in the classical case σ = identity, any such $\mathbf{T}$ gives rise to another system with the same i/o behavior when starting with any given triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.

These classical facts essentially apply only when σ is linear, as we will discuss in this article. We will show that for nonlinear activations (under weak assumptions) the natural group of symmetries is far smaller than that of arbitrary nonsingular matrices, being instead just a finite group. We will prove that if two nets give rise to the same i/o behavior then a matrix $\mathbf{T}$ will exist, satisfying (9), but having the special form of a permutation matrix composed with a diagonal matrix performing at most a sign reversal at each neuron. (Moreover, the activation function itself is identifiable, up to certain symmetries, in a precise sense.) In concrete terms, this will mean that *the input/output behavior uniquely determines all the weights, except for a reordering of the variables and, for odd activation functions, sign reversals of all incoming and outgoing weights at some units.*

After some thought, this result is not surprising—see the intuitive discussion given below for the single-neuron case—but technically it requires a certain amount of effort to establish.

1.2. Intuition Behind the Results

It is useful to consider our results in trivial situations (one neuron) to develop an intuition as to their nature.

Consider first the case of two one-neuron, one-input, one-output systems (i.e., $n = p = m = 1$) with zero $\mathbf{A}$ matrix, as follows:

$$\dot{x} = \sigma(bu), \quad \mathbf{y} = c\mathbf{x}$$
$$\dot{x} = \sigma(\bar{b}u), \quad \mathbf{y} = \bar{c}\mathbf{x}$$

[models of type (7)]. For the first equation, the zero-initial-state i/o behavior takes the form $\mathbf{u}(\cdot) \mapsto \mathbf{y}(\cdot)$, where

$$\mathbf{y}(t) = \int_0^t c\sigma(b\mathbf{u}(s))\, \mathrm{d}s$$

and similarly for the second. Note that in the case in which σ is an odd function (which is the case studied most often in applications) reversing the signs of weights leaves the i/o behavior invariant:

$$\int_0^t (-c)\sigma(-b\mathbf{u}(s))\, \mathrm{d}s = \int_0^t c\sigma(b\mathbf{u}(s))\, \mathrm{d}s$$

for all $\mathbf{u}(\cdot)$. Thus, both systems have the same i/o behavior if it holds that

$$b = -\bar{b}, \quad c = -\bar{c}. \tag{10}$$

Assume conversely that the zero-state i/o behaviors coincide; we wish to prove now that either both systems are the same or (10) holds.

Of course, the desired implication will not be true unless suitable nondegeneracy assumptions are made; for instance, if $c = \bar{c} = 0$ then both give the same i/o behavior but b and $\bar{b}$ may be different. Moreover, the case in which σ is linear must also be ruled out because otherwise $c\sigma(b) = cb\sigma(1)$ and the only constraint for equality is that $cb = \bar{c}\bar{b}$.

Note that the assumption of identical i/o behavior is equivalent to

$$c\sigma(b\mu) = \bar{c}\sigma(\bar{b}\mu) \quad \text{for all} \quad \mu \in \mathbb{R} \tag{11}$$

(equality as functions $\mathbb{R} \to \mathbb{R}$). If σ has three derivatives at zero, and if we assume that

$$\sigma'(0) \neq 0 \quad \text{and} \quad \sigma'''(0) \neq 0 \tag{12}$$

then taking first- and third-order derivatives with respect to μ in (11), and evaluating at $\mu = 0$, we conclude that

$$cb = \bar{c}\bar{b} \quad \text{and} \quad cb^3 = \bar{c}\bar{b}^3, \tag{13}$$

from which it follows, if $cb \neq 0$, that $b^2 = \bar{b}^2$ and hence that $|b| = |\bar{b}|$. Thus, either (10) holds or $b = \bar{b}$, $c = \bar{c}$, as desired. Instead of $\sigma'''(0) \neq 0$, we could have assume merely that *some* derivative $\sigma^{(q)}(0) \neq 0$, for some $q > 1$, because this would give by taking enough derivatives in (11) that $cb^q = \bar{c}\bar{b}^q$ and hence that $b^{(q-1)} = \bar{b}^{(q-1)}$ and again $|b| = |\bar{b}|$. (For analytic functions, $\sigma^{(q)}(0) \neq 0$ for some $q > 1$ is equivalent to nonlinearity of σ.)

For models of type (4), the equations in this special case of $\mathbf{A} = 0$ and $p = m = n = 1$ become

$$\dot{x} = -\mathbf{x} + \sigma(b\mathbf{u}), \quad \mathbf{y} = c\mathbf{x}.$$

Here, the output for control $u(\cdot)$ is

$$\mathbf{y}(t) = \int_0^t \mathrm{e}^{t-s} c\sigma(b\mathbf{u}(s))\, \mathrm{d}s,$$

so again the assumption of identical i/o behavior is equivalent to (11) and the result is as before.

The same result is valid for models of type (5) as well except that now the nondegeneracy assumptions must be slightly different because merely requiring $cb \neq 0$ is not sufficient to force uniqueness up to sign. Indeed, for $\mathbf{A} = 0$ the system (5) is linear, and thus the only invariant for i/o behavior is the matrix product $\mathbf{CB}$, so $\mathbf{B}$ and $\mathbf{C}$ are highly nonunique. For instance, $b = 1$, $c = 2$ gives the same zero initial state i/o behavior as $b = 2$, $c = 1$, for

$$\dot{x} = -\mathbf{x} + \sigma(a\mathbf{x}) + b\mathbf{u}, \quad \mathbf{y} = c\mathbf{x} \tag{14}$$

when $a = 0$. But, under suitable nondegeneracy of $\mathbf{A}$ the result is still true. As an illustration, take the $p = m = n = 1$ case, but now with $a = 1$, in (14). Looking at the derivative $\mathbf{y}'(0)$, we know that two systems of this form with same i/o behavior must have $cb = \bar{c}\bar{b}$. Arguing purely formally, now take an impulsive control of the type $u = \mu\delta$, starting at $\mathbf{x}(0) = 0$. This results in $\mathbf{x}(0^+) = b\mu$ and $\bar{\mathbf{x}}(0^+) = \bar{b}\mu$. Applying now a further control identically equal to zero, and taking derivative with respect to time, we obtain the value $-c\mathbf{x}(0^+) + c\sigma(\mathbf{x}(0^+)) + cb.0 = -cb\mu + c\sigma(b\mu)$ for the first system and similarly for the second. This implies that (11) again holds, and the argument is completed as above. Without using delta functions, we can argue as follows: Consider the control $\mathbf{u}(t) \equiv 1$ and the derivative at $t = 0$ of the resulting output $\mathbf{y}(t)$. Under the assumptions that σ has three derivatives, $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma'''(0) \neq 0$, one obtains that

$$\mathbf{y}^{(4)}(0) = cb^3\sigma'''(0)$$

and similarly for the second system, from which it again follows that $|b| = |\bar{b}|$, as desired.

1.3. Other Motivations for This Study

There are many motivations besides mathematical interest for posing the questions that we study in this work. In many applications, feedback neural nets, or "recurrent nets" as they are also called, are used as models whose parameters are fit to input/output data. (The purpose may be to use these models instead of a real plant, for purposes of control, or for predictive purposes.) This is done, for instance, in certain approaches to grammatical inference and speech processing; see, for instance, Cleeremans, Servan-Schreiber, and McClelland (1989) and Robinson and Fallside (1988). Typically, gradient descent algorithms are used to fit parameters through the minimization of an error functional that penalizes mismatches between the desired outputs and those that a candidate net produces (the term "continuous backpropagation" is sometimes used for the gradient descent procedure). Our results imply that a dimensionality reduction of the parameter space—as is the case with linear systems, where canonical forms are ubiquitous in identification methods—is, perhaps surprisingly, not possible for neural nets. In terms of the error function being minimized, the set of global minima is finite.

For precisely the above reasons, but restricted to the particular case of feedforward (i.e., nondynamic) nets, the question of deciding if the only possible symmetries are indeed the ones that we find we asked by Hecht-Nielsen (1989). The question was partially answered (for so-called "single-hidden layer" nets, and using a particular activation function) by Sussmann (1992), who established a uniqueness result that, in our setting, would apply to systems of the special type $\dot{x} = \vec{\sigma}(\mathbf{Bu})$, $\mathbf{y} = \mathbf{Cx}$, with $\sigma = \tanh(\mathbf{x})$. [That is, there is no "**A**" matrix; the result does allow for a constant bias vector inside the sigmoid, however. Sussmann's result can be generalized to a somewhat more general classes of activations than $\tanh(\mathbf{x})$ by means of an analysis based upon complex variables techniques and residue computations (Albertini, Sontag, & Maillot, 1993).]

A different, though related, motivation for studying the problem that we consider in this article originates from synthesis considerations rather than identification. Given a specified i/o behavior, it is of interest to know how many possible different networks can be built that achieve the design objective. Our results show that there is basically only one way to do so, as long as this objective is specified in terms of a desired input/output map. In this sense, structure (weights) is uniquely determined by function (desired i/o behavior).

1.4. Remarks

Note that *nonlinear realization theory,* as described, for instance, in Isidori (1985), Nijmeijer and Van der Schaft (1990), and Sussmann (1977), can also be applied to the problem considered here. This theory would allow us to conclude that, under suitable assumptions of controllability and observability, there is some abstract diffeomorphism that relates two networks having the same i/o behavior. It is not clear how to exploit the special structure of neural nets to obtain, using these tools, the more precise result that we give; on the other hand, the techniques that are used in the standard geometric theory of nonlinear systems—such as the systematic use of Lie derivatives—are also central to the proof of our result.

2. DEFINITIONS AND STATEMENTS OF RESULTS

In general, we define a *system* Σ to be a continuous-time time-invariant system:

$$\dot{x} = f(\mathbf{x}, u)$$
$$y = h(\mathbf{x}), \qquad (15)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$ for all t. We assume that f is at least differentiable on $(\mathbf{x}, \mathbf{u})$. For any measurable essentially bounded control $\mathbf{u}(\cdot): [0, \mathrm{T}] \rightarrow \mathbb{R}^m$, we denote by $\phi(t, \xi, \mathbf{u})$ the solution at time t of (15) with initial state $\mathbf{x}(0) = \xi$; this is defined at least on a small enough interval $[0, \varepsilon)$, $\varepsilon > 0$. For the systems of interest in neural network theory, $f(\mathbf{x}, \mathbf{u})$ is always uniformly Lipschitz with respect to $\mathbf{x}$, so $\varepsilon = T$. [All results that we use on existence of solutions and continuous dependence are included in standard texts such as Sontag (1990).]

For each control, we let $\lambda(\mathbf{u}) = \lambda_\Sigma(\mathbf{u})$ be the output function corresponding to the initial state $\mathbf{x}(0) = 0$, that is,

$$\lambda(\mathbf{u})(t) := h(\phi(t, 0, \mathbf{u})),$$

defined at least on some interval $[0, \varepsilon)$.

Given two systems Σ and $\bar{\Sigma}$ with the same numbers of input and output channels, that is, with $p = \bar{p}$ and $m = \bar{m}$, we say that Σ and $\bar{\Sigma}$ are *i/o equivalent* if it holds that

$$\lambda_\Sigma = \lambda_{\bar{\Sigma}}.$$

To be more precise, we require that for each u the domains of definition of $\lambda_\Sigma(\mathbf{u})$ and $\lambda_{\bar{\Sigma}}(\mathbf{u})$ coincide and their values be equal for all t in the common domain. As mentioned above, in the network application these functions are most often everywhere defined.

Fix an infinitely differentiable function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ that satisfies the following basic assumptions:

$$\boxed{\sigma(0) = 0, \quad \sigma'(0) \neq 0, \quad \sigma''(0) = 0} \qquad (*)$$

As before, we let $\vec{\sigma}(\mathbf{x})$ denote the application of σ to each coordinate of the vector $\mathbf{x}$.

We let $\mathcal{S}_1(n, m, p)$ [respectively, $\mathcal{S}_2(n, m, p)$ and $\mathcal{S}_3(n, m, p)$] denote the class of all systems of type (4) [respectively, types (5) and (7)], with fixed n, m, p.

Given two systems Σ and $\bar{\Sigma}$ in $\mathcal{S}_1(n, m, p)$ and $\mathcal{S}_1(\bar{n}_1, m, p)$, respectively—note that we assume $m = \bar{m}$ and $p = \bar{p}$—defined by the triples of matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$, we will say that Σ and $\bar{\Sigma}$ are *sign-permutation equivalent* if the dimensions of the state spaces are the same, that is, $n = \bar{n}$, and there exists a matrix $\mathbf{T}$ such that (9) holds and $\mathbf{T}$ has the following special form:

$$\mathbf{T} = \mathbf{PD}$$

where $\mathbf{P}$ is a permutation matrix and $\mathbf{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where each $\lambda_i = \pm 1$.

The systems Σ and $\bar{\Sigma}$ are just *permutation equivalent* if the above holds with $\mathbf{D} = \mathbf{I}$, that is, $\mathbf{T}$ is a permutation matrix.

We also define equivalence in an analogous manner among systems of type (5) or systems of type (7).

Later, in the process of establishing the main results we introduce a more general class of systems that includes systems of the types (4), (5), and (7); we will also define a notion of equivalence, "σ-equivalence," that will encompass both permutation and sign-permutation equivalence.

We call a subset $\mathcal{C}$ of $\mathcal{S}_1(n, m, p)$ *generic* if there exists a nonempty subset

$$\mathcal{G} \subseteq \mathbb{R}^{n^2+nm+mp},$$

whose complement is the set of zeros of a finite number of polynomials in $n^2 + nm + mp$ variables, so that Σ is in $\mathcal{C}$ if and only if its defining triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is in $\mathcal{G}$. Observe that such a set $\mathcal{G}$ is open and dense and its complement has measure zero. (In the results to be given, the polynomials defining the complement of $\mathcal{G}$ are given explicitely, and they correspond to the vanishing of appropriate determinants.) We also define generic subsets of $\mathcal{S}_2(n, m, p)$ and $\mathcal{S}_3(n, m, p)$ in the same manner.

Now, we are ready to state the main results. We assume that $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies the above assumptions $(*)$ and also that there exists some integer $q > 2$ such that $\sigma^{(q)}(0) \neq 0$.

The assumptions are extremely weak and include a large class of interesting nonlinearities. Observe that, for instance, if σ is odd $(\sigma(-\mathbf{x}) = -\sigma(\mathbf{x}))$, analytic, nonlinear, and it satisfies $\sigma'(0) \neq 0$, then it satisfies all hypotheses (so the standard nonlinearity $\tanh(\mathbf{x})$ is included).

THEOREM 1. *Assume that σ is odd. For each m, n, p there exists generic subsets*

$$\tilde{\mathcal{S}}_1(n, m, p) \subseteq \mathcal{S}_1(n, m, p)$$
$$\tilde{\mathcal{S}}_2(n, m, p) \subseteq \mathcal{S}_2(n, m, p)$$
$$\tilde{\mathcal{S}}_3(n, m, p) \subseteq \mathcal{S}_3(n, m, p)$$

such that, for any two systems

$$\Sigma \in \tilde{\mathcal{S}}_1(n, m, p \quad and \quad \bar{\Sigma} \in \tilde{\mathcal{S}}_1(\bar{n}, m, p)$$

[*respectively,* $\Sigma \in \tilde{\mathcal{S}}_2(n, m, p)$, $\bar{\Sigma} \in \tilde{\mathcal{S}}_2(\bar{n}, m, p)$ *or* $\Sigma \in \tilde{\mathcal{S}}_3(n, m, p)$, $\bar{\Sigma} \in \tilde{\mathcal{S}}_3(\bar{n}, m, p)$] *it holds that* Σ *and* $\bar{\Sigma}$ *are i/o equivalent if and only if* Σ *and* Σ *are sign-permutation equivalent.*

THEOREM 2. *Assume that σ is not odd. Then, there are generic subsets as in Theorem* 1, *so that:*

Σ *and* $\bar{\Sigma}$ *are i/o equivalent if and only if*

Σ and $\bar{\Sigma}$ *are permutation equivalent.*

The rest of this article presents a far stronger technical result and then shows how to derive these theorems as simple consequences. The more general result will deal also with systems of the type $\dot{x} = \mathbf{Dx} + \sigma(\mathbf{Ax} + \mathbf{Bu}) + \mathbf{Gu}$, $\mathbf{y} = \mathbf{Cx}$. Also, Theorem 6 will extend the results to the case when the nonlinearities are not a priori known to be the same.

3. GENERAL SETUP

We consider an infinitely differentiable function $\sigma : \mathbb{R} \to \mathbb{R}$ that satisfies the basic assumptions $(*)$, namely, $\sigma(0) = 0$, $\sigma'(0) \neq 0$, and $\sigma''(0) = 0$. We will see later (cf. Remark 3.4) that for our purposes we may always assume $\sigma'(0) = 1$ in $(*)$.

At various parts, we will also impose one or both of the following conditions on the function σ, the second of which is assumed for Theorems 1 and 2:

$$\boxed{\sigma'(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty} \tag{A1}$$

$$\boxed{\exists\, q > 2 \quad \text{such that} \quad \sigma^{(q)}(0) \neq 0.} \tag{A2}$$

Remark 3.1. As mentioned earlier, any function σ that is nonlinear, odd, and analytic [such as $\tanh(\mathbf{x})$] satisfies our requirements $(*)$ and also assumption (A2). Property (A1) is often satisfied in examples of interest, as well. However, property (A1) is needed mainly to deal with the general class of systems we will introduce [see eq. (19)]. Notice that this assumption on σ is not necessary to prove Theorems 1 and 2.

Remark 3.2. We will remark later that for most results only infinite differentiability of σ in a neighborhood of zero is needed, as opposed to globally.

If σ satisfies (A2), then the following property is an immediate consequence:

$$\boxed{\begin{array}{l}\text{If } \sigma(a\mathbf{x}) \\ = a\sigma(\mathbf{x}) \text{ for all } \mathbf{x} \text{ in a neighborhood of } 0, \\ \text{then} \quad a \in \{\pm 1, 0\}.\end{array}} \tag{P}$$

We will denote by Λ_σ the following set:

$$\Lambda_\sigma = \{\lambda \in \mathbb{R} \mid \lambda \neq 0 \quad \text{and} \quad \sigma(\lambda \mathbf{x}) = \lambda\sigma(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}\}. \tag{16}$$

Note that Λ_σ is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$. The fact that inverses are again in Λ_σ follows from the defining equation in (16), applied to $(1/\lambda)\mathbf{x}$ instead of $\mathbf{x}$. Moreover, when property (P) holds we have that

$$\Lambda_\sigma \subseteq \{1, -1\}.$$

Given any function $\sigma : \mathbb{R} \to \mathbb{R}$ and any positive integer n, we define $\vec{\sigma}$ as follows (more precisely, we should denote this as $\vec{\sigma}_n$, but we omit the n for simplicity):

$$\vec{\sigma} : \mathbb{R}^n \to \mathbb{R}^n; \quad \vec{\sigma}(\mathbf{x}) = \text{col}(\sigma(x_1), \ldots, \sigma(x_n)) \tag{17}$$

that is,

$$\vec{\sigma}\left(\sum_{i=1}^{n} a_i e_i\right) := \sum_{i=1}^{n} \sigma(a_i) e_i \quad \text{for all} \quad a_1, \ldots, a_n \in \mathbb{R}, \tag{18}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$.

From now on, we fix one such σ.

3.1. Systems

We will study conditnuous-time systems whose state space, input-value space, and output-value space are $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$, respectively, and for which the dynamics are given by equations of the type

$$\begin{cases} \dot{x} = f(\mathbf{x}, \mathbf{u}) = \mathbf{D}\mathbf{x} + \vec{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) + \mathbf{G}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases} \quad (19)$$

for some matrices $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{D} \in \mathbf{R}^{n\times n}$, $\mathbf{B} \in \mathbf{R}^{n\times m}$, $\mathbf{G} \in \mathbf{R}^{n\times m}$, and $\mathbf{C} \in \mathbf{R}^{p\times n}$. These are continuous-time systems in the sense of Sontag (1990). We will call such a system of *σ-system* and denote it by $\Sigma = (\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{C})_\sigma$.

Observe that in the special case in which σ is the identity, or more generally is linear, we have a linear system in the usual sense. The same holds if σ is arbitrary but $\mathbf{A} = \mathbf{B} = 0$. This is useful to keep in mind when understanding the conditions to be imposed later for the various conclusions to hold.

Remark 3.3. Different matrices $(\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G})$ may give rise to the same function $f(\mathbf{x}, \mathbf{u}) = \mathbf{D}\mathbf{x} + \vec{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) + \mathbf{G}\mathbf{u}$ and thus to the same system. ($\mathbf{C}$, on the other hand, is uniquely defined.) For instance, if σ is the identity, f depends only upon the sums $\mathbf{D} + \mathbf{A}$ and $\mathbf{B} + \mathbf{G}$. But, for the types of functions σ that we consider this ambiguity will in general not happen. Indeed, assume that

$$(\mathbf{D}_1, \mathbf{A}_1, \mathbf{B}_1, \mathbf{G}_1, \mathbf{C})_\sigma = (\mathbf{D}_2, \mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2, \mathbf{G})_\sigma.$$

Then:

(a) if $\mathbf{D}_1 = \mathbf{D}_2$ and $\mathbf{G}_1 = \mathbf{G}_2$, then $\mathbf{A}_1 = \mathbf{A}_2$ and $\mathbf{B}_1 = \mathbf{B}_2$;
(b) if σ satisfies (A1), then $\mathbf{D}_1 = \mathbf{D}_2$ and $\mathbf{G}_1 = \mathbf{G}_2$ (and hence also $\mathbf{A}_1 = \mathbf{A}_2$ and $\mathbf{B}_1 = \mathbf{B}_2$).

To prove these facts we can argue as follows. Taking $\mathbf{x} = \beta e_j$ and $\mathbf{u} = 0$, we have

$$\mathbf{D}_1\beta e_j + \vec{\sigma}(\mathbf{A}_1\beta e_j) = \mathbf{D}_2\beta e_j + \vec{\sigma}(\mathbf{A}_2\beta e_j) \quad \text{for all} \quad \beta \in \mathbb{R},$$

which implies, denoting $(\mathbf{A}_l)_{i,j}$ by $a^l_{i,j}$ and $(\mathbf{D}_l)_{i,j}$ by $d^l_{i,j}$.

$$d^1_{i,j}\beta + \sigma(a^1_{i,j}\beta) = d^2_{i,j}\beta + \sigma(a^2_{i,j}\beta) \quad (20)$$

for all β real and all indices i, j. After taking the derivative with respect to β in the previous equation, we have:

$$d^1_{i,j} + a^1_{i,j}\sigma'(a^1_{i,j}\beta) = d^2_{i,j} + a^2_{i,j}\sigma'(a^2_{i,j}\beta). \quad (21)$$

If $\mathbf{D}_1 = \mathbf{D}_2$, then evaluating (21) at $\beta = 0$ we have

$$a^1_{i,j}\sigma'(0) = a^2_{i,j}\sigma'(0),$$

which implies $\mathbf{A}_1 = \mathbf{A}_2$.

Denoting $(\mathbf{B}_l)_{i,j}$ by $b^l_{i,j}$, and $(\mathbf{G}_l)_{i,j}$ by $g^l_{i,j}$, and using $\mathbf{x} = 0$ and $\mathbf{u} = \beta e_j$ we can conclude, by the same arguments, that

$$b^1_{i,j}\sigma'(b^1_{i,j}\beta) + g^1_{i,j} = b^2_{i,j}\sigma'(b^2_{i,j}\beta) + g^2_{i,j}. \quad (22)$$

Thus, arguing as before, if $\mathbf{G}_1 = \mathbf{G}_2$ we have $\mathbf{B}_1 = \mathbf{B}_2$.

Note that, in general, when (A1) holds, $\lim_{\beta\to\infty} a\sigma'(a\beta) = 0$ for every $a \in \mathbb{R}$ [either $a = 0$ and this is identically zero or we use (A1)]. So, by taking the limit as $\beta \to \infty$ in (21) and in (22), we conclude $\mathbf{D}_1 = \mathbf{D}_2$ and $\mathbf{G}_1 = \mathbf{G}_2$ if (A1) holds.

We will focus our attention on the following question: When are two given σ-systems Σ_1, Σ_2 equivalent (in the senses defined in Section 2)?

Remark 3.4. We now explain why $\sigma'(0) \neq 0$ can be replaced by the stronger assumption that $\sigma'(0) = 1$ without loss of generality. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the basic assumptions (*) and let $a = \sigma'(0) \neq 0$. Consider the function $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}$ defined by: $\tilde{\sigma}(x) = \sigma(x/a)$. Then, also $\tilde{\sigma}$ satisfies the basic assumptions (*), and moreover $\tilde{\sigma}'(0) = 1$.

Now, if $\Sigma = (\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{C})_\sigma$ is a σ-system we may define the new system having $\tilde{\Sigma} = (\mathbf{D}, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \mathbf{G}, \mathbf{C})_{\tilde{\sigma}}$ to be the $\tilde{\sigma}$-system with $\tilde{\mathbf{A}} = a\mathbf{A}$ and $\tilde{\mathbf{B}} = a\mathbf{B}$. Notice that if $\mathbf{u}(t)$ is a control function defined on the interval $[0, \mathbf{T}]$, and $\mathbf{x}(t)$ is the corresponding trajectory of Σ starting at $\mathbf{x}(0) = 0$, then the trajectories and outputs in $\tilde{\Sigma}$ corresponding to any given control will be the same.

If Σ_1 and Σ_2 are σ-systems, and we construct $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ in this manner, we have that Σ_1 and Σ_2 are i/o equivalent if and only if $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are i/o equivalent. Because our interest is in establishing the existence of various linear equations relating $\mathbf{A}_1$ and $\mathbf{A}_2$, $\mathbf{B}_1$ and $\mathbf{B}_2$, and so forth, and because these equations are not changed under multiplication by a scalar, it is clear that we can assume, without loss of generality, that $\sigma'(0) = 1$. So, from now on, when we consider a differentiable function σ, we implicitly assume that $\sigma'(0) = 1$.

4. EQUIVALENCE

In this section, we give a straightforward sufficient condition for equivalence, which will also turn out to be necessary in a generic sense. We again fix a function $\sigma : \mathbb{R} \to \mathbb{R}$, although for now we do not need to assume that assumptions (*) hold. Let π be any permutation of $\{1, \ldots, n\}$ and P be the permutation matrix that represents π, that is,

$$\mathbf{P}_{i,j} = \delta_{i,\pi(i)}, \quad (23)$$

where $\delta_{i,k}$ is the Kronecker delta. In other words, $\mathbf{P}e_j = e_{\pi(j)}$ for the canonical basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$. Note that

$$\vec{\sigma}(\mathbf{P}\mathbf{x}) = \mathbf{P}\vec{\sigma}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n \quad (24)$$

because

$$\vec{\sigma}(\mathbf{P}\mathbf{x}) = \vec{\sigma}\left(\sum_{i=1}^{n} a_i e_{\pi(j)}\right) = \sum_{i=1}^{n} \sigma(a_j) e_{\pi(j)}$$

and

$$\mathbf{P}\vec{\sigma}(\mathbf{x}) = \mathbf{P}\left(\sum_{i=1}^{n} \sigma(a_j)e_j\right) = \sum_{j=1}^{n} \sigma(a_j)e_{\pi(j)}.$$

Let $\mathbf{Q}$ be any diagonal matrix such that

$$\mathbf{Q}e_j = \lambda_j e_j, \quad \text{where} \quad \lambda_j \in \Lambda_\sigma \quad \text{for all} \quad j = 1, \ldots, n.$$

Then, we also have that

$$\vec{\sigma}(\mathbf{Q}\mathbf{x}) = \mathbf{Q}\vec{\sigma}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n. \tag{26}$$

We let

$$\Lambda_\sigma^n := \{\mathbf{T} \in Gl(n) \mid \vec{\sigma}(\mathbf{T}\mathbf{x}) = \mathbf{T}\vec{\sigma}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n\}.$$

The following lemma follows trivially from (24) and (26).

LEMMA 4.1. *Let σ, $\mathbf{P}$, $\mathbf{Q}$ be as above. Then, both $\mathbf{T} = \mathbf{PQ}$ and $\mathbf{T} = \mathbf{QP}$ are in Λ_σ^n.* ■

Observe that Λ_σ^n is a subgroup of $Gl(n)$. When σ is linear, obviously $\Lambda_\sigma^n = Gl(n)$. Otherwise, this group is proper, and if (A2) holds it has cardinality at most $2^n n!$, as follows from the following result.

LEMMA 4.2. *If σ is differentiable but it is not a linear function, then every element of Λ_σ^n is of the form $\mathbf{PQ}$, with $\mathbf{P}$ a permutation matrix [as in (23)] and $\mathbf{Q} = Diag(\lambda_1, \ldots, \lambda_n)$ with each $\lambda_i \in \Lambda_\sigma$ [as in (25)].*

Proof. Assume that $\vec{\sigma}(\mathbf{T}\mathbf{x}) = \mathbf{T}\vec{\sigma}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Consider the αth-row of $\mathbf{T}$ and denote this $(t_1, \ldots, t_n)$. Then,

$$\sigma\left(\sum_{i=1}^{n} t_i x_i\right) = \sum_{i=1}^{n} t_i \sigma(x_i) \tag{27}$$

for all $(x_1, \ldots, x_n)$. Because $\mathbf{T}$ is invertible, there exists an index l such that $t_l \neq 0$. Taking the derivative in (27) with respect to x_l, we have

$$t_l \sigma'\left(\sum_{i=1}^{n} t_i x_i\right) = t_l \sigma'(x_l). \tag{28}$$

Picking any $j \neq l$, and taking the derivative in (27) with respect to x_j, we get

$$t_j \sigma'\left(\sum_{i=1}^{n} t_i x_i\right) = t_j \sigma'(x_j). \tag{29}$$

Multiplying both sides of (28) by t_j/t_l, and comparing (28) with (29), we have

$$t_j \sigma'(x_j) = t_j \sigma'(x_l).$$

This must hold for all choices of x_j and x_l in $\mathbb{R}$. Because σ is not a linear function, σ' is not constant so we must have $t_j = 0$. The index j was arbitrary; thus, we have that, in the αth-row of $\mathbf{T}$ there exists only one index l such that $t_l \neq 0$. Because also α was arbitrary, the matrix $\mathbf{T}$ has at most one nonzero entry in each row, and, by (27), this entry is in Λ_σ; thus, the conclusion follows. ■

DEFINITION 4.3. *Let $\Sigma_1 = (\mathbf{D}_1, \mathbf{A}_1, \mathbf{B}_1, \mathbf{G}_1, \mathbf{C}_1)_\sigma$, $\Sigma_2 = (\mathbf{D}_2, \mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2, \mathbf{C}_2)_\sigma$ be two σ-systems and n_1, n_2 be the dimensions of the state spaces in Σ_1, Σ_2, respectively. We say that Σ_1 and Σ_2 are σ-equivalent if $n_1 = n_2 = n$ and if there exists an invertible matrix $\mathbf{T} \in \Lambda_\sigma^n$ such that*

$$\mathbf{A}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T},$$
$$\mathbf{D}_2 = \mathbf{T}^{-1}\mathbf{D}_1\mathbf{T},$$
$$\mathbf{C}_2 = \mathbf{C}_1\mathbf{T},$$
$$\mathbf{B}_2 = \mathbf{T}^{-1}\mathbf{B}_1,$$
$$\mathbf{G}_2 = \mathbf{T}^{-1}\mathbf{G}_1.$$

The next property is trivial, but we state it for further reference.

Proposition 4.4. Let $\Sigma_1 = (\mathbf{D}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{G}_1, \mathbf{C}_1)_\sigma$, $\Sigma_2 = (\mathbf{D}_2, \mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2, \mathbf{C}_2)_\sigma$ be two σ-systems. If they are σ-equivalent, then they are also i/o-equivalent.

Proof. Denote by $\mathbf{x}$ the state variable for Σ_1 and by $\mathbf{z}$ the state variable for Σ_2. We have, for any measurable essentially bounded control u,

$$\begin{aligned}\dot{x} &= \mathbf{D}_1\mathbf{x} = \vec{\sigma}(\mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u}) + \mathbf{G}_1\mathbf{u} \\ &= \mathbf{T}\mathbf{D}_2\mathbf{T}^{-1}\mathbf{x} + \vec{\sigma}(\mathbf{T}(\mathbf{A}_2\mathbf{T}^{-1}\mathbf{x} + \mathbf{B}_2\mathbf{u})) + \mathbf{T}\mathbf{G}_2\mathbf{u}.\end{aligned}$$

Thus, letting $z = \mathbf{T}^{-1}\mathbf{x}$, we have

$$\begin{aligned}\dot{z} &= \mathbf{T}^{-1}(\mathbf{T}\mathbf{D}_2\mathbf{z} + \vec{\sigma}(\mathbf{T}(\mathbf{A}_2\mathbf{z} + \mathbf{B}_2\mathbf{u})) + \mathbf{T}\mathbf{G}_2\mathbf{u}) \\ &= \mathbf{T}^{-1}\mathbf{T}(\mathbf{D}_2\mathbf{z} + \vec{\sigma}(\mathbf{A}_2\mathbf{z} + \mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}) \\ &= \mathbf{D}_2\mathbf{z} + \vec{\sigma}(\mathbf{A}_2\mathbf{z} + \mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u},\end{aligned}$$

where, in the previous equations, we have used the fact that $\mathbf{T} \in \Lambda_\sigma^n$. So, as $\mathbf{z}(0) = 0 = \mathbf{T}^{-1}\mathbf{x}(0)$, uniqueness of solutions implies that

$$\mathbf{z}(t) = \mathbf{T}^{-1}\mathbf{x}(t)$$

for all t, for the trajectory of the second system corresponding to the same control, which implies

$$\mathbf{C}_2\mathbf{z}(t) = \mathbf{C}_1\mathbf{T}\mathbf{z}(t) = \mathbf{C}_1\mathbf{x}(t)$$

for all t. ■

5. TECHNICAL RESULTS

Again, we fix a $\sigma : \mathbb{R} \to \mathbb{R}$, assumed differentiable and so that (8) holds, more precisely so that

$$\sigma(0) = \sigma''(0) = 1 - \sigma'(0) = 0.$$

In this section, we assume that two fixed systems

$$\Sigma_1 = (\mathbf{D}_1, \mathbf{A}_1, \mathbf{B}_1, \mathbf{G}_1, \mathbf{C}_1)_\sigma$$

and

$$\Sigma_2 = (\mathbf{D}_2, \mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2, \mathbf{C}_2)_\sigma$$

with m inputs and p outputs are given and we would like to find necessary conditions for these two systems to have the same i/o behavior when both are initialized at $x_0^1 = x_0^2 = 0$.

5.1. Some Basic Identities

Let $\{u_1, \ldots, u_k\}$ be control values, that is, $u_i \in \mathbf{R}^m$ for each i, and let t_i, $i = 1, \ldots, k$ be positive real numbers. Consider the piecewise constant control $\mathbf{u}(t)$ on the interval $[0, \mathbf{T})$ with

$$\mathbf{T} = \sum_{i=1}^{k} t_i,$$

which is equal to u_1 on $[0, t_1)$, u_2 on $[t_1, t_1 + t_2)$, and so on. Denote by $y_j^i(t)$ the jth-component of the output of the system Σ_i, $i = 1, 2$, corresponding to the control $\mathbf{u}(t)$, and by $\mathbf{X}_{u_l}^i$, $l = 1, \ldots, k$, the following vector field:

$$\mathbf{X}_{u_l}^i(\mathbf{x}) = \mathbf{D}_i\mathbf{x} + \vec{\sigma}(\mathbf{A}_i\mathbf{x} + \mathbf{B}_i u_l) + \mathbf{G}_i u_l. \tag{30}$$

We denote $h_j^i(\mathbf{x}) = (\mathbf{C}_i\mathbf{x})_j$. The following is a well-known formula (see, e.g., Sontag, 1990, p. 210):

$$\left.\frac{\partial^k}{\partial t_1 \ldots \partial t_k}\right|_{t_1=\ldots=t_k=0} y_j^i(t) = L_{\mathbf{X}_{u_k}^i} \ldots L_{\mathbf{X}_{u_1}^i} h_j^i(x_0^i) \tag{31}$$

for all $x_0 \in \mathbb{R}^n$, where $L_{\mathbf{X}}h$ denotes the Lie-derivative of the function h along the vector field $\mathbf{X}$. Thus, if Σ_1 and Σ_2 have the same i/o behavior from (31) we have

$$L_{\mathbf{X}_{u_k}^1} \ldots L_{\mathbf{X}_{u_1}^1} h_j^1(0) = L_{\mathbf{X}_{u_k}^2} \ldots L_{\mathbf{X}_{u_1}^2} h_j^2(0) \tag{32}$$

for all $j = 1, \ldots, p_k$, $k \geq 0$, and vectors $u_j \in \mathbf{R}^m$.

The following technical lemma will be used later to derive some necessary conditions for the two systems to have the same i/o behavior. If h is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, and $\mathbf{Z}$ is a smooth vector field on $\mathbb{R}^n$, expressed in the natural coordinates, we will use the following notations:

1. $\nabla h = (\partial h/\partial x_1, \ldots, \partial h/\partial x_n)$, the usual gradient,
2. $(\mathbf{Z})_* =$ the usual Jacobian matrix of $\mathbf{Z}$, that is, if $(\mathbf{Z})_k = z^k$ then $[(\mathbf{Z})_*]_{lk} = \partial z^k/\partial x_l$.

When we omit arguments, we understand all Jacobians and gradients as being evaluated at the same point x.

LEMMA 5.1. *Let h be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, and $\mathbf{Z}_i$, $i \geq 1$, be smooth vector fields on $\mathbb{R}^n$. Then for all $k \geq 1$ following formula holds:*

$$L_{\mathbf{Z}_k} \ldots L_{\mathbf{Z}_1} h = \nabla h(\mathbf{Z}_1)_* \ldots (\mathbf{Z}_{k-1})_* \mathbf{Z}_k + \mathbf{G}^k(h, \mathbf{Z}_1, \ldots, \mathbf{Z}_k), \tag{33}$$

where the expression $\mathbf{G}^k(h, \mathbf{Z}_1, \ldots, \mathbf{Z}_k)$, which is a scalar function of x, is a sum of terms, each of which is a product of $k+1$ factors, and each of these factors is a derivative of order at most k either of the function h or of the components of the vector fields $\mathbf{Z}_i$. Moreover, in each term there are at least three factors where the derivative is of order $\{0$ or $2\}$ and at least two of them are of order 0.

Proof. We will prove the statement by induction on k. Let $k = 1$; then, we have

$$L_{\mathbf{Z}_1} h = \nabla h \mathbf{Z}_1 \tag{34}$$

by definition of Lie derivative. So, in this case the lemma holds with $\mathbf{G}^1(h, \mathbf{Z}_1) \equiv 0$. To give an idea of what the expression of the function $\mathbf{G}^k$ is, before dealing with the general induction step we also study explicitly the case $k = 2$. In this case, we have

$$L_{\mathbf{Z}_2} L_{\mathbf{Z}_1} h = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(L_{\mathbf{Z}_1} h)(\mathbf{Z}_2)_i = \nabla h(\mathbf{Z}_1)_* \mathbf{Z}_2 + \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j}(\mathbf{Z}_1)_i(\mathbf{Z}_2)_j; \tag{35}$$

thus, letting $\mathbf{G}^2(h, \mathbf{Z}_1, \mathbf{Z}_2) = \sum_{i,j=1}^{n} (\partial^2 h/\partial x_i \partial x_j) \times (\mathbf{Z}_1)_i(\mathbf{Z}_2)_j$, $\mathbf{G}^2$ satisfies the desired properties and so (33) holds for $k = 2$. Let now $k > 2$. By inductive assumption, we have

$$L_{\mathbf{Z}_k}(L_{\mathbf{Z}_{k-1}} \ldots L_{\mathbf{Z}_1} h) = \nabla[\nabla h(\mathbf{Z}_1)_* \ldots (\mathbf{Z}_{k-2})_* \mathbf{Z}_{k-1}]\mathbf{Z}_k + \nabla[\mathbf{G}^{k-1}(h, \mathbf{Z}_1, \ldots, \mathbf{Z}_{k-1})]\mathbf{Z}_k.$$

First, we discuss the first term of this sum. Let $L = \nabla h(\mathbf{Z}_1)_* \ldots (\mathbf{Z}_{k-2})_*$, a row vector. So, the first term is

$$\nabla[L\mathbf{Z}_{k-1}]\mathbf{Z}_k = L(\mathbf{Z}_{k-1})_* \mathbf{Z}_k + \mathbf{Z}_{k-1}^T (L^T)_* \mathbf{Z}_k,$$

where T indicates transpose. Thus, we have

$$L_{\mathbf{Z}_k} \ldots L_{\mathbf{Z}_1} h = \nabla h(\mathbf{Z}_1)_* \ldots (\mathbf{Z}_{k-1})_* \mathbf{Z}_k + \mathbf{G}^k(h, \mathbf{Z}_1, \ldots, \mathbf{Z}_k),$$

where

$$\mathbf{G}^k(h, \mathbf{Z}_1, \ldots, \mathbf{Z}_k) = \underbrace{\mathbf{Z}_{k-1}^T (L^T)_* \mathbf{Z}_k}_{L_1} + \underbrace{\nabla[\mathbf{G}^{k-1}(h, \mathbf{Z}_1, \ldots, \mathbf{Z}_{k-1})]\mathbf{Z}_k}_{L_2}. \tag{36}$$

Now we need to show that $\mathbf{G}^k$ has the right properties. It is clear from the expression of $\mathbf{G}^k$, using rules for differentiation of products, that it is a sum of terms in which each term is a product of $k + 1$ factors. Moreover, the terms arising from L_1 in (36) involve only derivatives of at most order two, and those coming from L_2 involve derivatives of order at most k (because by induction $\mathbf{G}^{k-1}$ has derivatives of order at most $k - 1$). So, we only need to show that in each term there are at least three derivatives of order 0 or 2 and two are of order 0. Each term of L_1 starts and ends in factors of order zero (the ones arising from $\mathbf{Z}_{k-1}$ and $\mathbf{Z}_k$), and exactly one of the factors in between includes a derivative of order 2 because in L there are only first-order derivatives. For the terms coming from L_2, notice that in each term the last factor is of order zero and using the inductive assumption we can conclude that there are two other factors, one of order zero and one of order 0 or 2. ∎

Notice that if $\mathbf{X}_u^i$ is any vector field of the type defined by eq. (30) then

$$(\mathbf{X}_u^i(\mathbf{x}))_* = \mathbf{D}_i + \hat{\sigma}(\mathbf{A}_i\mathbf{x} + \mathbf{B}_i\mathbf{u})\mathbf{A}_i, \quad (37)$$

where for any vector $\mathbf{v} = (v_1, \ldots, v_n)$, $\hat{\sigma}(\mathbf{v})$ denotes the $n \times n$ diagonal matrix

$$\hat{\sigma}(\mathbf{v}) = \mathrm{Diag}(\sigma'(v_1), \ldots, \sigma'(v_n)).$$

LEMMA 5.2. *Let* $\Sigma = (\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{C})_\sigma$ *be a* σ*-system and pick* $u_i \in \mathbb{R}^m$, $i = 1, \ldots, k$. *Denoting by* $\mathbf{C}_j$ *the* jth*-row of* $\mathbf{C}$, *we have that, for all* $j = 1, \ldots, p$

$$L_{\mathbf{X}_{u_k}} \ldots L_{\mathbf{X}_{u_1}}\mathbf{C}_j(\mathbf{x}) = \mathbf{C}_j(\mathbf{D} + \hat{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}u_1)\mathbf{A}) \ldots (\mathbf{D} + \hat{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}u_{k-1})\mathbf{A})(\mathbf{D}\mathbf{x} + \vec{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}u_k) + \mathbf{G}u_k) + f_j^k(\mathbf{x}, u_1, \ldots, u_k), \quad (38)$$

where f_j^k *is a function that is identically zero if* $k = 1$ *or* 2 *and for* $k \geq 3$ *satisfies* $f_j^k(0, 0, \ldots, 0, u, 0, \ldots, 0, \mathbf{v}) = 0$ *for all* $u, \mathbf{v} \in \mathbb{R}^m$.

Proof. We apply the result in Lemma 5.1. By (37), we have that the first term in the right-hand side of (38) is exactly the first term in (33). If $k = 1$ or 2, then we get the desired conclusions from eq. (34) and (35) because any derivative of $\mathbf{C}_j\mathbf{x}$ of order greater than 2 is identically zero. For $k \geq 3$, we only need to prove that

$$\mathbf{G}^k(\mathbf{C}_j\mathbf{x}, \mathbf{X}_{u_1}, \ldots, \mathbf{X}_{u_k}) = 0 \quad (39)$$

at $\mathbf{x} = 0$, when $u_k = \mathbf{v}$ and all the other u_is but one equal zero. Because the second derivative of $\mathbf{C}_j\mathbf{x}$ is zero, $\vec{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}u)|_{\mathbf{x}=0,\mathbf{u}=0} = 0$, and $\sigma''(0) = 0$, all the zero- and second-order derivatives vanish identically for entries of $\mathbf{X}_0$ and $\mathbf{C}$. Thus, (39) holds by Lemma 5.1. ■

Proposition 5.3. Let Σ_1 and Σ_2 be the two σ-systems. If they are i/o equivalent, then for all $l, k \geq 0$ and for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ it holds that

$$\mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^l(\mathbf{D}_1 + \hat{\sigma}(\mathbf{B}_1\mathbf{v})\mathbf{A}_1)(\mathbf{D}_1 + \mathbf{A}_1)^k[\vec{\sigma}(\mathbf{B}_1\mathbf{u}) + \mathbf{G}_1\mathbf{u}] = \mathbf{C}_2(\mathbf{D}_2 + \mathbf{A}_2)^l(\mathbf{D}_2 + \hat{\sigma}(\mathbf{B}_2\mathbf{v})\mathbf{A}_2)(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}]. \quad (40)$$

Proof. The statement follows from Lemma 5.2 applied to the sequence of control values

$$(\underbrace{0, \ldots, 0}_{l}, \mathbf{v}, \underbrace{0, \ldots, 0}_{k}, \mathbf{u})$$

because $\hat{\sigma}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})|_{\mathbf{x}=0,\mathbf{u}=0} = \mathbf{I}$ and $f_j^k(0, 0, \ldots, 0, \mathbf{v}, 0, \ldots, 0, \mathbf{u}) = 0$. ■

Remark 5.4. Let Σ_1 and Σ_2 be as before.

1. Applying (40) to the sequence of control values $(\underbrace{0, \ldots, 0}_{k}, \mathbf{u})$, we have

$$\mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^k[\vec{\sigma}(\mathbf{B}_1\mathbf{u}) + \mathbf{G}_1\mathbf{u}] = \mathbf{C}_2(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}] \quad (41)$$

for all $\mathbf{u} \in \mathbb{R}^m$.

2. Let v_l be the lth-coordinate of $\mathbf{v} \in \mathbb{R}^m$. Then

$$\frac{\partial}{\partial v_l}[\mathbf{C}(\mathbf{D} + \mathbf{A})^k(\vec{\sigma}(\mathbf{B}\mathbf{v}) + \mathbf{G}\mathbf{v})] = \mathbf{C}(\mathbf{D} + \mathbf{A})^k[\hat{\sigma}(\mathbf{B}\mathbf{v})\mathbf{B}_l + \mathbf{G}_l],$$

where $\mathbf{B}_l$ and $\mathbf{G}_l$ are the lth columns of $\mathbf{B}$ and $\mathbf{G}$, respectively. Thus, from the equality (41), and evaluating the previous derivative at $\mathbf{v} = 0$, we have that

$$\mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^k(\mathbf{B}_1 + \mathbf{G}_1) = \mathbf{C}_2(\mathbf{D}_2 + \mathbf{A}_2)^k(\mathbf{B}_2 + \mathbf{G}_2) \quad (42)$$

for all integers $k \geq 0$.

5.2. Finding the Equivalence T

For each fixed positive integer n, m, p, let

$$\mathbf{S}_{n,m,p} = \{(\mathbf{A}, \mathbf{B}, \mathbf{C}) \mid \mathbf{A} \in \mathbf{R}^{n\times n}, \mathbf{B} \in \mathbf{R}^{n\times m}, \mathbf{C} \in \mathbf{R}^{p\times n}\}.$$

We let $\mathbf{S}^c_{n,m,p}$ be the subset of $\mathbf{S}_{n,m,p}$ consisting of those triples $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ that are canonical, that is, observable:

$$\mathrm{rank}[\mathbf{C}^T, \mathbf{A}^T\mathbf{C}^T, \ldots, (\mathbf{A}^T)^{n-1}\mathbf{C}^T] = n$$

and controllable:

$$\mathrm{rank}[\mathbf{B}, \mathbf{A}\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}] = n;$$

see Sontag (1990, sec. 5.5). This is a generic, in the sense of the introduction, subset of $\mathbf{S}_{n,m,p}$, for each n, m, and p.

Proposition 5.5. Assume Σ_1 and Σ_2 are i/o equivalent. Assume, also, that

$$(\mathbf{D}_i + \mathbf{A}_i, \mathbf{B}_i + \mathbf{G}_i, \mathbf{C}_i) \in \mathbf{S}^c_{n_i,m,p}$$

for $i = 1, 2$. Then, $n_1 = n_2 = n$ and there exists a unique matrix $\mathbf{T} \in Gl(n)$ such that

$$\mathbf{C}_2 = \mathbf{C}_1\mathbf{T},$$
$$\mathbf{D}_2 + \mathbf{A}_2 = \mathbf{T}^{-1}(\mathbf{D}_1 + \mathbf{A}_1)\mathbf{T},$$
$$\mathbf{B}_2 + \mathbf{G}_2 = \mathbf{T}^{-1}(\mathbf{B}_1 + \mathbf{G}_1),$$
$$\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u} = \mathbf{T}^{-1}(\vec{\sigma}(\mathbf{B}_1\mathbf{u}) + \mathbf{G}_1\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}^m. \quad (43)$$

Proof. Note that eq. (42) says that the two linear systems given by the triples

$$(\mathbf{D}_i + \mathbf{A}_i, \mathbf{B}_i + \mathbf{G}_i, \mathbf{C}_i), \quad i = 1, 2$$

have the same i/o behavior; thus, because they are both canonical we know by the linear theory that $n_1 = n_2$ and there exists a unique invertible matrix $\mathbf{T}$ that satisfies the first three equations of (43). Thus, in particular, we have

$$\mathbf{C}_2(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}] = \mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^k\mathbf{T}[\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}] \quad \forall k \geq 0, \quad \forall \mathbf{u} \in \mathbb{R}^m.$$

Combined with eq. (41), this gives

$$\mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^k[\vec{\sigma}(\mathbf{B}_1\mathbf{u}) + \mathbf{G}_1\mathbf{u}] = \mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^k\mathbf{T}[\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}], \quad \forall k \geq 0, \quad \forall \mathbf{u} \in \mathbb{R}^m,$$

and thus by the observability assumption applied to the pair $((\mathbf{A}_1 + \mathbf{D}_1), \mathbf{C}_1)$ we have

$$\vec{\sigma}(\mathbf{B}_1\mathbf{u}) + \mathbf{G}_1\mathbf{u} = \mathbf{T}(\vec{\sigma}(\mathbf{B}_2\mathbf{u}) + \mathbf{G}_2\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}^m,$$

as desired. ∎

Remark 5.6. Let v_l be the lth-coordinate of $\mathbf{v} \in \mathbb{R}^m$. Then, by taking the derivative with respect to v_l in the last equation of (43), we have

$$\hat{\sigma}(\mathbf{B}_2\mathbf{v})(\mathbf{B}_2)_l + (\mathbf{G}_2)_l = \mathbf{T}^{-1}(\hat{\sigma}(\mathbf{B}_1\mathbf{v})(\mathbf{B}_1)_l + (\mathbf{G}_1)_l),$$

where, as usual, $(\mathbf{B}_i)_l$ and $(\mathbf{G}_i)_l$, $i = 1, 2$, are the lth columns of $\mathbf{B}$ and $\mathbf{G}$, respectively. This implies

$$\hat{\sigma}(\mathbf{B}_2\mathbf{v})\mathbf{B}_2 + \mathbf{G}_2 = \mathbf{T}^{-1}[\hat{\sigma}(\mathbf{B}_1\mathbf{v})\mathbf{B}_1 + \mathbf{G}_1] \quad \forall \mathbf{v} \in \mathbb{R}^m. \quad (44)$$

Proposition 5.7. Assume that Σ_1, Σ_2 satisfy the same assumptions as in Proposition 5.5. Then, the invertible matrix $\mathbf{T}$ satisfies also the following equation:

$$(\mathbf{D}_1 + \hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{A}_1)\mathbf{T} = \mathbf{T}(\mathbf{D}_2 + \hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{A}_2) \quad \forall \mathbf{u} \in \mathbb{R}^m. \quad (45)$$

Proof. Notice first that by (43) we have

$$\mathbf{C}_2(\mathbf{D}_2 + \mathbf{A}_2)^l = \mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^l\mathbf{T} \quad \forall l \geq 0.$$

Now, we apply (40) to get, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and any $l \geq 0$:

$$\mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^l(\mathbf{D}_1 + \hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{A}_1)(\mathbf{D}_1 + \mathbf{A}_1)^k[\vec{\sigma}(\mathbf{B}_1\mathbf{v}) + \mathbf{G}_1\mathbf{v}] = \mathbf{C}_1(\mathbf{D}_1 + \mathbf{A}_1)^l\mathbf{T}(\mathbf{D}_2 + \hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{A}_2)(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{v}) + \mathbf{G}_2\mathbf{v}].$$

Thus, by the observability assumption on $(\mathbf{A}_1 + \mathbf{D}_1, \mathbf{C}_1)$ we have

$$(\mathbf{D}_1 + \hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{A}_1)(\mathbf{D}_1 + \mathbf{A}_1)^k[\vec{\sigma}(\mathbf{B}_1\mathbf{v}) + \mathbf{G}_1\mathbf{v}] = \mathbf{T}(\mathbf{D}_2 + \hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{A}_2)(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{v}) + \mathbf{G}_2\mathbf{v}].$$

Applying (43) to the left-hand side of this, we have

$$(\mathbf{D}_1 + \hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{A}_1)\mathbf{T}(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{v}) + \mathbf{G}_2\mathbf{v}] = \mathbf{T}(\mathbf{D}_2 + \hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{A}_2)(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{v}) + \mathbf{G}_2\mathbf{v}].$$

Because for all $l = 1, \ldots, m$ we have

$$\frac{\partial}{\partial v_l}(\mathbf{D}_2 + \mathbf{A}_2)^k[\vec{\sigma}(\mathbf{B}_2\mathbf{v}) + \mathbf{G}_2\mathbf{v}]|_{\mathbf{v}=0} = (\mathbf{D}_2 + \mathbf{A}_2)^k[(\mathbf{B}_2)_l + (\mathbf{G}_2)_l],$$

where $(\mathbf{B}_2)_l$ and $(\mathbf{G}_2)_l$ are the lth columns of $\mathbf{B}_2$ and $\mathbf{G}_2$, respectively, we can conclude that, for all $k \geq 0$

$$(\mathbf{D}_1 + \hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{A}_1)\mathbf{T}(\mathbf{D}_2 + \mathbf{A}_2)^k(\mathbf{B}_2 + \mathbf{G}_2) = \mathbf{T}(\mathbf{D}_2 + \hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{A}_2)(\mathbf{D}_2 + \mathbf{A}_2)^k(\mathbf{B}_2 + \mathbf{G}_2).$$

This equation, together with the controllability assumption on $(\mathbf{A}_2 + \mathbf{D}_2, \mathbf{B}_2 + \mathbf{G}_2)$, gives (45). ∎

5.3. Characterization of T

Now, we want to prove that the sufficient condition stated in Proposition 4.4 is also necessary in a generic sense. For this purpose, we first let

$$\mathbf{B}^{\sigma}_{n,m} = \{\mathbf{B} \in \mathbb{R}^{n\times m} \mid b_{i,j} \neq 0 \quad \forall i, j \quad \text{and} \quad \forall i \neq j \, \exists \, k \text{ such that } b_{i,k}/b_{j,k} \notin \Lambda_\sigma\}. \quad (46)$$

Notice that if condition (P) holds, which in turn would be the case if (A2) holds, then the second condition in the definition of $\mathbf{B}^{\sigma}_{n,m}$ says that for all $i \neq j$ there exists k such that $|b_{i,k}| \neq |b_{j,k}|$. As an illustration, if (P) holds and $m = 1$ the conditions defining $\mathbf{B}^{\sigma}_{n,1}$ say that all the components of the vector $\mathbf{B}$ are nonzero and they have different absolute values.

We let

$$\tilde{\mathbf{S}}_{n,m,p} = \left\{(\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{C})_\sigma \,\middle|\, \begin{array}{l} \mathbf{B} \in \mathbf{B}^{\sigma}_{n,m}, \quad \text{rank}[\mathbf{A}, \mathbf{B}] = n \\ \text{and} \quad (\mathbf{A} + \mathbf{D}, \mathbf{B} + \mathbf{G}, \mathbf{C}) \in \mathbf{S}^c_{n,m,p} \end{array}\right\}. \quad (47)$$

Notice that $\tilde{\mathbf{S}}_{n,m,p}$ is a generic subset of the set of all σ-systems, when we identify the latter with $\mathbb{R}^{n^2+n^2+np+2mn}$—cf. Remark 3.3—provided that Λ_σ is finite.

THEOREM 3. *Let* $\Sigma_i = (\mathbf{D}_i, \mathbf{A}_i, \mathbf{B}_i, \mathbf{G}_i, \mathbf{C}_i)_\sigma \in \tilde{\mathbf{S}}_{n_i,m,p}$ *for* $i = 1, 2$, *and assume that* σ *satisfies* $(A1)$. *Then the two systems are i/o equivalent if and only if they are* σ*-equivalent.*

Proof. The sufficiency part is given by Proposition 4.4; thus, we need only to prove the necessary part. From Proposition 5.5, we know that $n_1 = n_2 = n$; let $\mathbf{T}$ be the matrix obtained in that proposition. By eq. (45), we have

$$(\mathbf{D}_1\mathbf{T})_{i,j} - (\mathbf{D}_2\mathbf{T})_{i,j} = -\sigma'\left(\sum_{l=1}^{m} b^1_{i,l}u_l\right)(\mathbf{A}_1\mathbf{T})_{i,j} + \sum_{k=1}^{n} t_{i,k}a^2_{k,j}\sigma'\left(\sum_{l=1}^{m} b^2_{k,l}u_l\right).$$

For a control value of the special form $\mathbf{u} = (\mathbf{v}, 0, \ldots, 0)$, with $\mathbf{v} \in \mathbb{R}$, this equation becomes

$$(\mathbf{D}_1\mathbf{T})_{i,j} - (\mathbf{D}_2\mathbf{T})_{i,j} = -\sigma'(b^1_{i,1}\mathbf{v})(\mathbf{A}_1\mathbf{T})_{i,j} + \sum_{k=1}^{n} t_{i,k}a^2_{k,j}\sigma'(b^2_{k,1}\mathbf{v}). \quad (48)$$

If we take the limit as $\mathbf{v} \to \infty$, then, because all the entries of $\mathbf{B}_1$ and $\mathbf{B}_2$ are nonzero and (A1) holds, the left-hand side of eq. (48) goes to zero. So, we can conclude that

$$\mathbf{D}_2 = \mathbf{T}^{-1}\mathbf{D}_1\mathbf{T}, \quad (49)$$

which in turn implies

$$\mathbf{A}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T}$$

by the second equation in (43). Using, again, the control value $\mathbf{u}$ of the special form $(\mathbf{v}, 0, \ldots, 0)$, eq. (44) becomes

$$(\mathbf{T}\mathbf{G}_2)_{i,j} - (\mathbf{G}_1)_{i,j} = \sigma'(b^1_{i,1}\mathbf{v})(\mathbf{B}_1)_{i,j} - \sum_{k=1}^{n} t_{i,k}b^2_{k,j}\sigma'(b^2_{k,1}\mathbf{v}).$$

Taking the limit as $\mathbf{v} \to \infty$, we have, as before, that the left-hand side of the previous equation goes to zero. Thus

$$\mathbf{T}\mathbf{G}_2 = \mathbf{G}_1,$$

which in turn implies

$$\mathbf{T}\mathbf{B}_2 = \mathbf{B}_1$$

by the third equation in (43). Notice, for further reference, that assumption (A1) is used only in these two parts of the proof.

Thus, the matrix $\mathbf{T}$ satisfies all the desired interlacing equations. It remains to show that $\mathbf{T} \in \Lambda_\sigma^n$, for which it suffices to show that $\mathbf{T}$ is of the form $\mathbf{PQ}$, where $\mathbf{P}$ and $\mathbf{Q}$ are defined, respectively, as in (23) and (25). From (45) and (49), we get

$$\hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{T}\mathbf{A}_2 = \mathbf{T}\hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{A}_2 \quad \forall \mathbf{u} \in \mathbb{R}^m. \qquad (50)$$

On the other hand, because we have proved that $\mathbf{G}_2 = \mathbf{T}^{-1}\mathbf{G}_1$ the last equation in (43) becomes

$$\vec{\sigma}(\mathbf{B}_2\mathbf{u}) = \mathbf{T}^{-1}\vec{\sigma}(\mathbf{B}_1\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}^m. \qquad (51)$$

Taking the partial derivative with respect to u_l in the previous equation, we conclude, for all $\mathbf{u} \in \mathbb{R}^m$:

$$\hat{\sigma}(\mathbf{B}_2\mathbf{u})(\mathbf{B}_2)_l = \mathbf{T}^{-1}\hat{\sigma}(\mathbf{B}_1\mathbf{u})(\mathbf{B}_1)_l,$$

where $(\mathbf{B}_i)_l$ indicates the lth column of $\mathbf{B}_i$ and hence also $\hat{\sigma}((\mathbf{B}_1\mathbf{u})\mathbf{B}_1 = \mathbf{T}\hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{B}_2$. Thus, using the other interlacing equations, we conclude

$$\hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{T}\mathbf{B}_2 = \mathbf{T}\hat{\sigma}(\mathbf{B}_2\mathbf{u})\mathbf{B}_2 \qquad (52)$$

for all $\mathbf{u} \in \mathbb{R}^m$. From (50) and (52), we have

$$\hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{T}[\mathbf{A}_2, \mathbf{B}_2] = \mathbf{T}\hat{\sigma}(\mathbf{B}_2\mathbf{u})[\mathbf{A}_2, \mathbf{B}_2].$$

Thus, by the full rank assumption on the matrix $[\mathbf{A}_2, \mathbf{B}_2]$, we can conclude

$$\hat{\sigma}(\mathbf{B}_1\mathbf{u})\mathbf{T} = \mathbf{T}\hat{\sigma}(\mathbf{B}_2\mathbf{u}), \quad \forall \mathbf{u} \in \mathbb{R}^m. \qquad (53)$$

We can rephrase (53) as follows:

$$\sigma'\left(\sum_{l=1}^{m} b_{i,l}^1 u_l\right) t_{i,j} = \sigma'\left(\sum_{l=1}^{m} b_{j,l}^2 u_l\right) t_{i,j}.$$

By taking $\mathbf{u}$ of the special form $w_k = (0, \ldots, 0, \mathbf{v}, 0, \ldots, 0)$ ($\mathbf{v} \in \mathbb{R}$ in the kth-position), we conclude, for each pair i, j

$$t_{i,j} \neq 0 \Rightarrow \sigma'(b_{i,k}^1 v) = \sigma'(b_{j,k}^2 v) \qquad \forall k = 1, \ldots, m, \quad \forall \mathbf{v} \in \mathbb{R}. \quad (54)$$

Because $\sigma(0) = 0$, we may integrate and conclude that

$$b_{j,k}^2 \sigma(b_{i,k}^1 \mathbf{v}) = b_{i,k}^1 \sigma(b_{j,k}^2 \mathbf{v})$$

for any such i, j and for all $\mathbf{v}$, or equivalently that $\sigma(s\mathbf{x}) = s\sigma(\mathbf{x})$ for all $\mathbf{x}$, where $s = b_{j,k}^2 / b_{i,k}^1$, that is,

$$t_{i,j} \neq 0 \Rightarrow b_{j,k}^2 / b_{i,k}^1 \in \Lambda_\sigma \quad \forall k = 1, \ldots, m. \qquad (55)$$

For $i = 1, \ldots, n$, let

$$L_i = \{j \mid t_{i,j} \neq 0\};$$

because $\mathbf{T}$ is invertible, each $L_i \neq \varnothing$. Moreover, if $L_i \cap L_j \neq \varnothing$ for some pair $i \neq j$, then we have (using any $k \in L_i \cap L_j$)

$$\frac{b_{k,l}^2}{b_{i,l}^1} \in \Lambda_\sigma \quad \text{and} \quad \frac{b_{k,l}^2}{b_{j,l}^1} \in \Lambda_\sigma \quad \forall l = 1, \ldots, m.$$

Because Λ_σ is a subgroup, the previous equation implies

$$\frac{b_{i,l}^1}{b_{j,l}^1} = \frac{b_{i,l}^1}{b_{k,l}^2}\frac{b_{k,l}^2}{b_{j,l}^1} \in \Lambda_\sigma \quad \forall l = 1, \ldots, m,$$

which contradicts the assumption that $\mathbf{B}_1 \in \mathbf{B}_{n,m}^\sigma$. So, we conclude that L_i consists of only one element; thus, for each $i = 1, \ldots, n$ there exists only one index $j = \pi(i)$ such that $t_{i,\pi(i)} \neq 0$. Let $\mathbf{P}$ be the permutation matrix representing π; then, we can write the matrix $\mathbf{T}$ as follows:

$$\mathbf{T} = \mathbf{PQ},$$

where $\mathbf{Q} = \mathrm{Diag}(t_{1,\pi(1)}, \ldots, t_{n,\pi(n)})$.

To complete the proof, we need to see only that $t_{i,\pi(i)} \in \Lambda_\sigma$ for all $i = 1, \ldots, n$. From eq. (51) and because $\mathbf{B}_1 = \mathbf{T}\mathbf{B}_2$, we have that, for each $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$.

$$\sigma\left(\sum_{l=1}^{m} t_{i,\pi(i)} b_{\pi(i),l}^2 u_l\right) = t_{i,\pi(i)}\sigma\left(\sum_{l=1}^{m} b_{\pi(i),l}^2 u_l\right), \quad \text{for each} \quad i = 1, \ldots, n.$$

Again, using $\mathbf{u}$ of the special type $w_k = (0, \ldots, 0, \mathbf{v}, 0, \ldots, 0)$ ($\mathbf{v} \in \mathbb{R}$ in the kth-position) we get

$$\sigma(t_{i,\pi(i)} b_{\pi(i),k}^2 \mathbf{v}) = t_{i,\pi(i)}\sigma(b_{\pi(i),k}^2 \mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{R}.$$

As $b_{\pi(i),k}^2 \neq 0$, this means $t_{i,\pi(i)} \in \Lambda_\sigma$ as desired. ■

Remark 5.8. We can conclude the same results if only we require that the function σ be infinitely differentiable in a neighborhood of zero, instead of on all of $\mathbb{R}$, and if we change (A1) into

$$\sigma \in \mathbf{C}^1(\mathbb{R}) \quad \text{and} \quad \sigma'(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty. \qquad (\text{A1}')$$

For the case $\mathbf{D} = \alpha\mathbf{I}$ mentioned in Remarks 5.4.1 and 5.4.2, just smoothness about $\mathbf{x} = 0$ is sufficient.

Remark 5.9. Assume given n different functions σ_i that satisfy our assumptions (∗) instead of a fixed σ. Define

$$\vec{\sigma}(\mathbf{x}) = (\sigma_1(x_1), \ldots, \sigma_n(x_n)).$$

If $\sigma_i'(0) = \beta \neq 0$ for all i, for some fixed β, then all our results still hold for two generic systems of the same dimension.

5.4. Special Cases

We show in this section how to deal with a few special cases of the general class of systems considered.

5.4.1. *When D is Diagonal and $G = 0$.* Assume now that we fix an $\alpha \in \mathbb{R}$ and restrict our attention to the subclass of systems of the form

$$\Sigma = (\alpha\mathbf{I}, \mathbf{A}, \mathbf{B}, 0, \mathbf{C})_\sigma,$$

where α is this fixed real number (the same for all members of the class). In this case, the result of Theorem 3 applies *even if we drop the assumption* (A1). In fact, from eq. (43), because $\mathbf{D}_1 = \mathbf{D}_2 = \alpha\mathbf{I}$, and $\mathbf{G}_1 = \mathbf{G}_2 = 0$, we can conclude, without using assumption (A1), that

$$\mathbf{A}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T}, \quad \mathbf{B}_1 = \mathbf{T}\mathbf{B}_2, \quad \text{and} \quad \vec{\sigma}(\mathbf{B}_1\mathbf{u}) = \mathbf{T}\vec{\sigma}(\mathbf{B}_2\mathbf{u}) \quad \forall \mathbf{u} \in \mathbb{R}^m.$$

As assumption (A1) was used only in this part of the proof, our statement follows. In particular, this applies for $\alpha = 0$, which gives the class of systems in eq. (7), or $\alpha = -1$, which gives the class in eq. (4).

Observe that in this case ($\mathbf{D}$ = diagonal and $\mathbf{G} = 0$) the assumption rank $[\mathbf{A}, \mathbf{B}] = n$ is redundant as it follows from controllability of the pair $(\mathbf{A} + \alpha\mathbf{I}, \mathbf{B})$ or, equivalently, of the pair $(\mathbf{A}, \mathbf{B})$; this is just the case $\lambda = -\alpha$ of the Hautus condition (cf. Sontag, 1990, Lemma 3.3.7).

5.4.2. *When D is diagonal and $B = 0$.* Assume that we fix again an $\alpha \in \mathbb{R}$ and this time restrict our attention to the subclass of systems of the form

$$\Sigma = (\alpha\mathbf{I}, \mathbf{A}, 0, \mathbf{G}, \mathbf{C})_\sigma,$$

where α is the fixed real number (the same for all members of the class). Notice that when $\alpha = -1$ this class of systems is the class described in eq. (5). A typical system Σ of this class is of the type

$$\dot{x} = \alpha\mathbf{x} + \sigma(\mathbf{A}\mathbf{x}) + \mathbf{G}\mathbf{u}$$
$$y = \mathbf{C}\mathbf{x} \tag{56}$$

In this case, we cannot apply the results of Theorem 3 directly because for this class $\mathbf{B} = 0$ and the zero matrix does not belong to $\mathbf{B}^\sigma_{n,m}$, so we argue as follows.

Given Σ, as before, we consider the following system $\tilde{\Sigma}$:

$$\dot{z} = \alpha\mathbf{z} + \sigma(\mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{G}\mathbf{u})$$
$$w = \mathbf{C}\mathbf{z} \tag{57}$$

The next Lemma is easily checked.

LEMMA 5.10. *Let $\mathbf{u}(t)$ be an input function defined on $[0, \mathrm{T}]$ with $\mathbf{u} \in \mathbf{C}^1[0, \mathrm{T}]$ and $u(0) = 0$. If $\mathbf{z}(t)$ is the trajectory of $\tilde{\Sigma}$ corresponding to $\mathbf{u}(t)$, with $\mathbf{z}(0) = 0$, then*

$$\mathbf{x}(t) = \mathbf{z}(t) + \mathbf{G}\mathbf{u}(t)$$

is the trajectory of Σ corresponding to the input $-\alpha\mathbf{u}(t) + \dot{u}(t)$, with $\mathbf{x}(0) = 0$. ■

Let $\mathbf{A}$ be the following class of control functions:

$$\mathbf{A} = \{\mathbf{v} \mid \mathbf{v} \in \mathbf{C}^1[0, \mathrm{T}] \quad \text{for some} \quad \mathrm{T} > 0, \quad \mathbf{v}(0) = 0\}.$$

LEMMA 5.11. *If Σ_1 and Σ_2 are two σ-systems that are not i/o equivalent, then there exists $\mathbf{v} \in \mathbf{A}$ and $t > 0$ such that, denoting by y_i, $i = 1, 2$ the respective output functions corresponding to $\mathbf{v}$, $y_1(t) \neq y_2(t)$.*

Proof. Given an admissible control $\mathbf{u}(\cdot)$, we can find a sequence $u_n(\cdot) \in \mathbf{A}$ such that the controls $u_n(\cdot)$ are equibounded and converge to $\mathbf{u}(\cdot)$ almost everywhere. Now, we need only apply the approximation results in Theorem 1 of Sontag (1990) to conclude the desired result. ■

Let Σ_1 and Σ_2 be two systems of type (56) and $\tilde{\Sigma}_i$ for $i = 1, 2$ their corresponding systems of type (57).

Proposition 5.12. If Σ_1 and Σ_2 are i/o equivalent, then $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are also i/o equivalent.

Proof. We first notice that if Σ_1 and Σ_2 are i/o equivalent then, for any value $\mathbf{v} \in \mathbb{R}^m$, we have

$$\mathbf{C}_1\mathbf{G}_1\mathbf{v} = \mathbf{C}_2\mathbf{G}_2\mathbf{v}. \tag{58}$$

Indeed, let $\mathbf{u}(t) = \mathbf{v}t$ be a linear-in-time control function. Then, for t small enough this control is admissible for both Σ_1 and Σ_2. Moreover, denoting by $y_{i2}(\cdot)$ the respective output functions we have

$$\dot{y}_i(0) = \mathbf{C}_i\mathbf{G}_i\mathbf{v},$$

which, given the i/o equivalence, implies (58).

We now prove the statement by way of contradiction. Assume that $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are not i/o equivalent. Then, by Lemma 5.11 there exist $\mathbf{u}(\cdot) \in \mathbf{A}$ and $\bar{t} > 0$ such that, denote by $\tilde{y}_i(\cdot)$ the respective output functions corresponding to $\mathbf{u}(\cdot)$, we have

$$\tilde{y}_1(\bar{t}) \neq \tilde{y}_2(\bar{t}). \tag{59}$$

Because $\mathbf{u}(\cdot) \in \mathbf{A}$, by Lemma 5.10 we have that $x_i(t) = z_i(t) + \mathbf{G}_i\mathbf{u}(t)$ for $i = 1, 2$, are the trajectories in Σ_i corresponding to the control $-\alpha\mathbf{u}(\cdot) + \dot{u}(\cdot)$. Moreover, their corresponding outputs are

$$y_i(t) = \tilde{y}_i(t) + \mathbf{C}_i\mathbf{G}_i\mathbf{u}(t). \tag{60}$$

From the previous equation, and from eqs. (58) and (59), we have

$$y_1(\bar{t}) \neq y_2(\bar{t}),$$

which contradicts the fact that the systems Σ_1 are i/o equivalent. ■

The proof in this case will rely upon the transformation

$$\Sigma = (\alpha\mathbf{I}, \mathbf{A}, 0, \mathbf{G}, \mathbf{C})_\sigma \to \tilde{\Sigma} = (\alpha\mathbf{I}, \mathbf{A}, \mathbf{A}\mathbf{G}, 0, \mathbf{C})_\sigma.$$

This will require reinterpreting the genericity conditions for the latter system in terms of the former one. From this purpose, we first observe that for any two matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{G} \in \mathbb{R}^{n\times m}$ it holds that:

1. rank $[\mathbf{A}, \mathbf{AG}] = n$ if and only if $\mathbf{A}$ is invertible.
2. $(\mathbf{A}, \mathbf{AG})$ is a controllable pair if and only if $\mathbf{A}$ is invertible and the pair $(\mathbf{A}, \mathbf{G})$ is controllable.

The first property follows from the fact that $[\mathbf{A}, \mathbf{AG}] = \mathbf{A}[\mathbf{I}, \mathbf{G}]$, while the second assertion follows from $[\mathbf{AG}, \mathbf{A}^2\mathbf{G}, \ldots, \mathbf{A}^n\mathbf{G}] = \mathbf{A}[\mathbf{G}, \mathbf{AG}, \ldots, \mathbf{A}^{n-1}\mathbf{G}]$.

Now, let

$$\hat{\mathbf{S}}_{n,m,p} = \left\{ (\alpha\mathbf{I}, \mathbf{A}, 0, \mathbf{G}, \mathbf{C})_\sigma \middle| \begin{matrix}\mathbf{AG} \in \mathbf{B}^\sigma_{n,m}, \mathbf{A} \text{ invertible,} \\ \text{and } (\mathbf{A}, \mathbf{G}, \mathbf{C}) \in \mathbf{S}^c_{n,m,p}\end{matrix} \right\}.$$

Theorem 4. *Let* $\Sigma_i = (\alpha\mathbf{I}, \mathbf{A}_i, 0, \mathbf{G}_i, \mathbf{C}_i)_\sigma \in \hat{\mathbf{S}}_{n_i,m,p}$ *for* $i = 1, 2$. *Then, the two systems are i/o equivalent if and only if they are σ-equivalent.*

Proof. We need only to prove the necessary part. By Proposition 5.12, if Σ_1 and Σ_2 are i/o equivalent then their corresponding systems $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ of type (57) are also i/o equivalent. It is easy to see that the condition $\Sigma_i \in \hat{\mathbf{S}}_{n_i,m,p}$ implies $\tilde{\Sigma}_i \in \tilde{\mathbf{S}}_{n_i,m,p}$. So, we can apply Theorem 3 [notice that this theorem holds without assumption (A1) by the remark in Section 5.4.1] and conclude that $n_1 = n_2 = n$, and there exists $\mathbf{T} \in \Lambda^n_\sigma$ such that

$$\mathbf{C}_2 = \mathbf{C}_1\mathbf{T},$$
$$\mathbf{A}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T},$$
$$\mathbf{A}_2\mathbf{G}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{G}_1.$$

Because the matrices $\mathbf{A}_i$ are invertible, the last two equations imply

$$\mathbf{G}_2 = \mathbf{T}^{-1}\mathbf{G}_1,$$

as desired. ■

5.4.3. *Yet Another Class.*. Under a change of coordinates $\mathbf{z} = \mathbf{Ax}$, eq. (56) becomes

$$\dot{z} = \alpha\mathbf{z} + \mathbf{A}\vec{\sigma}(\mathbf{z}) + \mathbf{G}'\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}'\mathbf{z}, \tag{61}$$

where $\mathbf{G}' = \mathbf{AG}$ and $\mathbf{C}' = \mathbf{CA}^{-1}$ (assuming invertibility of $\mathbf{A}$). This type of system appears often in the literature as well. Again, the uniqueness result holds; a genericity condition can be derived from the above change of coordinates:

$$\mathbf{G}' \in \mathbf{B}^\sigma_{n,m}, \quad \mathbf{A} \text{ invertible, and } \quad (\mathbf{A}, \mathbf{G}', \mathbf{C}') \in \mathbf{S}^c_{n,m,p}.$$

Several other classes of systems can also be treated by coordinate transformations on the classes considered here.

6. PROOF OF MAIN RESULTS

Theorems 1 and 2 are now proved as follows.

First observe that because property (P) holds, and using Lemma 4.2, every element of Λ^n_σ is of the form $\mathbf{PQ}$, with $\mathbf{P}$ a permutation matrix and $\mathbf{Q}$ a diagonal matrix with entries in $\{-1, 1\}$, and:

- When σ is odd, Λ^n_σ is precisely the set of such elements.
- When σ is not odd, $-1 \notin \Lambda_\sigma$, so Λ^n_σ is exactly the set of permutation matrices.

The results now follow from Theorem 3, via the remarks in Sections 5.4.1 and 5.4.2. Notice that for Theorem 1 one can take

$$\mathcal{S}_1(n, m, p) = \mathcal{S}_3(n, m, p) = \{(\mathbf{A}, \mathbf{B}, \mathbf{C}) \mid \mathbf{B} \in \mathbf{B}^\sigma_{n,m} \text{ and } (\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathbf{S}^c_{n,m,p}\}$$

and

$$\mathcal{S}_2(n, m, p) = \{(\mathbf{A}, \mathbf{B}, \mathbf{C}) \mid \mathbf{AB} \in \mathbf{B}^\sigma_{n,m}, \mathbf{A} \text{ invertible, and } (\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathbf{S}^c_{n,m,p}\}.$$

Moreover, here $\mathbf{B}^\sigma_{n,m}$ is just the class of matrices $\mathbf{B}$ for which

1. $b_{i,j} \neq 0$ for all i, j,
2. for each $i \neq j$, there exists some k such that $|b_{i,k}| \neq |b_{j,k}|$.

For Theorem 2, the sets can be taken in the same way except that now the set $\mathbf{B}^\sigma_{n,m}$ consists of the matrices $\mathbf{B}$ that satisfy that all entries are nonzero and this property holds: For each $i \neq j$, there exists some k such that $b_{i,k} \neq b_{j,k}$.

7. UNIVERSAL INPUTS FOR IDENTIFICATION

The results in this article imply that i/o experiments completely determine the internal weights of networks (up to a relabeling of units and sign reversals.) This means that if two nets are not equal then there is some input to which they respond differently. Given any pair of distinct nets—and assuming the genericity conditions apply to both—there is some such diagnostic input, which depends upon the pair that has to be distinguished. In principle, however, it may have been the case that there is no "universal" input that, given *any* pair of distinct nets, serves to distinguish this particular pair. It turns out, however, that such universal testing inputs do exist, and in fact inputs with such properties are "generic" in a precise mathematical sense. This is summarized in the following statement.

We assume that σ is an analytic function and that its derivative is bounded. (The last condition is imposed simply to obtain a statement not involving domains of definition of solutions of differential equations and can be relaxed; the analyticity assumption is critical, on the other hand.) For each input $\mathbf{u}(\cdot) : [0, \mathbf{T}] \to \mathbb{R}^m$, consider the mapping

$$\psi_\mathbf{u} : \Sigma \mapsto \mathbf{y}(\cdot)$$

where $\mathbf{y}(\cdot) : [0, \mathbf{T}] \to \mathbb{R}^p$ is the (zero-initial-state) output of Σ when the input $\mathbf{u}(\cdot)$ is applied to the system. We view the domain of $\psi_\mathbf{u}$ as the union of the sets $\mathcal{S}_1(n, m, p)$ over all positive integers n, that is, the set of all systems of the first class and with m input channels and p outputs. (An entirely analogous result holds for all the other classes considered in this article as long as the above assumptions on σ hold.)

THEOREM 5. *Let* $T > 0$ *be arbitrary. Then, there exists an input* $\mathbf{u} : [0, \mathbf{T}] \to \mathbb{R}^m$ *such that* $\psi_{\mathbf{u}}$ *is one-to-one.*

This result can be interpreted as saying that it is theoretically possible, just from the information contained in the output $\mathbf{y}(\cdot)$, and from the assumption that the system is of the class considered, to uniquely determine all the weights in the network. It does not say anything about actual algorithms for determining the weights from this output data; of course, this latter one is an issue for further research. Moreover, the class of infinitely differentiable inputs satisfying the conclusions of the theorem can be shown to be generic in the sense that this class contains a countable intersection of open dense subsets, in the standard $\mathbf{C}^\infty$ topology, of the set of all smooth controls $\mathbf{u} : [0, \mathbf{T}] \to \mathbb{R}^m$, and there are even analytic inputs with this universality property.

The proof of the above theorem is immediate from the general results for control systems given in Sontag (1979) and Sussmann (1979), which imply that identifiability is equivalent to "single experiment" identifiability, for systems defined by analytic differential equations and depending analytically upon parameters (here, the weights). We omit details of the application as this would involve introducing considerable extra terminology, but the proof is totally routine (one proves the result for each fixed n and then takes the intersection of the set of all good inputs over all n, which gives still a generic set.)

8. LESS KNOWLEDGE OF FUNCTION

It is interesting to point out that the nontrivial (necessity) parts of the results presented in this article often do not really require precise knowledge of the nonlineary σ but merely of some of its derivatives—or, as discussed later, slightly weaker results hold even with almost no information at all. In this section, we sketch these generalizations.

Assume that we have two infinitely differentiable nonlinear functions σ_1, and σ_2, such that

$$\sigma_i(0) = \sigma_i''(0) = 1 - \sigma_i'(0) = 0 \quad \text{for} \quad i = 1, 2 \tag{62}$$

and they satisfy assumption A2 with the same $q > 2$, that is,

$$\exists\, q > 2 \quad \text{such that} \quad \sigma_i^{(q)}(0) = \alpha \neq 0. \tag{63}$$

Notice that we have assumed, in particular, that $\sigma_1^{(q)}(0) = \sigma_2^{(q)}(0)$. Under these assumptions, it is easy to see that if $\sigma_1(a\mathbf{x}) = a\sigma_2(\mathbf{x})$ for all $\mathbf{x}$ in a neighborhood of zero then $a \in \{\pm 1, 0\}$.

We let

$$\Lambda^n = \left\{ \mathbf{T} \in Gl(n) \,\middle|\, \mathbf{T} = \mathbf{PQ}, \text{ where } \begin{array}{l} \mathbf{P} \text{ is a permutation matrix, and} \\ \mathbf{Q} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n) \quad \text{with} \quad \lambda_i = \pm 1 \end{array} \right\}. \tag{64}$$

It can be proved that, arguing as in Lemma 4.2,

$$\vec{\sigma}_1(\mathbf{Tx}) = \mathbf{T}\vec{\sigma}_2(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n \quad \text{implies} \quad \mathbf{T} \in \Lambda^n.$$

We denote by $\mathbf{F}_{\alpha,q}$ the set of all infinitely differentiable functions that satisfy the conditions expressed in (62) and (63) and also assumption A1.

Now, we give a definition that parallels Definition 4.3.

DEFINITION 8.1. *Let* $\Sigma_1 = (\mathbf{D}_1, \mathbf{A}_1, \mathbf{B}_1, \mathbf{G}_1, \mathbf{C}_1)_{\sigma_1}$, $\Sigma_2 = (\mathbf{D}_2, \mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2, \mathbf{C}_2)_{\sigma_2}$ *be two systems* n_1, n_2 *be the dimensions of the state spaces of* Σ_1, Σ_2, *respectively. We say that* Σ_1 *and* Σ_2 *are equivalent if* $n_1 = n_2 = n$ *and if there exists an invertible matrix* $\mathbf{T} \in \Lambda^n$ *such that*

$$\mathbf{A}_2 = \mathbf{T}^{-1}\mathbf{A}_1\mathbf{T},$$
$$\mathbf{D}_2 = \mathbf{T}^{-1}\mathbf{D}_1\mathbf{T},$$
$$\mathbf{C}_2 = \mathbf{C}_1\mathbf{T},$$
$$\mathbf{B}_2 = \mathbf{T}^{-1}\mathbf{B}_1,$$
$$\mathbf{G}_2 = \mathbf{T}^{-1}\mathbf{G}_1.$$

Given the previous definition, we would like to conclude that (again generically) two systems that are i/o equivalent must be necessarily equivalent. Notice that all the results presented in Sections 5.1 and 5.2 are based upon the properties expressed by eq. (62) for the functions σ_i. Thus, it is easy to see that those results, in particular the equalities expressed by eqs. (40), (43), and (45), hold if we substitute σ, $\vec{\sigma}$, and $\hat{\sigma}$ with σ_i, $\vec{\sigma}_i$, and $\hat{\sigma}_i$ in a consistent way. Now, we fix two values for the parameters α and q of eq. (63) and define two sets that will play the same role as the sets $\mathbf{B}^{\sigma}_{n,m}$ and $\tilde{\mathbf{S}}_{n,m,p}$ defined in eqs. (46) and (47), respectively. We let

$$\mathbf{B}_{n,m} = \{\mathbf{B} \in \mathbb{R}^{n\times m} \,|\, b_{i,j} \neq 0 \quad \forall i, j \quad \text{and} \quad \forall i \neq j\ \exists\, k \quad \text{such that} \quad b_{i,k}/b_{j,k} \neq \pm 1 \tag{65}$$

and

$$\mathbf{S}_{n,m,p} = \left\{ (\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{C})_\sigma \,\middle|\, \begin{array}{c} \sigma \in \mathbf{F}_{\alpha,q} \\ \mathbf{B} \in \mathbf{B}_{n,m},\ \mathrm{rank}[A, B] = n \\ \text{and } (\mathbf{A} + \mathbf{D}, \mathbf{B} + \mathbf{G}, \mathbf{C}) \in \mathbf{S}^c_{n,m,p} \end{array} \right\}. \tag{66}$$

Given these definitions, we can now prove the following result, which corresponds to the result presented in Theorem 3 for the case of only one function σ. (Of course, now there is no reason for equivalence to imply i/o equivalence, as the nonlinearities may be different.)

THEOREM 6. *If the systems* $\Sigma_i = (\mathbf{D}_i, \mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i)_{\sigma_i} \in \mathbf{S}_{n_i,m,p}$ *for* $i = 1, 2$ *are i/o equivalent, then they are equivalent. Moreover, either* $\sigma_1(\mathbf{x}) = \sigma_2(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}$, *or* $\sigma_1(\mathbf{x}) = -\sigma_2(-\mathbf{x}) \ \forall \mathbf{x} \in \mathbb{R}$.

Proof. We will only give a brief sketch of the proof, describing only the critical steps, following the proof of Theorem 3. Using the same arguments as in the mentioned proof, we can conclude that $n_1 = n_2 = n$ and there exists an invertible matrix $\mathbf{T}$ that satisfies all the desired interlacing equations. Thus, it remains to show that $\mathbf{T} \in \Lambda^n$. Notice that the following equation holds:

$$\hat{\sigma}_1(\mathbf{B}_1\mathbf{u})\mathbf{T} = \mathbf{T}\hat{\sigma}_2(\mathbf{B}_2\mathbf{u}), \quad \forall \mathbf{u} \in \mathbb{R}^m.$$

So, we find

$$t_{i,j} \neq 0 \Longrightarrow \sigma_1(s\mathbf{v}) = s\sigma_2(\mathbf{v}),$$

where $s = b^2_{j,k}/b^1_{i,k}$, and the previous equation holds for all $k = 1, \ldots, m$, and $\forall \mathbf{v} \in \mathbb{R}$. Thus, we have

$$t_{i,j} \neq 0 \Rightarrow b^2_{j,k}/b^1_{i,k} = \pm 1 \quad \forall k = 1, \ldots, m.$$

From now on, we argue as in the last part of the proof of Theorem 3. For $i = 1, \ldots, n$, let

$$L_i = \{ j \mid t_{i,j} \neq 0 \};$$

because $\mathbf{T}$ is invertible, each $L_i \neq \emptyset$. Moreover, if $L_i \cap L_j \neq 0$ for some pair $i \neq j$ then we have, (using any $k \in L_i \cap L_j$)

$$\frac{b^2_{k,l}}{b^1_{i,l}} = \pm 1 \quad \text{and} \quad \frac{b^2_{k,l}}{b^1_{j,l}} = \pm 1 \quad \forall l = 1, \ldots, m.$$

Clearly, the previous equation implies

$$\frac{b^1_{i,l}}{b^1_{j,l}} = \pm 1 \quad \forall l = 1, \ldots, m,$$

which contradicts the assumption that $\mathbf{B}_1 \in \mathbf{B}_{n,m}$. So, we conclude that L_i consists of only one element; thus, for each $i = 1, \ldots, n$ there exists only one index $j = \pi(i)$ such that $t_{i,\pi(i)} \neq 0$. Let $\mathbf{P}$ be the permutation matrix representing π; then, we can write the matrix $\mathbf{T}$ as follows:

$$\mathbf{T} = \mathbf{P}\mathbf{Q},$$

where $\mathbf{Q} = \mathrm{Diag}(t_{1,\pi(1)}, \ldots, t_{n,\pi(n)})$.

To complete the proof, we need to see that $t_{i,\pi(i)} = \pm 1$ for all $i = 1, \ldots, n$. Notice that we have

$$\sigma_1(t_{i,\pi(i)} b^2_{\pi(i),k}\mathbf{v}) = t_{i,\pi(i)}\sigma_2(b^2_{\pi(i),k}\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{R}.$$

As $b^2_{\pi(i),k} \neq 0$, this means $t_{i,\pi(i)} = \pm 1$ as desired. Notice that this last equation says, in particular, that

$$\sigma_1(t_{i,\pi(i)}\mathbf{x}) = t_{i,\pi(i)}\sigma_2(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}. \tag{67}$$

Thus, we can have these following three cases:

1. if $t_{i,\pi(i)} = 1$ for all i, then $\sigma_1(\mathbf{x}) = \sigma_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}$;
2. if $t_{i,\pi(i)} = -1$ for all i, then $\sigma_1(\mathbf{x}) = -\sigma_2(-\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}$.
3. if the $t_{i,\pi(i)}$s take both values ± 1, then, necessarily, $\sigma_1(\mathbf{x}) = \sigma_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}$; moreover, in this case we can also conclude that σ_1 must be an odd function. ■

8.1. No Information on Derivatives

Finally, we point out that even with no knowledge of derivatives partial results can be obtained. Indeed, assume we are given a function σ satisfying the following properties:

$$\begin{aligned} \sigma(0) &= \sigma''(0) = 0 \\ \sigma'(0) &= a > 0 \\ \sigma^q(0) &= b > 0, \end{aligned} \tag{68}$$

together with assumption A1.

Let $\tilde{\sigma}(t) := r\sigma(st)$, where $s^{q-1} = a/b$ and $r = 1/as$. Then, $\tilde{\sigma} \in \mathbf{F}_{1,q}$. If $\Sigma = (\mathbf{D}, \mathbf{A}, \mathbf{B}, \mathbf{G}, \mathbf{C})_\sigma$ is a σ-system, then the $\tilde{\sigma}$-system with the following matrices:

$$\tilde{\mathbf{D}} = \mathbf{D}, \quad \tilde{\mathbf{A}} = \frac{1}{rs}\mathbf{A}, \quad \tilde{\mathbf{B}} = \frac{1}{s}\mathbf{B},$$

$$\tilde{\mathbf{G}} = r\mathbf{G}, \quad \tilde{\mathbf{C}} = \frac{1}{r}\mathbf{C}$$

has the same i/o behavior.

Assume now, that we are given two systems Σ_i, with $i = 1, 2$, and with functions σ_i satisfying the assumptions of eq. (68) (now with possibly different parameters a and b). Then, the following proposition is a direct consequence of Theorem 6 and of the previous observation.

Proposition 8.2. Let $\Sigma_i = (\mathbf{D}_i, \mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i)_{\sigma_i}$ be two systems as before. Assume, for $i = 1, 2$, that

$$\begin{cases} \dfrac{1}{s_i}\mathbf{B}_i \in \mathbf{B}_{n_i,m}, \\ \mathrm{rank}\left[\dfrac{1}{r_i}\mathbf{A}_i, \mathbf{B}_i\right] = n_i, \\ \left(\dfrac{1}{s_i r_i}\mathbf{A}_i + \mathbf{D}_i, \dfrac{1}{s_i}\mathbf{B}_i + r_i\mathbf{G}_i, \dfrac{1}{r_i}\mathbf{C}_i\right) \in \mathbf{S}^c_{n_i,m,p}. \end{cases}$$

If the two systems are i/o equivalent, then $n_1 = n_2 = n$ and there exists a matrix $\mathbf{T} \in \Lambda^n$ such that

$$\mathbf{A}_2 = \frac{s_2 r_2}{s_1 r_1}\mathbf{T}^{-1}\mathbf{A}_1\mathbf{T},$$

$$\mathbf{D}_2 = \mathbf{T}^{-1}\mathbf{D}_1\mathbf{T},$$

$$\mathbf{C}_2 = \frac{r_2}{r_1}\mathbf{C}_1\mathbf{T},$$

$$\mathbf{B}_2 = \frac{s_2}{s_1}\mathbf{T}^{-1}\mathbf{B}_1,$$

$$\mathbf{G}_2 = \frac{r_2}{r_1}\mathbf{T}^{-1}\mathbf{G}_1.$$

Moreover, either $\sigma_1(\mathbf{x}) = (r_2/r_1)\sigma_2([s_2/s_1]\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}$ or $\sigma_1(\mathbf{x}) = -(r_2/r_1)\sigma_2(-[s_2/s_1]\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}$. ■

9. CONCLUSIONS

We proved that, generally on nets, the i/o behavior uniquely determines the internal form, up to simple symmetries. The sets where this conclusion does not hold are "thin" in the sense that they are included in sets defined by algebraic equalities. (Note that the particular sets $\tilde{\mathcal{S}}(n, m, p)$ that we construct may be slightly larger than strictly needed for the conclusions to hold.)

The results imply unique identifiability of parameters under all possible input/output experiments. We also gave a result showing that *single* experiments are (generically) sufficient for identification in the analytic case.

Finally, the last section showed that if the precise nonlinearities are not known a result can still be obtained, essentially providing uniqueness up to four parameters.

As problems for further research, we may mention the following:

- Obtain precise sample complexity bounds.
- Study the effect of noise. Here, one may expect a connection to optimal Hankel approximation as well as other linear control theory issues.
- Design algorithms for parameter identification using the techniques introduced here.

An analogous result for discrete-time networks has been recently obtained (see Albertini & Sontag, 1993).

REFERENCES

Albertini, F., & Sontag, E. D. (1993). Identifiability of discrete-time neural networks. *Proceedings European Control Conference*, Groningen, The Netherlands.

Albertini, F., Sontag, E. D., & Maillot, V. (1993). Uniqueness of weights for neural networks. In R. Mammone (Ed.), *Artificial neural networks with applications in speech and vision*. London: Chapman and Hall.

Cleeremans, A., Servan-Schreiber, D., & McClelland, J. L. (1989). Finite state automata and simple recurrent networks. *Neural Computation*, **1**, 372–381.

Cohen, M. A., & Grossberg, S. (1983). Absolute stability of global pattern formation and parallel memory storage by competitive neural networks. *IEEE Transactions on Systems, Man, and Cybernetics*, **13**, 815–826.

Cowan, J. D. (1968). Statistical mechanics of neural nets. In E. R. Caianiello (Ed.), *Neural networks* (pp. 181–188). Berlin: Springer.

Hecht-Nielsen, R. (1989). Theory of the backpropagation neural network. In *Proceedings of the International Joint Conference on Neural Networks*. (pp. 593–605). New York: IEEE Publications.

Hirsch, M. W. (1989). Convergent activation dynamics in continuous-time networks. *Neural Networks*, **2**, 331–349.

Hopfield, J. J. (1984). Neurons with graded responses have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences USA*, **81**, 3088–3092.

Isidori, A. (1985). *Nonlinear control systems: An introduction*. Berlin: Springer-Verlag.

Michel, A. N., Farrell, J. A., & Porod, W. (1989). Qualitative analysis of neural networks. *IEEE Transactions of Circuits and Systems*, **36**, 229–243.

Nijmeijer, H., & Van der Schaft, A. V. (1990). *Nonlinear dynamical control systems*. New York: Springer-Verlag.

Robinson, A. J., & Fallside, F. (1988). Static and dynamic error propagation networks with application to speech coding. In D. Z. Anderson (Ed.), *Neural information processing systems* (pp. 632–641). New York: American Institute of Physics.

Schwarzschild, R., & Sontag, E. D. (1991). Algebraic theory of sign-linear systems. *Proceedings of the Automatic Control Conference* (pp. 799–804) Piscataway, NJ: IEEE.

Siegelmann, H. T., & Sontag, E. D. (1991). Turning computability with neural nets. *Applied Mathematics Letters*, **4**(6), 77–80.

Siegelmann, H. T., & Sontag, E. D. (1992). On the computational power of neural nets. In *Proceedings of the Fifth ACM Workshop on Computational Learning Theory* (pp. 440–449). New York: ACM.

Siegelmann, H. T., & Sontag, E. D. (1992a). Some results on computing with 'neural nets.' In *Proceedings of the IEEE Conference on Decision and Control* (pp. 1476–1481). Piscataway, NJ: IEEE.

Sontag, E. D. (1979). On the observability of polynomial systems. *SIAM Journal of Control and Optimization*, **17**, 139–151.

Sontag, E. D. (1990). *Mathematical control theory: Deterministic finite dimensional systems*. New York: Springer.

Sussmann, H. J. (1979). Single-input observability of continuous-time systems. *Mathematical Systems Theory*, **12**, 371–393.

Sussmann, H. J. (1977). Existence and uniqueness of minimal realizations of nonlinear systems. *Mathematical Systems Theory*, **10**, 263–284.

Sussmann, H. J. (1992). Uniqueness of the weights for minimal feedforward nets with a given input–output map. *Neural Networks*, **5**, 589–593.