Chapter 11
Checkable Conditions for Contraction After Small Transients in Time and Amplitude

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Abstract Contraction theory is a powerful tool for proving asymptotic properties of nonlinear dynamical systems including convergence to an attractor and entrainment to a periodic excitation. We consider generalizations of contraction with respect to a norm that allow contraction to take place after small transients in time and/or amplitude. These generalized contractive systems (GCSs) are useful for several reasons. First, we show that there exist simple and checkable conditions guaranteeing that a system is a GCS, and demonstrate their usefulness using several models from systems biology. Second, allowing small transients does not destroy the important asymptotic properties of contractive systems like convergence to a unique equilibrium point, if it exists, and entrainment to a periodic excitation. Third, in some cases as we change the parameters in a contractive system it becomes a GCS just before it looses contractivity with respect to a norm. In this respect, generalized contractivity is the analogue of marginal stability in Lyapunov stability theory.

11.1 Introduction

Differential analysis studies nonlinear dynamical systems based on the time evolution of the distance between trajectories emanating from different initial conditions. A dynamical system is called *contractive* if any two trajectories converge to one

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other at an exponential rate. A contractive system has many important properties including convergence to a unique attractor (if it exists), and entrainment to periodic excitations [2, 21, 34]. These properties can be proven even when the equilibrium point or attractor are not known explicitly. Contraction theory found applications in control theory [22], observer design [10], synchronization of coupled oscillators [3, 44], and more. It has also been extended in many directions including the notion of partial contraction [38], analysis of networks of interacting agents using contraction theory [6, 35], and Lyapunov and Lyapunov-Finsler characterizations of incremental stability [4] and contraction [18]. The latter also leads to a LaSalle-type principle for contractive systems [18]. There is also a growing interest in design techniques for controllers that render control systems contractive or incrementally stable; see, e.g. [45] and the references therein, and also the incremental ISS condition in [15].

A contractive system with added diffusion terms or random noise still satisfies certain asymptotic properties [1, 28]. Also, there exist explicit bounds on the deviations between trajectories of the system and those of its discretization [15]. In this respect, contraction is a *robust* property.

Contraction can in general be defined with respect to a norm that depends on time and/or space [21]. However, establishing that a given dynamical systems is contractive with respect to such a norm may be difficult (see, e.g. [8]). There are, however, *easy to check* conditions for establishing contraction with respect to a fixed norm that are based on the corresponding matrix measure.

Since contraction is usually used to prove asymptotic properties, i.e. properties that hold as time goes to infinity, it is natural to consider systems that are *eventually contractive*, i.e. that become contractive after some time $T > 0$. However, finding *checkable* conditions that guarantee this property seems difficult.

In this chapter, we consider three forms of generalized contractive systems (GCSs). These are motivated by requiring contraction, with respect to a fixed norm, to take place after arbitrarily small transients in time and/or amplitude. We give easy to check sufficient conditions for GSC that are based on matrix measures. In some cases as we change the parameters in a contractive system it becomes a GCS just before it looses contractivity. In this respect, a GCS is the analogue of marginal stability in Lyapunov stability theory. We demonstrate the usefulness of these generalizations using examples of systems that are *not* contractive with respect to any norm, yet are GCSs.

The remainder of this chapter is organized as follows. The next section provides a brief review of some ideas from contraction theory. Section 11.3 presents three generalizations of contraction with respect to a fixed norm. Section 11.4 details sufficient conditions for their existence and describes their implications. The proofs of all the results are placed in Sect. 11.5. The GSCs reviewed here were introduced in [42] (see also [24]). Due to space constraints, [24, 42] did not include the proofs of the main results. These are included here, as well as several new results and examples.

11.2 Preliminaries

We begin with a brief review of some ideas from contraction theory. For more details, including the historic development of contraction theory, and the relation to other notions, see e.g. [20, 33, 40].

Consider the time-varying dynamical system

$$\dot{x} = f(t, x), \tag{11.1}$$

with the state x evolving on a positively invariant convex set $\Omega \subseteq \mathbb{R}^n$. We assume that $f(t, x)$ is differentiable with respect to x, and that both $f(t, x)$ and $J(t, x) := \frac{\partial f}{\partial x}(t, x)$ are continuous in (t, x). Let $x(t, t_0, x_0)$ denote the solution of (11.1) at time $t \geq t_0$ with $x(t_0) = x_0$. For the sake of simplicity, we assume from here on that $x(t, t_0, x_0)$ exists and is unique for all $t \geq t_0 \geq 0$ and all $x_0 \in \Omega$.

We say that (11.1) is *contractive* on Ω with respect to a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if there exists $c > 0$ such that

$$|x(t_2, t_1, a) - x(t_2, t_1, b)| \leq \exp(-(t_2 - t_1)c)|a - b| \tag{11.2}$$

for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$. This means that any two trajectories contract to one another at an exponential rate. This implies in particular that the initial condition is "quickly forgotten."

Note that Ref. [21] provides a more general definition, where contraction is with respect to a time- and state-dependent metric $M(t, x)$ rather than to a fixed norm (see also [37] for a general treatment of contraction on a Riemannian manifold). Some of the results below may be stated using this more general framework. But, for a given dynamical system finding such a metric may be difficult. Another extension of contraction is incremental stability [4].

Our approach is based on the fact that there exists a simple sufficient condition guaranteeing (11.2), so generalizing (11.2) appropriately leads to *checkable* sufficient conditions for a system to be a GCS. Another advantage of our approach is that a GCS retains the important property of entrainment to periodic signals.

Recall that a vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ induces a matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $\mu(A) := \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}(||I + \epsilon A|| - 1)$, where $||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is the matrix norm induced by $|\cdot|$. It is well known (see, e.g. [34]) that if there exist a vector norm $|\cdot|$ and $c > 0$ such that the induced matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies

$$\mu(J(t, x)) \leq -c, \tag{11.3}$$

for all $t \geq 0$ and all $x \in \Omega$ then (11.2) holds. This is in fact a particular case of using a Lyapunov-Finsler function to prove contraction [18].

We list here the matrix measures corresponding to some vector norms (see, e.g. [43, Chap. 3]). The matrix measure induced by the L_1 vector norm is

$$\mu_1(A) = \max\{c_1(A), \dots, c_n(A)\}, \tag{11.4}$$

where

$$c_j(A) := A_{jj} + \sum_{\substack{1 \le i \le n \\ i \ne j}} |A_{ij}|, \tag{11.5}$$

i.e., the sum of the entries in column j of A, with non diagonal elements replaced by their absolute values. The matrix measure induced by the L_∞ norm is

$$\mu_\infty(A) = \max\{d_1(A), \dots, d_n(A)\}, \tag{11.6}$$

where

$$d_j(A) := A_{jj} + \sum_{\substack{1 \le i \le n \\ i \ne j}} |A_{ji}|, \tag{11.7}$$

i.e., the sum of the entries in row j of A, with non diagonal elements replaced by their absolute values.

Often it is useful to work with scaled norms. Let $|\cdot|_*$ be some vector norm, and let $\mu_* : \mathbb{R}^{n \times n} \to \mathbb{R}$ denote its induced matrix measure. If $P \in \mathbb{R}^{n \times n}$ is an invertible matrix, and $|\cdot|_{*,P} : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm defined by $|z|_{*,P} := |Pz|_*$ then the induced matrix measure is $\mu_{*,P}(A) = \mu_*(PAP^{-1})$.

One important implication of contraction is *entrainment* to a periodic excitation. Recall that $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is called *T-periodic* if

$$f(t, x) = f(t + T, x)$$

for all $t \ge 0$ and all $x \in \Omega$. Note that for the system $\dot{x}(t) = f(u(t), x(t))$, with u an input (or excitation) function, f will be T periodic if u is a T-periodic function. It is well known [21, 34] that if (11.1) is contractive and f is T-periodic then for any $t_1 \ge 0$ there exists a unique periodic solution $\alpha : [t_1, \infty) \to \Omega$ of (11.1), of period T, and every trajectory converges to α. Entrainment is important in various applications ranging from biological systems [23, 34] to the stability of a power grid [17]. Note that for the particular case where f is time-invariant, this implies that if Ω contains an equilibrium point e then it is unique and all trajectories converge to e.

11.3 Definitions of Contraction After Small Transients

We begin by defining three generalizations of (11.2).

Definition 11.1 The time-varying system (11.1) is said to be *contractive after a small overshoot and short transient* (SOST) on Ω w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if for each $\varepsilon > 0$ and each $\tau > 0$ there exists $\ell = \ell(\tau, \varepsilon) > 0$ such that

$$|x(t_2+\tau,t_1,a)-x(t_2+\tau,t_1,b)| \leq (1+\varepsilon)\exp(-(t_2-t_1)\ell)|a-b| \tag{11.8}$$

for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$.

This definition is motivated by requiring contraction at an exponential rate, but only after an (arbitrarily small) time τ, and with an (arbitrarily small) overshoot $(1+\varepsilon)$. However, as we will see below when the convergence rate ℓ may depend on ε a somewhat richer behavior may occur.

Definition 11.2 The time-varying system (11.1) is said to be *contractive after a small overshoot* (SO) on Ω w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if for each $\varepsilon > 0$ there exists $\ell = \ell(\varepsilon) > 0$ such that

$$|x(t_2,t_1,a)-x(t_2,t_1,b)| \leq (1+\varepsilon)\exp(-(t_2-t_1)\ell)|a-b| \tag{11.9}$$

for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$.

The definition of SO is thus similar to that of SOST, yet now the convergence rate ℓ depends only on ε, and there is no time transient τ (i.e., $\tau = 0$). In other words, SO is a uniform (in τ) version of SOST.

Definition 11.3 The time-varying system (11.1) is said to be *contractive after a short transient* (ST) on Ω w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if for each $\tau > 0$ there exists $\ell = \ell(\tau) > 0$ such that

$$|x(t_2+\tau,t_1,a)-x(t_2+\tau,t_1,b)| \leq \exp(-(t_2-t_1)\ell)|a-b| \tag{11.10}$$

for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$.

This definition allows the contraction to "kick in" only after a time transient of length τ.

It is clear that every contractive system is SOST, SO, and ST. Thus, all these notions are generalizations of contraction. Also, both SO and ST imply SOST and, as we will see below, under a mild technical condition on (11.1) SO and SOST are equivalent. Figure 11.2 on p. 16 summarizes the relations between these GCSs (as well as other notions defined below).

The next simple example demonstrates a system that does not satisfy (11.2), but is a GCS.

Example 11.1 Consider the *scalar* time-varying system

$$\dot{x}(t) = -\alpha(t)x(t), \tag{11.11}$$

where the state x evolves on $\Omega := [-1, 1]$, and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a class K function (i.e. α is continuous and strictly increasing, with $\alpha(0) = 0$). It is straightforward to show that this system does not satisfy (11.2) w.r.t. *any* norm (consider the trajectories emanating from $x(0) = 0$ and from $x(0) = \varepsilon$, with $\varepsilon > 0$ sufficiently small), yet it is ST, with $\ell(\tau) = \alpha(\tau) > 0$, for any given $\tau > 0$.

The next section analyzes the properties of the three forms of GCSs introduced above, with an emphasis on checkable conditions for establishing that a system is a GCS based on matrix measures. We assume from here on that the state space $\Omega \subset \mathbb{R}^n$ is compact and convex. The proofs of all the results are placed in Sect. 11.5.

11.4 Properties of GCSs

The next three subsections study the three forms of GCSs defined above.

11.4.1 Contraction After a Small Overshoot and Short Transient (SOST)

Just like contraction, SOST implies entrainment to a periodic excitation. To demonstrate this, assume for example that the vector field f in (11.1) is T periodic. Pick $t_0 \geq 0$. Define $m : \Omega \to \Omega$ by $m(a) := x(T + t_0, t_0, a)$. In other words, m maps a to the solution of (11.1) at time $T + t_0$ for the initial condition $x(t_0) = a$. Then m is continuous and maps the convex and compact set Ω to itself, so by the Brouwer fixed point theorem (see, e.g. [11, Chap. 6]) there exists $\zeta \in \Omega$ such that $m(\zeta) = \zeta$, i.e., $x(T + t_0, t_0, \zeta) = \zeta$. This implies that (11.1) admits a periodic solution $\gamma : [t_0, \infty) \to \Omega$ with period T. Assuming that the system is also SOST, pick $\tau, \varepsilon > 0$. Then there exists $\ell = \ell(\tau, \varepsilon) > 0$ such that

$$|x(t - t_0 + \tau, t_0, a) - x(t - t_0 + \tau, t_0, \zeta)| \leq (1 + \varepsilon) \exp(-(t - t_0)\ell)|a - \zeta|,$$

for all $a \in \Omega$ and all $t \geq t_0$. Taking $t \to \infty$ implies that every solution converges to γ. In particular, there cannot be two distinct periodic solutions. Thus, we proved the following.

Proposition 11.1 *Suppose that the time-varying system (11.1), with state x evolving on a compact and convex state-space $\Omega \subset \mathbb{R}^n$, is SOST, and that the vector field f is T-periodic. Then for any $t_0 \geq 0$ it admits a unique periodic solution $\gamma : [t_0, \infty) \to \Omega$ with period T, and $x(t, t_0, a)$ converges to γ for any $a \in \Omega$.*

Since both SO and ST imply SOST, Proposition 11.1 holds for all three forms of GCSs.

Our next goal is to derive a sufficient condition for SOST. One may naturally expect that if (11.1) is contractive w.r.t. a set of norms $|\cdot|_\zeta$, with, say $\zeta \in (0, p]$, $p > 0$, and that $\lim_{\zeta \to 0} |\cdot|_\zeta = |\cdot|$ then (11.1) is a GCS w.r.t. the norm $|\cdot|$. In fact, this can be further generalized by requiring (11.1) to be contractive w.r.t. $|\cdot|_\zeta$ only on suitable subsets Ω_ζ of the state-space. This leads to the following definition.

Definition 11.4 System (11.1) is said to be *nested contractive* (NC) on Ω with respect to a norm $|\cdot|$ if there exist convex sets $\Omega_\zeta \subseteq \Omega$, and norms $|\cdot|_\zeta : \mathbb{R}^n \to \mathbb{R}_+$, where $\zeta \in (0, 1/2]$, such that the following conditions hold.

(a) $\cup_{\zeta \in (0,1/2]} \Omega_\zeta = \Omega$, and

$$\Omega_{\zeta_1} \subseteq \Omega_{\zeta_2}, \quad \text{for all } \zeta_1 \geq \zeta_2. \tag{11.12}$$

(b) For every $\tau > 0$ there exists $\zeta = \zeta(\tau) \in (0, 1/2]$, with $\zeta(\tau) \to 0$ as $\tau \to 0$, such that for every $a \in \Omega$ and every $t_1 \geq 0$

$$x(t, t_1, a) \in \Omega_\zeta, \quad \text{for all } t \geq t_1 + \tau, \tag{11.13}$$

and (11.1) is contractive on Ω_ζ with respect to $|\cdot|_\zeta$.

(c) The norms $|\cdot|_\zeta$ converge to $|\cdot|$ as $\zeta \to 0$, i.e., for every $\zeta > 0$ there exists $s = s(\zeta) > 0$, with $s(\zeta) \to 0$ as $\zeta \to 0$, such that

$$(1 - s)|y| \leq |y|_\zeta \leq (1 + s)|y|, \quad \text{for all } y \in \Omega.$$

Equation (11.13) means that after an arbitrarily short time ζ every trajectory enters and remains in a subset Ω_ζ of the state space on which we have contraction with respect to $|\cdot|_\zeta$. We can now state the main result in this subsection. Recall that the proofs of all the results are placed in Sect. 11.5.

Theorem 11.1 *If the system (11.1) is NC w.r.t. the norm $|\cdot|$ then it is SOST w.r.t. the norm $|\cdot|$.*

The next result is an application of Theorem 11.1 to systems with a cyclic structure (see, e.g. [6, 7] and the references therein). It also shows that as we change the parameters in a contractive system, it may become a GCS when it hits the "verge" of contraction (as defined in 11.2). This is reminiscent of an asymptotically stable system that becomes marginally stable as it looses stability.

Proposition 11.2 *Consider the system*

$$\begin{aligned}
\dot{x}_1 &= -f_1(x_1) + g(x_n), \\
\dot{x}_2 &= -f_2(x_2) + k_1 x_1, \\
\dot{x}_3 &= -f_3(x_3) + k_2 x_2, \\
&\vdots \\
\dot{x}_n &= -f_n(x_n) + k_{n-1} x_{n-1}.
\end{aligned} \tag{11.14}$$

Suppose that the following properties hold for all i: $k_i > 0$, $f_i(0) = 0$, $f_i'(s)$ is a non-decreasing function of s with $f_i'(0) > 0$, $g(0) > 0$, and $g'(s)$ is a strictly decreasing function of s with $g'(s) > 0$ for all $s \geq 0$. (Note that these properties imply in particular that $\mathbb{R}^n_+$ is an invariant set of the dynamics. We further assume that there *exists a compact and convex set $\Omega \subset \mathbb{R}^n_+$ that is an invariant set of the dynamics.) Let $k := \prod_{i=1}^{n-1} k_i$. For $\varepsilon > 0$, let*

$$D_\varepsilon := \operatorname{diag}\left(1, \frac{f_1'(0)-\varepsilon}{k_1}, \frac{(f_1'(0)-\varepsilon)(f_2'(0)-\varepsilon)}{k_1 k_2}, \dots, \prod_{i=1}^{n-1} \frac{f_i'(0)-\varepsilon}{k_i}\right).$$

If

$$\prod_{i=1}^{n} f_i'(0) > k g'(0) \tag{11.15}$$

then (11.14) is contractive on Ω w.r.t. the scaled norm $|\cdot|_{1,D_\varepsilon}$ for all $\varepsilon > 0$ sufficiently small. If $\prod_{i=1}^{n} f_i'(0) = kg'(0)$ then (11.14) does not satisfy (11.2), w.r.t. any (fixed) norm on Ω, yet it is SOST on Ω w.r.t. the norm $|\cdot|_{1,D_0}$.

Example 11.2 Consider the cyclic system

$$\begin{aligned}
\dot{x}_1 &= -\alpha_1 x_1 + g(x_n),\\
\dot{x}_2 &= -\alpha_2 x_2 + x_1,\\
\dot{x}_3 &= -\alpha_3 x_3 + x_2,\\
&\vdots\\
\dot{x}_n &= -\alpha_n x_n + x_{n-1},
\end{aligned} \tag{11.16}$$

where $\alpha_i > 0$, and

$$g(u) := \frac{1+u}{c+u}, \quad \text{with } c > 1.$$

As explained in [39, Chap. 4] this is a model for a simple biochemical feedback control circuit for protein synthesis in the cell. The x_i's represent concentrations of various macromolecules in the cell and are therefore non-negative. It is straightforward to see that this system satisfies all the properties in Proposition 11.2 with $f_i(s) = \alpha_i s$, and $k_i = 1$. Using the fact that $g(u) < 1$ for all $u \geq 0$ it is straightforward to show that the set $\Omega_r := r([0, \alpha_1^{-1}] \times [0, (\alpha_1\alpha_2)^{-1}] \times \cdots \times [0, \alpha^{-1}])$ is an invariant set of the dynamics for all $r \geq 1$.

Let $\alpha := \prod_{i=1}^{n} \alpha_i$. We conclude that if

$$c - 1 < c^2 \alpha$$

then (11.16) is contractive on Ω_r w.r.t. the scaled norm $|\cdot|_{1,D_\varepsilon}$ for all $\varepsilon > 0$ sufficiently small, where $D_\varepsilon := \operatorname{diag}\left(1, \alpha_1 - \varepsilon, (\alpha_1 - \varepsilon)(\alpha_2 - \varepsilon), \dots, \prod_{i=1}^{n-1}(\alpha_i - \varepsilon)\right)$. On the other hand, if $c - 1 = c^2\alpha$ then (11.16) does not satisfy (11.2), w.r.t. any (fixed) norm on Ω_r, yet it is SOST on Ω_r w.r.t. the norm $|\cdot|_{1,D_0}$. Intuitively speaking, this means that the system is contractive when the "total dissipation" α is strictly larger than the maximal value of the feedback's derivative $g'(0)$, and looses contractivity to become SOST when these two values are equal.

Thus, (11.16), with $c - 1 \le c^2\alpha$, admits a unique equilibrium point $e \in \Omega_1$ and

$$\lim_{t\to\infty} x(t,a) = e, \quad \text{for all } a \in \mathbb{R}^n_+.$$

This property also follows from a more general result [39, Prop. 4.2.1] that is proved using the theory of irreducible cooperative dynamical systems. Yet the contraction approach leads to new insights. For example, it implies that the distance between trajectories can only decrease, and can also be used to prove entrainment to suitable generalizations of (11.16) that include periodically varying inputs.

We now describe another application of Theorem 11.1 to a model from systems biology. Cells often respond to stimulus by modification of proteins. One mechanism for doing this is *phosphorelay* (also called phosphotransfer) in which a phosphate group is transferred through a serial chain of proteins from an initial histidine kinase (HK) down to a final response regulator (RR). The next example uses Theorem 11.1 to analyze a model for phosphorelay studied in [13].

Example 11.3 Consider the system

$$\begin{aligned}
\dot{x}_1 &= (p_1 - x_1)c - \eta_1 x_1(p_2 - x_2),\\
\dot{x}_2 &= \eta_1 x_1(p_2 - x_2) - \eta_2 x_2(p_3 - x_3),\\
&\vdots\\
\dot{x}_{n-1} &= \eta_{n-2}x_{n-2}(p_{n-1} - x_{n-1}) - \eta_{n-1}x_{n-1}(p_n - x_n),\\
\dot{x}_n &= \eta_{n-1}x_{n-1}(p_n - x_n) - \eta_n x_n,
\end{aligned} \tag{11.17}$$

where $\eta_i, p_i > 0$, and $c : [t_1, \infty) \to \mathbb{R}_+$. In the context of phosphorelay [13], $c(t)$ is the strength at time t of the stimulus activating the HK, $x_i(t)$ is the concentration of the phosphorylated form of the protein at the ith layer at time t, and p_i denotes the total protein concentration at that layer. Note that $\eta_n x_n$ is the flow of the phosphate group to an external receptor molecule.

In the particular case where $p_i = 1$ for all i (11.17) becomes the *ribosome flow model* (RFM) [32]. This is the mean-field approximation of an important model from nonequilibrium statistical physics called the *totally asymmetric simple exclusion process* (TASEP) [9]. In the RFM, $x_i \in [0,1]$ is the normalized occupancy at site i, where $x_i = 0$ [$x_i = 1$] means that site i is completely free [full], and η_i is the capacity of the link that connects site i to site $i+1$. This has been used to model mRNA translation, where every site corresponds to a group of codons on the mRNA strand, $x_i(t)$ is the normalized occupancy of ribosomes at site i at time t, $c(t)$ is the initiation rate at time t, and η_i is the elongation rate from site i to site $i+1$.

Our original motivation for introducing GCSs was to prove entrainment in the RFM [23]. For more on the analysis of the RFM, and networks of interconnected RFMs, using tools from systems and control theory, see [25–27, 29–31, 46].

Assume that there exists $\eta_0 > 0$ such that $c(t) \geq \eta_0$ for all $t \geq t_1$. Let $\Omega := [0, p_1] \times \cdots \times [0, p_n]$ denote the state-space of (11.17). Then, as shown in Sect. 11.5, (11.17) does not satisfy (11.2), w.r.t. any norm, on Ω, yet it is SOST on Ω w.r.t. the L_1 norm.

Systems in which every state variable describes the amount of "material" in a compartment, and the dynamics describes the flow between the compartments and the environment are called compartmental systems [19]. Both (11.16) and (11.17) are thus compartmental systems. Analysis of contraction in such systems using the matrix measure corresponding to the scaled L_1 norm goes back at least to the work of Sandberg [36].

Considering Theorem 11.1 in the special case where all the sets Ω_ζ in Definition 11.4 are equal to Ω yields the following result.

Corollary 11.1 *Suppose that (11.1) is contractive on Ω w.r.t. a set of norms $|\cdot|_\zeta$, $\zeta \in (0, 1/2]$, and that condition (c) in Definition 11.4 holds. Then (11.1) is SOST on Ω w.r.t. $|\cdot|$.*

Corollary 11.1 may be useful in cases where some matrix measure of the Jacobian J of (11.1) turns out to be non positive on Ω, but not strictly negative, suggesting that the system is "on the verge" of satisfying (11.2). The next result demonstrates this for the time-invariant system

$$\dot{x} = f(x), \tag{11.18}$$

and the particular case of the matrix measure $\mu_1 : \mathbb{R}^{n\times n} \to \mathbb{R}$ induced by the L_1 norm. Recall that this is given by (11.4) with the c_js defined in (11.5).

Proposition 11.3 *Consider the Jacobian $J(\cdot) : \Omega \to \mathbb{R}^{n\times n}$ of the time-invariant system (11.18). Suppose that Ω is compact and convex, and that the set $\{1, \ldots, n\}$ can be divided into two nonempty disjoint sets S_0 and S_- such that the following properties hold for all $x \in \Omega$:*

1. *for any $k \in S_0$, $c_k(J(x)) \leq 0$;*
2. *for any $j \in S_-$, $c_j(J(x)) < 0$;*
3. *for any $i \in S_0$ there exists an index $z = z(i) \in S_-$ such that $J_{zi}(x) > 0$.*

Then (11.18) is SOST on Ω w.r.t. the L_1 norm.

The proof of Proposition 11.3 is based on the following idea. By compactness of Ω, there exists $\delta > 0$ such that

$$c_j(J(x)) < -\delta, \quad \text{for all } j \in S_- \text{ and all } x \in \Omega. \tag{11.19}$$

The conditions stated in the proposition imply that there exists a diagonal matrix P such that $c_k(PJP^{-1}) < 0$ for all $k \in S_0$. Furthermore, there exists such a P with diagonal entries *arbitrarily close* to 1, so $c_j(PJP^{-1}) < -\delta/2$ for all $j \in S_-$. Thus, $\mu_1(PJP^{-1}) < 0$. Now Corollary 11.1 implies SOST. Note that this implies that the compactness assumption may be dropped if for example it is known that (11.19) holds.

Example 11.4 Consider the system:

$$\begin{aligned}\dot{x} &= -\delta x + k_1 y - k_2(e_T - y)x,\\ \dot{y} &= -k_1 y + k_2(e_T - y)x,\end{aligned} \tag{11.20}$$

where $\delta, k_1, k_2, e_T > 0$, and $\Omega := [0, \infty) \times [0, e_T]$. This is a basic model for a transcriptional module that is ubiquitous in both biology and synthetic biology (see, e.g. [14, 34]). Here $x(t)$ is the concentration at time t of a transcriptional factor X that regulates a downstream transcriptional module by binding to a promoter with concentration $e(t)$ yielding a protein-promoter complex Y with concentration $y(t)$. The binding reaction is reversible with binding and dissociation rates k_2 and k_1, respectively. The linear degradation rate of X is δ, and as the promoter is not subject to decay, its total concentration, e_T, is conserved, so $e(t) = e_T - y(t)$. The Jacobian of (11.20) is $J = \begin{bmatrix} -\delta - k_2(e_T - y) & k_1 + k_2 x \\ k_2(e_T - y) & -k_1 - k_2 x \end{bmatrix}$, and all the properties in Proposition 11.3 hold with $S_- = \{1\}$ and $S_0 = \{2\}$. Indeed, $J_{12} = k_1 + k_2 x > k_1 > 0$ for all $\begin{bmatrix} x & y \end{bmatrix}^T \in \Omega$. Thus, (11.20) is SOST on Ω w.r.t. the L_1 norm. Note that Ref. [34] showed that (11.20) is contractive w.r.t. a certain *weighted* L_1 norm. Here we showed SOST w.r.t. the (unweighted) L_1 norm.

Example 11.5 A more general example studied in [34] is where the transcription factor regulates several independent downstream transcriptional modules. This leads to the following model:

$$\begin{aligned}\dot{x} &= -\delta x + k_{11} y_1 - k_{21}(e_{T,1} - y_1)x + k_{12} y_2 - k_{22}(e_{T,2} - y_2)x\\ &\quad + \cdots + k_{1n} y_n - k_{2n}(e_{T,n} - y_n)x,\\ \dot{y}_1 &= -k_{11} y_1 + k_{21}(e_{T,1} - y_1)x,\\ &\ \vdots\\ \dot{y}_n &= -k_{1n} y_n + k_{2n}(e_{T,n} - y_n)x,\end{aligned} \tag{11.21}$$

where n is the number of regulated modules. The state-space is $\Omega = [0, \infty) \times [0, e_{T,1}] \times \cdots \times [0, e_{T,n}]$. The Jacobian of (11.21) is

$$J = \begin{bmatrix} -\delta - \sum_{i=1}^{n} k_{2i}(e_{T,i} - y_i) & k_{11} + k_{21}x & k_{12} + k_{22}x & \dots & k_{1n-1} + k_{2n-1}x & k_{1n} + k_{2n}x \\ k_{21}(e_{T,1} - y_1) & -k_{11} - k_{21}x & 0 & \dots & 0 & 0 \\ k_{22}(e_{T,2} - y_2) & 0 & -k_{12} - k_{22}x & 0 & \dots & 0 \\ & \vdots & & & & \\ k_{2n}(e_{T,n} - y_n) & 0 & 0 & \dots & 0 & -k_{1n} - k_{2n}x \end{bmatrix},$$

and all the properties in Proposition 11.3 hold with $S_- = \{1\}$ and $S_0 = \{2, 3, \dots, n\}$. Thus, this system is SOST on Ω w.r.t. the L_1 norm.

Arguing as in the proof of Proposition 11.3 for the matrix measure μ_∞ induced by the L_∞ norm (see 11.7) yields the following result.

Proposition 11.4 *Consider the Jacobian $J(\cdot) : \Omega \to \mathbb{R}^{n\times n}$ of the time-invariant system (11.18). Suppose that Ω is compact and that the set $\{1, \dots, n\}$ can be divided into two nonempty disjoint sets S_0 and S_- such that the following properties hold for all $x \in \Omega$:*

1. *$d_j(J(x)) \leq 0$ for all $j \in S_0$;*
2. *$d_k(J(x)) < 0$ for all $k \in S_-$;*
3. *for any $j \in S_0$ there exists an index $z = z(j) \in S_-$ such that $J_{jz}(x) \neq 0$.*

Then (11.18) is SOST on Ω w.r.t. the L_∞ norm.

11.4.2 Contraction After a Small Overshoot (SO)

A natural question is under what conditions SO and SOST are equivalent. To address this issue, we introduce the following definition.

Definition 11.5 We say that (11.1) is *weakly expansive* (WE) if for each $\delta > 0$ there exists $\tau_0 > 0$ such that for all $a, b \in \Omega$ and all $t_0 \geq 0$

$$|x(t, t_0, a) - x(t, t_0, b)| \leq (1 + \delta)|a - b|, \quad \text{for all } t \in [t_0, t_0 + \tau_0]. \tag{11.22}$$

Proposition 11.5 *Suppose that (11.1) is WE. Then (11.1) is SOST if and only if it is SO.*

Remark 11.1 Suppose that f in (11.1) is Lipschitz globally in Ω uniformly in t, i.e., there exists $L > 0$ such that

$$|f(t, x) - f(t, y)| \leq L|x - y|, \quad \text{for all } x, y \in \Omega,\ t \geq 0.$$

Then by Gronwall's Lemma (see, e.g. [41, Appendix C])

$$|x(t, t_0, a) - x(t, t_0, b)| \leq \exp\left(L(t - t_0)\right) |a - b|,$$

for all $t \geq t_0 \geq 0$, and this implies that (11.22) holds for $\tau_0 := \frac{1}{L}\ln(1 + \delta) > 0$. In particular, if Ω is compact and f is periodic in t then WE holds under rather weak continuity arguments on f.

To explore the connection of SO with other notions related to contraction, we require the following definitions.

Definition 11.6 We say that (11.1) is *non expansive* (NE) w.r.t. a norm $|\cdot|$ if for all $a, b \in \Omega$ and all $s_2 > s_1 \geq 0$

$$|x(s_2, s_1, a) - x(s_2, s_1, b)| \leq |a - b|. \tag{11.23}$$

We say that (11.1) is *weakly contractive* (WC) if (11.23) holds with $\leq$ replaced by $<$.

One may perhaps expect that any of the three generalizations of contraction also implies WC. However, the next example shows that SO does not imply WC.

Example 11.6 Consider the scalar system

$$\dot{x} = \begin{cases} -2x, & 0 \le |x| < 1/2, \\ -\frac{x}{|x|}, & \frac{1}{2} \le |x| \le 1, \end{cases} \tag{11.24}$$

with x evolving on $\Omega := [-1, 1]$. Clearly, this system is not WC. However, it is not difficult to show that it satisfies the definition of SO with $\ell = \ell(\varepsilon) := \min\{\ln(1 + \varepsilon), 1\}$.

11.4.3 Contraction After a Short Transient (ST)

For *time-invariant* systems whose state evolves on a convex and compact set Ω it is possible to give a simple sufficient condition for ST. Let $\operatorname{Int}(S)$ $[\partial S]$ denote the interior [boundary] of a set S.

Definition 11.7 The time-invariant system (11.18) with the state x evolving on a compact and convex set $\Omega \subset \mathbb{R}^n$, is said to be *interior contractive* (IC) w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if the following properties hold:

(a) for every $x_0 \in \partial\Omega$,
$$x(t, x_0) \notin \partial\Omega, \quad \text{for all } t > 0; \tag{11.25}$$

(b) for every $x \in \operatorname{Int}(\Omega)$,
$$\mu(J(x)) < 0, \tag{11.26}$$

where $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the matrix measure induced by $|\cdot|$.

In other words, the matrix measure is negative in the interior of Ω, and the boundary of Ω is "repelling". Note that these conditions do not necessarily imply that the system satisfies (11.2) on Ω, as it is possible that $\mu(J(x)) = 0$ for some $x \in \partial\Omega$. Yet, (11.26) does imply that (11.18) is NE on Ω. We can now state the main result in this subsection.

Theorem 11.2 *If the system (11.18) is IC w.r.t. a norm $|\cdot|$ then it is ST w.r.t. $|\cdot|$.*

The proof of this result is based on showing that IC implies that for each $\tau > 0$ there exists $d = d(\tau) > 0$ such that

$$\operatorname{dist}(x(t, x_0), \partial\Omega) \ge d, \quad \text{for all } x_0 \in \Omega \text{ and all } t \ge \tau,$$

and then using this to conclude that for any $t \ge \tau$ all the trajectories of the system are contained in a convex and compact set $D \subset \operatorname{Int}(\Omega)$. In this set the system is contractive with rate $c := \max_{x \in D} \mu(J(x)) < 0$. The next example, that is a variation of a system studied in [34], demonstrates this reasoning.

Example 11.7 Consider a transcriptional factor X that regulates a downstream transcriptional module by irreversibly binding, at a rate $k_2 > 0$, to a promoter E yielding a protein-promoter complex Y. The promoter is not subject to decay, so its total concentration, denoted by $e_T > 0$, is conserved. Assume also that X is obtained from an inactive form X_0, for example through a phosphorylation reaction that is catalyzed by a kinase with abundance $u(t)$ satisfying $u(t) \geq u_0 > 0$ for all $t \geq 0$. The sum of the concentrations of X_0, X, and Y is constant, denoted by z_T, with $z_T > e_T$. Letting $x_1(t), x_2(t)$ denote the concentrations of X, Y at time t yields the model

$$\begin{aligned} \dot{x}_1 &= (z_T - x_1 - x_2)u - \delta x_1 - k_2(e_T - x_2)x_1, \\ \dot{x}_2 &= k_2(e_T - x_2)x_1, \end{aligned} \tag{11.27}$$

with the state evolving on $\Omega := [0, z_T] \times [0, e_T]$. Here $\delta \geq 0$ is the dephosphorylation rate $X \to X_0$. Let $P := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and consider the matrix measure $\mu_{\infty,P}$. A calculation yields

$$\begin{aligned} \tilde{J} &:= PJP^{-1} \\ &= \begin{bmatrix} -u - \delta & \delta \\ k_2(e_T - x_2) & k_2(x_2 - x_1 - e_T) \end{bmatrix}, \end{aligned}$$

so $d_1(\tilde{J}) = -u - \delta + |\delta| \leq -u_0 < 0$, and

$$\begin{aligned} d_2(\tilde{J}) &= k_2(x_2 - x_1 - e_T) + |k_2(e_T - x_2)| \\ &= -k_2 x_1. \end{aligned}$$

Letting $S := \{0\} \times [0, e_T]$, we conclude that $\mu_{\infty,P}(x) < 0$ for all $x \in (\Omega \setminus S)$. For any $x \in S$, $\dot{x}_1 = (z_T - x_2)u \geq (z_T - e_T)u_0 > 0$, and arguing as in the proof of Theorem 11.2 (see Sect. 11.5), we conclude that for any $\tau > 0$ there exists $d = d(\tau) > 0$ such that

$$x_1(t, a) \geq d, \quad \text{for all } a \in \Omega \text{ and all } t \geq \tau.$$

In other words, after time τ all the trajectories are contained in the closed and convex set $D = D(\tau) := [d, z_T] \times [0, e_T]$. Letting $c := c(\tau) = \max_{x \in D} \mu_{\infty,P}(J(x))$ yields $c < 0$ and

$$|x(t + \tau, a) - x(t + \tau, b)|_{\infty,P} \leq \exp(ct)|a - b|_{\infty,P}, \quad \text{for all } a, b \in \Omega \text{ and all } t > 0,$$

so (11.27) is ST w.r.t. $|\cdot|_{\infty,P}$.

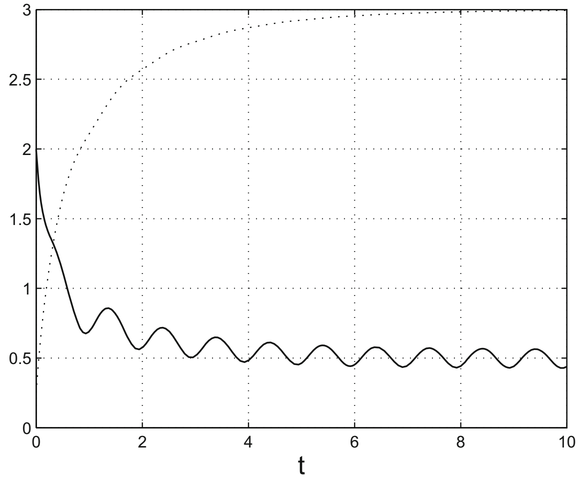

Fig. 11.1 Solution $x_1(t)$ (*solid line*) and $x_2(t)$ (*dashed line*) of the system in Example 11.8 as a function of t

As noted above, one motivation for GCSs is that contraction is used to deduce asymptotic results, so allowing initial transients should increase the class of systems that can be analyzed. The next result demonstrates this.

Corollary 11.2 *If (11.18) is IC with respect to some norm then it admits a unique equilibrium point $e \in \operatorname{Int}(\Omega)$, and $\lim_{t\to\infty} x(t,a) = e$ for all $a \in \Omega$.*

Remark 11.2 The proof of Corollary 11.2, given in the Appendix, is based on Theorem 11.2. It is possible to give another proof using the *variational system* (see, e.g. [18]) associated with (11.18):

$$\begin{aligned}\dot{x} &= f(x),\\ \dot{\delta x} &= J(x)\delta x.\end{aligned} \tag{11.28}$$

The function $V(x, \delta x) := |\delta x|$, where $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm corresponding to the matrix measure μ in (11.26), is a Lyapunov-Finsler function of (11.28), and Corollary 11.2 follows from the LaSalle invariance principle described in [18].

Since IC implies ST and this implies SOST, it follows from Proposition 11.1 that IC implies entrainment to T-periodic vector fields.[1] The next example demonstrates this.

Example 11.8 Consider again the system in Example 11.7, and assume that the kinase abundance $u(t)$ is a strictly positive and periodic function of time with period T. Since we already showed that this system is ST, it admits a unique periodic solution γ, of period T, and any trajectory of the system converges to γ. Figure 11.1

[1]Note that the proof that IC implies ST used a result for time-invariant systems, but an analogous argument holds for the time-varying case as well.

depicts the solution of (11.27) for $\delta = 2, k_2 = 1, z_T = 4, e_T = 3, u(t) = 2 + \sin(2\pi t)$, and initial condition $x_1(0) = 2, x_2(0) = 1/4$. It may be seen that both state variables converge to a periodic solution with period $T = 1$. (In particular, x_2 converges to the constant function $x_2(t) \equiv e_T$ that is of course periodic with period T.)

Contraction can be characterized using a Lyapunov-Finsler function [18]. The next result describes a similar characterization for ST. For simplicity, we state this for the time-invariant system (11.18).

Proposition 11.6 *The following two conditions are equivalent.*

(a) The time-invariant system (11.18) is ST w.r.t. a norm $|\cdot|$.
(b) For any $\tau > 0$ *there exists* $\ell = \ell(\tau) > 0$ *such that for any* $a, b \in \Omega$ *and any* c *on the line connecting* a *and* b *the solution of (11.28) with* $x(0) = c$ *and* $\delta x(0) = b - a$ *satisfies*

$$|\delta x(t + \tau)| \le \exp(-\ell t)|\delta x(0)|, \quad \text{for all } t \ge 0. \tag{11.29}$$

Note that (11.29) implies that the function $V(x, \delta x) := |\delta x|$ *is a* generalized *Lyapunov-Finsler function in the following sense. For any* $\tau > 0$ *there exists* $\ell = \ell(\tau) > 0$ *such that along solutions of the variational system:*

$$V\left(x(t + \tau, x(0)), \delta x(t + \tau, \delta x(0), x(0))\right) \le \exp(-\ell t) V(x(0), \delta x(0)),$$

for all $t \ge 0$.

Figure 11.2 summarizes the relations between the various contraction notions.

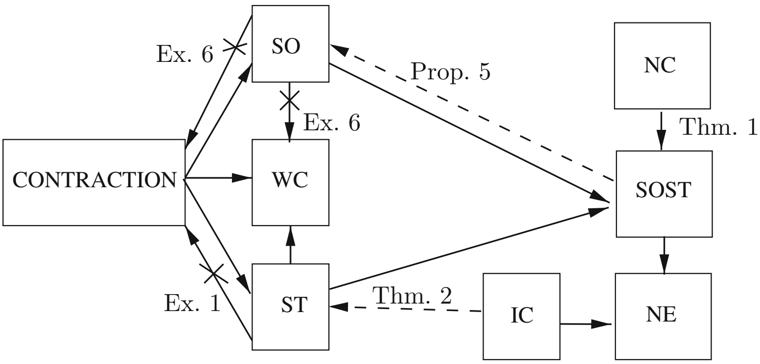

Fig. 11.2 Relations between various contraction notions. A *solid arrow* denotes implication; a *crossed out arrow* denotes that the implication is in general false; and a *dashed arrow* denotes an implication that holds under some additional conditions. Some of the relations are immediate. Others follow from the results marked near the *arrows*

11.5 Proofs

Proof of Theorem 11.1 Fix arbitrary $t_1 \geq 0$. The function $\zeta = \zeta(\tau) \in (0, 1/2]$ is as in the statement of the Theorem. For each $\tau > 0$, let $c_\zeta > 0$ be a contraction constant on Ω_ζ, where we write $\zeta = \zeta(\tau)$ here and in what follows. Pick $a, b \in \Omega$ and $\tau > 0$. By (11.13), $x(t, t_1, a), x(t, t_1, b) \in \Omega_\zeta$ for all $t \geq t_1 + \tau$, so

$$\begin{aligned}|x(t, t_1, a)-&x(t, t_1, b)|_\zeta \\ &\leq \exp(-c_\zeta(t - t_1 - \tau))|x(t_1 + \tau, t_1, a) - x(t_1 + \tau, t_1, b)|_\zeta,\end{aligned}$$

for all $t \geq t_1 + \tau$. In particular,

$$|x(t, t_1, a) - x(t, t_1, b)|_\zeta < |x(t_1 + \tau, t_1, a) - x(t_1 + \tau, t_1, b)|_\zeta, \tag{11.30}$$

for all $t > t_1 + \tau$. From the convergence property of norms in the Theorem statement, there exist $v_\zeta, w_\zeta > 0$ such that

$$|y| \leq v_\zeta |y|_\zeta \leq w_\zeta v_\zeta |y|, \quad \text{for all } y \in \Omega, \tag{11.31}$$

and $v_\zeta \to 1$, $w_\zeta \to 1$ as $\tau \to 0$. Combining this with (11.30) yields

$$|x(t, t_1, a) - x(t, t_1, b)| < v_\zeta w_\zeta |x(t_1 + \tau, t_1, a) - x(t_1 + \tau, t_1, b)|,$$

for all $t > t_1 + \tau$. Note that taking $\tau \to 0$ yields

$$|x(t, t_1, a) - x(t, t_1, b)| \leq |a - b|, \quad \text{for all } t > t_1. \tag{11.32}$$

Now for $t \geq t_1 + \tau$ let $p := t - t_1 - \tau$. Then

$$\begin{aligned}|x(t, t_1, a) - x(t, t_1, b)| &\leq v_\zeta |x(t, t_1, a) - x(t, t_1, b)|_\zeta \\ &\leq v_\zeta \exp(-c_\zeta p)|x(t_1 + \tau, t_1, a) - x(t_1 + \tau, t_1, b)|_\zeta \\ &\leq v_\zeta w_\zeta \exp(-c_\zeta p)|x(t_1 + \tau, t_1, a) - x(t_1 + \tau, t_1, b)| \\ &\leq v_\zeta w_\zeta \exp(-c_\zeta p)|a - b|,\end{aligned}$$

where the last inequality follows from (11.32). Now pick $\varepsilon > 0$. Since $v_\zeta \to 1$, $w_\zeta \to 1$ as $\tau \to 0$, $v_\zeta w_\zeta \leq 1 + \varepsilon$ for $\tau > 0$ small enough. We conclude that there exists $\tau_m > 0$ sufficiently small such that for all $\bar{\tau} \in [0, \tau_m]$

$$|x(t + \bar{\tau}, t_1, a) - x(t + \bar{\tau}, t_1, b)| \leq (1 + \varepsilon) \exp(-c_\zeta(t - t_1))|a - b|, \tag{11.33}$$

for all $a, b \in \Omega$ and all $t \geq t_1$. Now pick $\bar{\tau} > \tau_m$. For any $t \geq t_1$, let $s := t + \bar{\tau} - \tau_m$. Then

$$\begin{aligned}|x(t+\bar{\tau},t_1,a)-x(t+\bar{\tau},t_1,b)| &= |x(s+\tau_m,t_1,a)-x(s+\tau_m,t_1,b)|\\ &\le (1+\varepsilon)\exp(-c_\zeta(s-t_1))|a-b|\\ &\le (1+\varepsilon)\exp(-c_\zeta(t-t_1))|a-b|,\end{aligned}$$

and this completes the proof of Theorem 11.1. □

Proof of Proposition 11.2 The Jacobian of (11.14) is

$$J(x)=\begin{bmatrix} -f_1'(x_1) & 0 & 0 & \dots & 0 & g'(x_n)\\ k_1 & -f_2'(x_2) & 0 & \dots & 0 & 0\\ 0 & k_2 & -f_3'(x_3) & \dots & 0 & 0\\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & k_{n-1} & -f_n'(x_n)\end{bmatrix}, \tag{11.34}$$

so

$$D_\varepsilon J(x) D_\varepsilon^{-1}=\begin{bmatrix} -f_1'(x_1) & 0 & 0 \dots & 0 & \frac{g'(x_n)}{\prod_{i=1}^{n-1}\frac{f_i'(0)-\varepsilon}{k_i}}\\ f_1'(0)-\varepsilon & -f_2'(x_2) & 0 \dots & 0 & 0\\ 0 & f_2'(0)-\varepsilon & 0 \dots & 0 & 0\\ & & \vdots & & \\ 0 & 0 & 0 \dots & f_{n-1}'(0)-\varepsilon & -f_n'(x_n)\end{bmatrix}.$$

Thus, for any sufficiently small $\varepsilon>0$, $\mu_{1,D_\varepsilon}(J(x))$ is the maximum of the n values:

$$v_1 := f_1'(0)-f_1'(x_1)-\varepsilon,\dots,v_{n-1} := f_{n-1}'(0)-f_{n-1}'(x_{n-1})-\varepsilon,$$

and

$$v_n := \frac{kg'(x_n)-f_n'(x_n)\prod_{i=1}^{n-1}(f_i'(0)-\varepsilon)}{\prod_{i=1}^{n-1}(f_i'(0)-\varepsilon)}.$$

Since f_i' is nondecreasing, $v_i\le-\varepsilon$ for all $i=1,\dots,n-1$. Suppose that $\prod_{i=1}^n f_i'(0)>kg'(0)$. Then since $f_i'(x_n)\ge f_i'(0)$ and $g'(x_n)\le g'(0)$, there exists a sufficiently small $\varepsilon>0$ such that $v_n\le-\varepsilon/2$, so $\mu_{1,D_\varepsilon}(J(x))\le-\varepsilon/2$ for all $x\in\mathbb{R}^n_+$, and thus the system is contractive on $\mathbb{R}^n_+$ w.r.t. $|\cdot|_{1,D_\varepsilon}$.

Now assume that

$$\prod_{i=1}^n f_i'(0)=kg'(0). \tag{11.35}$$

By (11.34),

$$\det(J(x))=(-1)^n\left(\prod_{i=1}^n f_i'(x_i)-kg'(x_n)\right),$$

so (11.35) implies that $\det(J(0)) = 0$, and thus the system does not satisfy (11.2) w.r.t. any (fixed) norm on $\mathbb{R}^n_+$.

We now use Theorem 11.1 to prove that (11.14) is SOST on $\mathbb{R}^n_+$. For $\zeta \in (0, 1/2]$, let

$$\Omega_\zeta := \{x \in \mathbb{R}^n_+ : x \geq \zeta\}.$$

It is straightforward to verify that (11.14) satisfies condition (BR) in [23, Lemma 1], and this implies that for every $\tau > 0$ there exists $\varepsilon(\tau) > 0$ such that $x(t) \in \Omega_\varepsilon$ for all $t \geq \tau$. Then $g'(x_n) < g'(0)$, and $f_n'(x_n) \geq f_n'(0)$ so for any sufficiently small $\varepsilon > 0$,

$$kg'(x_n) - f_n'(x_n) \prod_{i=1}^{n-1} (f_i'(0) - \varepsilon) < kg'(0) - f_n'(0) \prod_{i=1}^{n-1} f_i'(0) = 0.$$

We already showed that this implies that there exists a $\zeta > 0$ and a norm $|\cdot|_{1,D_\zeta}$ such that (11.14) is contractive on Ω_ε w.r.t. this norm. Summarizing, all the conditions in Theorem 11.1 hold, and we conclude that (11.14) is SOST on $\mathbb{R}^n_+$ w.r.t. $|\cdot|_{1,D_0}$. □

Analysis of the system in Example 11.3. For $a \in \Omega$, let $x(t, t_1, a)$ denote the solution of (11.17) at time $t \geq t_1$ for the initial condition $x(t_1) = a$. Pick $\tau > 0$. Equation (11.17) satisfies condition (BR) in [23, Lemma 1], and this implies that there exists $\varepsilon = \varepsilon(\tau) > 0$ such that for all $a \in \Omega$, all $i = 1, \ldots n$, and all $t \geq t_1 + \tau$

$$x_i(t, t_1, a) \geq \varepsilon.$$

Furthermore, if we define $y_i(t) := p_{n-i+1} - x_{n-i+1}(t)$, $i = 1, \ldots, n$, then the y system also satisfies condition (BR) in [23, Lemma 1], and this implies that there exists $\varepsilon_1 = \varepsilon_1(\tau) > 0$ such that for all $a \in \Omega$, all $i = 1, \ldots n$, and all $t \geq t_1 + \tau$

$$y_i(t, t_1, a) \geq \varepsilon_1.$$

We conclude that after an arbitrarily short time $\tau > 0$ every state variable $x_i(t)$, $t \geq \tau + t_1$, is separated from 0 and from p_i. This means the following. For $\zeta \in [0, 1/2]$, let

$$\Omega_\zeta := \{x \in \Omega : \ \zeta p_i \leq x_i \leq (1-\zeta) p_i, \ i = 1, \ldots, n\}.$$

Note that $\Omega_0 = \Omega$, and that Ω_ζ is a strict subcube of Ω for all $\zeta \in (0, 1/2]$. Then for any $t_1 \geq 0$, and any $\tau > 0$ there exists $\zeta = \zeta(\tau) \in (0, 1/2)$, with $\zeta(\tau) \to 0$ as $\tau \to 0$, such that

$$x(t, t_1, a) \in \Omega_\zeta, \quad \text{for all } t \geq t_1 + \tau \text{ and all } a \in \Omega. \tag{11.36}$$

The Jacobian of (11.17) satisfies $J(t, x) = L(x) - \operatorname{diag}(c(t), 0, \ldots, 0, \eta_n)$, where

$$L(x) = \begin{bmatrix} -\eta_1(p_2-x_2) & \eta_1 x_1 & 0 & 0 \\ \eta_1(p_2-x_2) & -\eta_1 x_1-\eta_2(p_3-x_3) & \dots & 0 \\ 0 & \eta_2(p_3-x_3) & \dots & 0 \\ & & \ddots & \\ 0 & \dots & -\eta_{n-2}x_{n-2}-\eta_{n-1}(p_n-x_n) & \eta_{n-1}x_{n-1} \\ 0 & \dots & \eta_{n-1}(p_n-x_n) & -\eta_{n-1}x_{n-1} \end{bmatrix}.$$

Note that $L(x)$ is Metzler, tridiagonal, and has zero sum columns for all $x \in \Omega$. Note also that for any $x \in \Omega_\zeta$ every entry L_{ij} on the sub- and superdiagonal of L satisfies $\zeta s_1 \le L_{ij} \le (1-\zeta)s_2$, with $s_2 := \max_i\{\eta_i p_i\} > s_1 := \min_i\{\eta_i p_i\} > 0$.

Note also that there exist $x \in \partial\Omega$ such that $J(x)$ is singular (e.g., when $x_1 = 0$ and $x_3 = p_3$ the second column of J is all zeros), and this implies that the system does not satisfy (11.2) on Ω w.r.t. any norm.

By [23, Theorem 4], for any $\zeta \in (0, 1/2]$ there exists $\varepsilon = \varepsilon(\zeta) > 0$, and a diagonal matrix $D = \operatorname{diag}(1, q_1, q_1q_2, \dots, q_1q_2 \dots q_{n-1})$, with $q_i = q_i(\varepsilon) > 0$, such that (11.17) is contractive on Ω_ζ w.r.t. the scaled L_1 norm defined by $|z|_{1,D} := |Dz|_1$. Furthermore, we can choose ε such that $\varepsilon(\zeta) \to 0$ as $\zeta \to 0$, and $D(\varepsilon) \to I$ as $\varepsilon \to 0$. Summarizing, all the conditions in Definition 11.4 hold, so (11.17) is NC on Ω and applying Theorem 11.1 concludes the analysis. □

Proof of Proposition 11.3 Without loss of generality, assume that $S_0 = \{1, \dots, k\}$, with $1 \le k < n-1$, so that $S_- = \{k+1, \dots, n\}$. Fix $\varepsilon \in (0, 1)$. Let $D = \operatorname{diag}(d_1, \dots, d_n)$ with the d_is defined as follows. For every $i \in S_0$, $d_i = 1$ and $d_{z(i)} = 1-\varepsilon$. All the other d_is are one. Let $\tilde{J} := DJD^{-1}$. Then $\tilde{J}_{ij} = \frac{d_i}{d_j} J_{ij}$. We now calculate $\mu_1(\tilde{J})$. Fix $j \in S_0$. Then $d_j = 1$, so

$$\begin{aligned} c_j(\tilde{J}) &= \tilde{J}_{jj} + \sum_{\substack{1\le i\le n \\ i\ne j}} |\tilde{J}_{ij}| \\ &= J_{jj} + \sum_{\substack{i\in S_0 \\ i\ne j}} d_i |J_{ij}| + \sum_{\substack{k\in S_- \\ k\ne j}} d_k |J_{kj}| \\ &= J_{jj} + \sum_{\substack{i\in S_0 \\ i\ne j}} |J_{ij}| + \sum_{\substack{k\in S_- \\ k\ne j}} d_k |J_{kj}| \\ &< c_j(J), \end{aligned}$$

where the inequality follows from the fact that $d_k \le 1$ for all k, and for the specific value $k = z(j) \in S_-$ we have $d_k = 1-\varepsilon$ and $|J_{kj}| > 0$. We conclude that for every $j \in S_0$, $c_j(\tilde{J}) < c_j(J) = 0$. It follows from property 11.3) in the statement of Proposition 11.3 and the compactness of Ω that there exists $\delta > 0$ such that $c_j(J(x)) < -\delta$ for all $j \in S_-$ and all $x \in \Omega$, so for $\varepsilon > 0$ sufficiently small we have $c_j(\tilde{J}(x)) < -\delta/2$ for all $j \in S_-$ and all $x \in \Omega$. We conclude that for all $\varepsilon > 0$ sufficiently small, $\mu_1(DJD^{-1}) = \max_j c_j(\tilde{J}) < 0$, i.e., the system is contractive w.r.t. $|\cdot|_{1,D}$. Clearly, $|\cdot|_{1,D} \to |\cdot|_1$ as $\varepsilon \to 0$, and applying Corollary 11.1 completes the proof. □

Proof of Proposition 11.5 Suppose that (11.1) is WE, and also SOST w.r.t. some norm $|\cdot|_v$. Pick $\varepsilon > 0$. Since the system is WE, there exists $\tau_0 = \tau_0(\varepsilon) > 0$ such that

$$|x(t, t_0, a) - x(t, t_0, b)|_v \leq \left(1 + \frac{\varepsilon}{2}\right) |a - b|_v,$$

for all $t \in [t_0, t_0 + \tau_0]$. Letting $\ell_2 := \frac{1}{\tau_0} \ln(\frac{1+\varepsilon}{1+(\varepsilon/2)})$ yields

$$|x(t, t_0, a) - x(t, t_0, b)|_v \leq (1 + \varepsilon) \exp(-(t - t_0)\ell_2)|a - b|_v, \tag{11.37}$$

for all $t \in [t_0, t_0 + \tau_0]$. It is not difficult to show that SOST implies that there exists $\ell_1 = \ell_1(\tau_0, \varepsilon) > 0$ such that

$$|x(t, t_0, a) - x(t, t_0, b)|_v \leq (1 + \varepsilon) \exp(-(t - t_0)\ell_1)|a - b|_v,$$

for all $t \geq t_0 + \tau_0$. Combining this with (11.37) yields

$$|x(t, t_0, a) - x(t, t_0, b)|_v \leq (1 + \varepsilon) \exp(-(t - t_0)\ell)|a - b|_v,$$

for all $t \geq t_0$, where $\ell := \min\{\ell_1, \ell_2\} > 0$. This proves SO. □

Proof of Theorem 11.2 We require the following result.

Lemma 11.1 *If system (11.18) is IC then for each $\tau > 0$ there exists $d = d(\tau) > 0$ such that*

$$\mathrm{dist}(x(t, x_0), \partial\Omega) \geq d, \quad \textit{for all } x_0 \in \Omega \textit{ and all } t \geq \tau.$$

Proof of Lemma 11.1 Pick $\tau > 0$ and $x_0 \in \Omega$. Since Ω is an invariant set, $\mathrm{Int}(\Omega)$ is also an invariant set (see, e.g. [5, Lemma III.6]), so (11.25) implies that $x(t, x_0) \notin \partial\Omega$ for all $t > 0$. Since $\partial\Omega$ is compact, $e_{x_0} := \mathrm{dist}(x(\tau, x_0), \partial\Omega) > 0$. Thus, there exists a neighborhood U_{x_0} of x_0, such that $\mathrm{dist}(x(\tau, y), \partial\Omega) \geq e_{x_0}/2$ for all $y \in U_{x_0}$. Cover Ω by such U_{x_0} sets. By compactness of Ω, we can pick a finite subcover. Pick smallest e in this subcover, and denote this by d. Then $d > 0$ and we have that $\mathrm{dist}(x(\tau, x_0), \partial\Omega) \geq d$ for all $x_0 \in \Omega$. Now, pick $t \geq \tau$. Let $x_1 := x(t - \tau, x_0)$. Then

$$\begin{aligned}\mathrm{dist}(x(t, x_0), \partial\Omega) &= \mathrm{dist}(x(\tau, x_1), \partial\Omega) \\ &\geq d,\end{aligned}$$

and this completes the proof of Lemma 11.1. □

We can now prove Theorem 11.2. We recall some definitions from the theory of convex sets. Let $B(x, r)$ denote the closed ball of radius r around x (in the Euclidean norm). Let K be a compact and convex set with $0 \in \mathrm{Int}(K)$. Let $s(K)$ denote the *inradius* of k, i.e., the radius of the largest ball contained in K. For $\lambda \in [0, s(K)]$ the *inner parallel set of K at distance λ* is

$$K_{-\lambda} := \{x \in K : B(x, \lambda) \subseteq K\}.$$

Note that $K_{-\lambda}$ is a compact and convex set; in fact, $K_{-\lambda}$ is the intersection of all the translated support hyperplanes of K, with each hyperplane translated “inwards” through a distance λ (see [12, Section 17]). Assume, without loss of generality, that $0 \in \mathrm{Int}(\Omega)$. Pick $\tau > 0$. Let $M = M(\tau) := \{x(t, x_0) : t \geq \tau,\ x_0 \in \Omega\}$. By Lemma 11.1, $M \subset \Omega$ and $\mathrm{dist}(y, \partial\Omega) \geq d > 0$ for all $y \in M$. Let $\lambda = \lambda(\tau) := \frac{1}{2}\min\{d, s(\Omega)\}$. Then $\lambda > 0$. Pick $z \in M$. We claim that $B(z, \lambda) \subseteq \Omega$. To show this, assume that there exists $v \in B(z, \lambda)$ such that $v \notin \Omega$. Then there is a point q on the line connecting v and z such that $q \in \partial\Omega$. Therefore,

$$\begin{aligned}
\mathrm{dist}(z, \partial\Omega) &\leq |z - q| \\
&\leq |z - v| \\
&\leq \lambda \\
&\leq d/2,
\end{aligned}$$

and this is a contradiction as $z \in M$. We conclude that $M \subseteq K_{-\lambda}$. Let $c = c(\tau) := \max_{x \in K_{-\lambda}} \mu(J(x))$. Then (11.26) implies that $c < 0$. Thus, the system is contractive on $K_{-\lambda}$, and for all $a, b \in \Omega$ and all $t \geq 0$

$$|x(t + \tau, a) - x(t + \tau, b)| \leq \exp(ct)|a - b|,$$

where $|\cdot|$ is the vector norm corresponding to the matrix measure μ. This establishes ST, and thus completes the proof of Theorem 11.2. □

Proof of Corollary 11.2. Since Ω is convex, compact, and invariant, it includes an equilibrium point e of (11.18). Clearly, $e \in \mathrm{Int}(\Omega)$. By Theorem 11.2, the system is ST. Pick $a \in \Omega$ and $\tau > 0$, and let $\ell = \ell(\tau) > 0$. Applying (11.10) with $b = e$ yields

$$|x(t + \tau, a) - e| \leq \exp(-\ell t)|a - e|,$$

for all $t \geq 0$. Taking $t \to \infty$ completes the proof. □

Remark 11.3 Another possible proof of Corollary 11.2 is based on defining $V : \Omega \to \mathbb{R}_+$ by $V(x) := |x - e|$. Then for any $a \in \Omega$, $V(x(t, a))$ is nondecreasing, and the LaSalle invariance principle tells us that $x(t, a)$ converges to an invariant subset of the set $\{y \in \Omega : |y - e| = r\}$, for some $r \geq 0$. If $r = 0$ then we are done. Otherwise, pick y in the omega limit set of the trajectory. Then $y \notin \partial\Omega$, so (11.26) implies that V is strictly decreasing. This contradiction completes the proof.

Proof of Proposition 11.6. Pick $a, b \in \Omega$. Let $\gamma : [0, 1] \to \Omega$ be the line $\gamma(r) := (1 - r)a + rb$. Note that since Ω is convex, $\gamma(r) \in \Omega$ for all $r \in [0, 1]$. Let

$$w(t, r) := \frac{d}{dr} x(t, \gamma(r)).$$

This measures the sensitivity of the solution at time t to a change in the initial condition along the line γ. Note that $w(0, r) = \frac{d}{dr}\gamma(r) = b - a$, and

$$\dot{w}(t, r) = J(x(t, \gamma(r)))w(t, r).$$

Comparing this to (11.28) implies that $w(t, r)$ is equal to the second component, $\delta x(t)$, of the solution of the variational system (11.28) with initial condition

$$\begin{aligned} x(0) &= (1-r)a + rb, \\ \delta x(0) &= b - a. \end{aligned} \tag{11.38}$$

Suppose that the time-invariant system (11.18) is ST. Pick $\tau > 0$. Let $\ell = \ell(\tau) > 0$. Then for any $r \in [0, 1)$ and any $\varepsilon \in [0, 1-r]$,

$$|x(t+\tau, \gamma(r+\varepsilon)) - x(t+\tau, \gamma(r))| \leq \exp(-t\ell)|\gamma(r+\varepsilon) - \gamma(r)|.$$

Dividing both sides of this inequality by ε and taking $\varepsilon \downarrow 0$ implies that

$$|w(t+\tau, r)| \leq \exp(-t\ell)|b - a|, \tag{11.39}$$

so

$$|\delta x(t+\tau)| \leq \exp(-t\ell)|\delta x(0)|.$$

This proves the implication (a) → (b). To prove the converse implication, assume that (11.29) holds. Then (11.39) holds and thus

$$\begin{aligned} |x(t+\tau, b) - x(t+\tau, a)| &= \left| \int_0^1 \frac{d}{dr} x(t+\tau, \gamma(r)) \mathrm{d}r \right| \\ &\leq \int_0^1 |w(t+\tau, r)| \, \mathrm{d}r \\ &\leq \int_0^1 \exp(-\ell t)|b-a| \mathrm{d}r \\ &= \exp(-\ell t)|b - a|, \end{aligned}$$

so the system is ST. □

Above, we have used several times the fact that singularity of the Jacobian implies that the system $\dot{x} = f(x)$ cannot be contractive (as defined in 11.2) w.r.t. any (fixed) norm. For the sake of completeness, we now show this.

Pick any point $a \in \mathrm{Int}(\Omega)$ and any fixed $\varepsilon > 0$ such that the sphere B of radius ε around a is contained in Ω. Pick any $b = a + q$, $q \in B$, and let $\gamma : [0, 1] \to \Omega$ be the line $\gamma(r) := (1-r)a + rb = a + rq$. Since Ω is convex, this line is contained in Ω. Let $w(t, r) := \frac{d}{dr} x(t, \gamma(r))$. Since $\dot{w}(t, r) = J(x(t, \gamma(r)))w(t, r)$, we have that for any vector norm and for any $\tau > 0$,

$$\begin{aligned} |w(\tau, 0)| - |w(0, 0)| &= |(I + \tau J(x(0, \gamma(0))) + o(\tau))w(0, 0)| - |w(0, 0)| \\ &= |(I + \tau J(a))q| - |q| + o(\tau)\,. \end{aligned}$$

Pick $r \in [0, 1)$, and $\varepsilon > 0$ sufficiently small. If the system is contractive then there exist a vector norm $|\cdot|$ and $\eta > 0$ such that for all $t \geq 0$,

$$|x(t, \gamma(r+\varepsilon)) - x(t, \gamma(r))| \leq \exp(-\eta t)|\gamma(r+\varepsilon) - \gamma(r)|.$$

Dividing both sides by ε and taking limits as $\varepsilon \to 0$ yields $|w(t, r)| \leq \exp(-\eta t)|q|$, for all $t \geq 0$, and all $r \in [0, 1)$. In particular,

$$|w(\tau, 0)| - |w(0, 0)| \leq (\exp(-\eta\tau) - 1)|q|.$$

Combining all this information, we have that

$$|(I + \tau J(a))q| - |q| + o(\tau) \leq (\exp(-\eta\tau) - 1)|q|$$

and therefore, dividing by $|q|$ and $\tau > 0$,

$$\frac{\frac{|(I+\tau J(a))q|}{|q|} - 1}{\tau} \leq -\eta + \frac{o(\tau)}{\tau}.$$

For each fixed τ, pick a $q = q(\tau)$ so that $\|I + \tau J(a)\| = \frac{|(I+\tau J(a))q|}{|q|}$, so the inequality gives

$$\frac{\|I + \tau J(a)\| - 1}{\tau} \leq -\eta + \frac{o(\tau)}{\tau}.$$

Taking the limit as $\tau \searrow 0$ gives that $\mu(J(a)) \leq -\eta$, where μ is the matrix measure associated to the given norm. It follows that the real part of every eigenvalue of $J(a)$ is also less than $-\eta$ [16, p. 35], so $J(a)$ is nonsingular. There remains the case when a is not in the interior of Ω. However, continuity of eigenvalues implies that the conclusion that the real part of every eigenvalue of $J(a)$ is $\leq -\eta$ is true as well.

11.6 Conclusions

Contraction theory is a powerful tool for studying nonlinear dynamical systems. Contraction implies several desirable asymptotic properties such as convergence to a unique attractor (if it exists), and entrainment to periodic excitation. This holds even if the equilibrium point or periodic attractor are not known in explicit form. However, proving contraction is in many cases nontrivial.

We considered three generalizations of contraction. These are motivated by allowing contraction to take place after an arbitrarily small transient in time and/or amplitude. In particular, this means that they have the same asymptotic properties as contractive systems. We provided checkable conditions guaranteeing that a dynamical system is a GCS, and demonstrated their usefulness by using them to analyze a number of models from systems biology. Some of these models do not satisfy (11.2), w.r.t. any (fixed) norm, yet are a GCS.

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