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Uniform stability properties of switched systems with switchings governed by digraphs

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Abstract

Characterizations of various uniform stability properties of switched systems described by differential inclusions, and whose switchings are governed by a digraph, are developed. These characterizations are given in terms of stability properties of the system with restricted switchings and also in terms of Lyapunov functions.

Keywords: Switched systems; Differential inclusions; Digraphs; Input-to-output stability; Lyapunov method

1. Introduction

Recently the study of the stability properties of switched systems described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

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with $\sigma : [0, +\infty) \to \Gamma$ an arbitrary *switching signal* and Γ the index set, has received a great deal of attention, mainly motivated by the rapid development of the area of intelligent control (see [8] and references therein for details). In particular, in [10,11,4] the existence of common Lyapunov functions for systems as in (1) was established for input-to-state stability and other related properties.

Although under mild regularity conditions, the differential equation (1) provides for each initial condition and each switching signal a complete description of the time evolution of the state $x(\cdot)$ and, in consequence, a tractable analysis of the closed loop system (with $u(t) = k(x(t))$), it lacks robustness in terms of the external disturbances and system uncertainties that are inevitable in practice. In order to take into account such disturbances and uncertainties, it is more appropriate to consider switched systems described by controlled differential inclusions of the form

$$\dot{x}(t) \in F_{\sigma(t)}(x(t), u(t)), \quad y(t) = h(x(t)). \tag{2}$$

Systems as in (2) form a very rich class of systems which include in particular control systems defined by differential inclusions (cf. [1,2]) and switched systems as in (1). In recent works [12,13], it was shown that a switched system as in (2) can be represented by a perturbed control system described by differential equations, driven by inputs consisting of the controls of the original systems and perturbations that evolve in compact sets, in the sense that the set of maximal trajectories of the system of differential inclusions is a dense subset of maximal trajectories of the representing system. The obtained representation theorems allowed one to extend previous results on Lyapunov characterizations of input/output stability and detectability properties for systems of differential equations to switched systems defined by differential inclusions.

In this work we will consider systems as in (2) with switchings governed by a given digraph $H^* \subseteq \Gamma \times \Gamma$. This restriction of the switchings enables us, for example, to model the restrictions imposed by the discrete process of a hybrid system whose continuous portion is as in (1) (see [3,9]).

The stability properties that will be studied are formulated in the theoretic framework of input-to-state stability (ISS). In the last decade, the notion of ISS has been generalized to systems with outputs of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{3}$$

yielding a number of useful concepts that deal with the output stability of these systems. The different notions on input/output stability introduced in [15] serve to formalize the idea of *stable* dependence of outputs *y* upon inputs *u*. They differ in the precise formulations of the decay estimates and the transient behavior characteristics of the output, though they all specialize to the ISS property when the output is the complete state.

Another important notion, the *input–output-to-state stability* (IOSS) introduced in [7], concerns the *stable* dependence of states *x* upon outputs *y* and inputs *u*. This is a notion of zero-detectability that incorporates the effect of a disturbance input, and characterizes the property that the information from the output and the input is sufficient to deduce stability of the state to the origin. Although for nonlinear systems the IOSS condition cannot guarantee the existence of a “complete” observer, it does guarantee the existence of norm observers (see [7]).

It is well understood that the existence of Lyapunov functions yields insight into stability properties and provides powerful tools in system design. In [16,7] necessary and sufficient characterizations of input/output stability and IOSS in terms of Lyapunov functions were, respectively, provided.

In this work we will consider the different input/output stability properties as well as the IOSS property for switched systems as in (2) with switchings governed by digraphs. The digraph H^* imposes a restriction on the switchings, which results in a family of subclasses of switching signals, so that a switching signal belongs to a certain subclass whenever it takes values in a strongly connected component of H^*. The behavior of system (2) when the switchings are restricted to these classes plays a significant role, as it will be shown in this work that several uniform stability properties of the system with switchings governed by H^* are equivalent to the same properties when the switchings are restricted to each of these subclasses. Combining these results with those obtained in [13], we will present converse Lyapunov theorems for the input/output stability and IOSS properties for systems as in (2) with switchings governed by digraphs.

The outline of the paper is as follows. In Section 2, the basic notation is presented. In Section 3, we introduce the formal notion of switched systems described by differential inclusions with a brief review of our previous results on the notions of uniform stability for switched systems. In Section 4 we present characterizations of uniform uOLIOS, uSIIOS, and the uIOSS stability properties of switched systems with switchings governed by a digraph, both in terms of the stability properties of the restricted system and in terms of Lyapunov functions for the restricted system. In Section 5 we introduce the notions of uniform input-measurement-to-error stability and strong uniform input-measurement-to-error stability, which allows us to deal with the uIOSS and the output stability properties in a unified manner. Section 6 contains proofs of the results presented in the previous sections, and some conclusions are given in Section 7.

2. Notation

Here we introduce some notations and definitions that will be used in the sequel. We use $|\cdot|$ to denote the Euclidean norm for any given $\mathbb{R}^q$ and with $\mathscr{B}_q$ we denote its closed unit ball, i.e., $\mathscr{B}_q := \{z \in \mathbb{R}^q : |z| \leqslant 1\}$. For a normed vector space X, we define $\|A\| := \sup\{\|a\| : a \in A\}$ if $A \subseteq X$.

Given a metric space E, we denote by $\mathscr{M}(E)$ the set of Lebesgue measurable functions $\eta : [0, +\infty) \to E$ that are *locally essentially bounded*, i.e., for each compact interval $\mathscr{J} \subset [0, +\infty)$ there is a compact subset $K \subseteq E$ such that $\eta(t) \in K$ for almost all $t \in \mathscr{J}$. For a function $g \in \mathscr{M}(\mathbb{R}^q)$ we denote by $\|g\|$ the (possibly infinite) L^q_∞-norm of g, i.e., $\|g\| := \text{ess sup}\{|g(t)| : t \geqslant 0\}$ and, for any $t \geqslant 0$, $\|g\|_{[0,t]}$ stands for the L^q_∞-norm of g restricted to the interval $[0, t]$, i.e., $\|g\|_{[0,t]} := \text{ess sup}\{|g(s)| : 0 \leqslant s \leqslant t\}$.

Let X be a metric space. We denote the distance from a point $\xi \in X$ to a set $A \subseteq X$ by $dist(\xi, A)$. The *Hausdorff distance* between two nonempty closed subsets of X, A and B, is defined as $d_{\mathrm{H}}(A, B) := \max\{\sup_{\xi \in B} dist(\xi, A), \sup_{\eta \in A} dist(\eta, B)\}$.

We denote by $\mathscr{K}(X)$ the set of nonempty compact subsets of X and we recall that the Hausdorff distance d_{H} is a metric on $\mathscr{K}(X)$. Given another metric space Z, we say that a set-valued map $G : Z \to \mathscr{K}(X)$ is locally Lipschitz if it is locally Lipschitz when the Hausdorff distance is considered in $\mathscr{K}(X)$.

As usual, by a $\mathscr{K}$-function we mean a function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ that is strictly increasing and continuous, and satisfies $\alpha(0)=0$, by a $\mathscr{K}_\infty$-function one that is in addition unbounded, and we let $\mathscr{K}\mathscr{L}$ be the class of functions $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ which are of class $\mathscr{K}_\infty$ on the first argument and decrease to zero on the second argument. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(0) = 0$ and $V(\xi) > 0$ for all $\xi \neq 0$; and is radially unbounded if it is proper, i.e., if $\{\xi : V(\xi) \in A\}$ is compact whenever A is compact.

3. Stability of switched systems

In this work we consider switched systems whose subsystems are described by forced differential inclusions. More precisely, given a family of locally Lipschitz set-valued maps $\mathscr{P} = \{F_\gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathscr{K}(\mathbb{R}^n),\ \gamma \in \Gamma\}$, where Γ is an index set and, without loss of generality, $F_\gamma \neq F_{\gamma'}$ if $\gamma \neq \gamma'$, and given a locally Lipschitz output-map $h : \mathbb{R}^n \to \mathbb{R}^p$, with $h(0) = 0$, we consider the switched system with inputs and outputs

$$\dot{x} \in F_\sigma(x, u), \quad y = h(x), \tag{4}$$

where x takes values in $\mathbb{R}^n$, $u \in \mathscr{U} := \mathscr{M}(\mathbb{R}^m)$, and the *switching signal* $\sigma : [0, +\infty) \to \Gamma$ is a piecewise constant function. Associated with each switching signal σ there are a strictly increasing sequence $\{t_i\}_{i=0}^{N_\sigma}$, with $N_\sigma \leqslant \infty$, $t_0 = 0$, and $\lim_{i\to\infty} t_i = \infty$ when $N_\sigma = \infty$, and a sequence of indexes $\{\gamma_i\}_{i=0}^{N_\sigma} \subseteq \Gamma$, with $\gamma_i \neq \gamma_{i+1}$ for all $0 \leqslant i < N_\sigma$, such that $\sigma(t) = \sigma_i$ for all $t_i \leqslant t < t_{i+1}$ with $0 \leqslant i < N_\sigma$, and $\sigma(t) = \sigma_{N_\sigma}$ for all $t \geqslant t_{N_\sigma}$ when N_σ is finite. We will denote by $\mathscr{S}$ the family of switching signals of a given switched system and by $\mathscr{S}(\Gamma^*)$ the subclass of switching signals that takes values in a subset $\Gamma^* \subseteq \Gamma$.

Given an input $u \in \mathscr{U}$ and a switching signal $\sigma \in \mathscr{S}$, we say that a locally absolutely continuous function $x : \mathscr{I} \to \mathbb{R}^n$ where $\mathscr{I} = [0, T]$ or $[0, T)$ with $0 < T \leqslant +\infty$ is a *trajectory* of (4) *corresponding* to $u \in \mathscr{U}$ and to $\sigma \in \mathscr{S}$ if $\dot{x}(t) \in F_{\sigma(t)}(x(t), u(t))$ a.e. $t \in \mathscr{I}$. Observe that, due to the assumptions about F_γ, for each $\xi \in \mathbb{R}^n$, each $u \in \mathscr{U}$ and each $\sigma \in \mathscr{S}$, there always exists a trajectory z corresponding to u and to σ that verifies $x(0) = \xi$ and that is defined on an interval $[0, T)$ for some $T > 0$ small enough. A trajectory x corresponding to $u \in \mathscr{U}$ and to $\sigma \in \mathscr{S}$ is called *maximal* if it does not have an extension which is a solution corresponding to u and to σ, i.e., if $[0, T_x)$ is the interval of definition of x, either $T_x = +\infty$ or there does not exist a trajectory $z : [0, T) \to \mathbb{R}^n$ corresponding to u and to σ with $T > T_x$ so that $z(t) = x(t)$ for all $t \in [0, T_x)$.

For any $\xi \in \mathbb{R}^n$, any $u \in \mathscr{U}$ and any $\sigma \in \mathscr{S}$, we denote with $\mathscr{T}^s(\xi, u, \sigma)$ the collection of all the maximal trajectories x of (4) corresponding to u and to σ that satisfy $x(0) = \xi$.

We will say that system (4) is *forward complete* with respect to a subclass of switching signals $\mathscr{S}_0 \subseteq \mathscr{S}$ if every maximal solution $x \in \mathscr{T}^s(\xi, u, \sigma)$ corresponding to any initial condition $\xi \in \mathbb{R}^n$, any input $u \in \mathscr{U}$ and any switching signal $\sigma \in \mathscr{S}_0$ is defined for all $t \geqslant 0$, i.e., $T_x = \infty$. We just say that system (4) is *forward complete* when it is forward complete with respect to $\mathscr{S}$. It is not hard to see that (4) is forward complete with respect to $\mathscr{S}(\Gamma^*)$

if and only if $\dot{x} \in F_\gamma(x, u)$ is forward complete for each parameter $\gamma \in \Gamma^*$. To derive the main results of this work, we assume the following:

- **C**1: The family $\mathscr{P}$ is uniformly locally Lipschitz, i.e., for each $N \in \mathbb{N}$, there exists $l_N \geqslant 0$ such that

$$d_{\mathrm{H}}(F_\gamma(\xi, \mu), F_\gamma(\xi', \mu')) \leqslant l_N(|\xi - \xi'| + |\mu - \mu'|)$$

  for all $(\xi, \mu), (\xi', \mu') \in N\mathscr{B}_n \times N\mathscr{B}_m$ and all $\gamma \in \Gamma$.
- **C**2: The family $\mathscr{P}$ is pointwise equibounded, i.e., for each $(\xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^m$ there exists $M_{(\xi,\mu)} \geqslant 0$ such that

$$\|F_\gamma(\xi, \mu)\| \leqslant M_{(\xi,\mu)} \quad \forall \gamma \in \Gamma.$$

Remark 3.1. We observe that Assumptions **C1** and **C2** are trivially satisfied when Γ is a finite set.

Below we briefly review several results on stability properties for switching systems developed in [13].

Definition 3.2. Given a subclass of switched signals $\mathscr{S}_0 \subseteq \mathscr{S}$, system (4) is *uniformly input–output-to-state stable with respect to* $\mathscr{S}_0$ (uIOSS w.r.t. $\mathscr{S}_0$) if there exist a function β of class $\mathscr{K}\mathscr{L}$ and functions α and θ of class $\mathscr{K}$ such that

$$|x(t)| \leqslant \beta(|\xi|, t) + \theta(\|y\|_{[0,t]}) + \alpha(\|u\|_{[0,t]}) \quad \forall t \in [0, T_x), \tag{5}$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$.

Definition 3.3. Given a subclass of switched signals $\mathscr{S}_0 \subseteq \mathscr{S}$, a system as in (4) is

- *uniformly input–output stable with respect to* $\mathscr{S}_0$ (uIOS w.r.t. $\mathscr{S}_0$), if it is forward complete w.r.t $\mathscr{S}_0$ and there exist a function β of class $\mathscr{K}\mathscr{L}$ and a function γ of class $\mathscr{K}$ such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$,

$$|y(t)| \leqslant \beta(|\xi|, t) + \gamma(\|u\|) \quad \forall t \geqslant 0; \tag{6}$$

- *uniformly output–Lagrange input to output stable with respect to* $\mathscr{S}_0$ (uOLIOS w.r.t. $\mathscr{S}_0$) if it is uIOS w.r.t. $\mathscr{S}_0$ and there exist some $\mathscr{K}$-functions σ_1, σ_2 such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$,

$$|y(t)| \leqslant \max\{\sigma_1(|h(\xi)|), \sigma_2(\|u\|)\} \quad \forall t \geqslant 0; \tag{7}$$

- *uniformly state independent input–output stable with respect to* $\mathscr{S}_0$ (uSIIOS w.r.t. $\mathscr{S}_0$) if it is forward complete w.r.t $\mathscr{S}_0$ and there exist a function β of class $\mathscr{K}\mathscr{L}$ and a function α of class $\mathcal{K}$ such that the following holds:

$$|y(t)| \leqslant \beta(|h(\xi)|, t) + \alpha(\|u\|) \quad \forall t \geqslant 0, \tag{8}$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}_0$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$.

Remark 3.4. Note that by causality, the same estimates (6), (7) and (8) will result if one replaces $\|u\|$ by $\|u\|_{[0,t]}$. Also, by causality, equivalent definitions of the properties of uIOSS, uIOS, uOLIOS and uSIIOS (w.r.t $\mathcal{S}_0$) will be obtained if one considers only inputs with finite norm instead of inputs in $\mathcal{U}$.

Remark 3.5. It was shown in [15] that for any $\mathcal{KL}$-function β and any $\mathcal{K}$-function α there are a $\mathcal{KL}$-function $\widehat{\beta}$ and a $\mathcal{K}$-function $\widehat{\gamma}$ so that $\min\{\alpha(s), \beta(r,t)\} \leqslant \widehat{\beta}(s, t/(1+\widehat{\gamma}(r)))$ for all s, r, t. It follows that a system is uOLIOS w.r.t $\mathcal{S}_0$ if and only if there exist a function β of class $\mathcal{KL}$ and two functions ρ and γ of class $\mathcal{K}$ such that

$$|y(t)| \leqslant \beta\left(|h(\xi)|, \frac{t}{1+\rho(|\xi|)}\right) + \gamma(\|u\|) \quad \text{for all } t \geqslant 0, \tag{9}$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{U}$, all $\sigma \in \mathcal{S}_0$ and all $x \in \mathcal{T}^s(\xi, u, \sigma)$.

We note that the stability properties introduced in Definitions 3.2–3.3 are natural extensions of those given, respectively, in [7] and [16] for systems described by differential equations.

Various stability properties are particular cases of these ones. For example, for systems without inputs, the global asymptotic stability of the system with respect to a forward invariant set $\mathcal{A}$ is equivalent to the SIIOS property, if we consider as output map $h(\xi) = \mathrm{dist}(\xi, \mathcal{A})$. The well-known ISS property is also a particular case of these properties. In fact, if we consider as output map $h(\xi) = 0$, then the IOSS property becomes the ISS property. On the other hand, if we consider $h(\xi) = \xi$, then both the SIIOS and the IOS properties are equivalent to ISS.

In our previous work [13], we have shown the following:

Lemma 3.6. *Assume that Assumptions* **C1–C2** *hold for system* (4). *Suppose* $F_\gamma(0,0) = \{0\}$ *for all* $\gamma \in \Gamma$. *Then the system is uIOSS w.r.t.* $\mathcal{S}(\Gamma)$ *if and only if there exists a smooth* ($\mathcal{C}^\infty$) *function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, *called a common uIOSS-Lyapunov function w.r.t.* $\mathcal{S}(\Gamma)$, *such that*

- *for some* $\mathcal{K}_\infty$*-functions* α_1, α_2,

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n; \tag{10}$$

- *for some* $\mathcal{K}_\infty$*-function* α *and* $\mathcal{K}$*-functions* σ_1, σ_2

$$DV(\xi)v \leqslant -\alpha(|\xi|) + \sigma_1(|h(\xi)|) + \sigma_2(|\mu|) \quad \forall\, \xi \in \mathbb{R}^n,$$

for all $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$ *and all* $v \in F_\gamma(\xi, \mu)$.

For the Lyapunov characterizations for the input/output stability properties, we need the following:

Definition 3.7. A system as in (4) is *uniformly bounded input bounded state stable with respect to* $\mathscr{S}_0 \subseteq \mathscr{S}$ (uBIBS w.r.t. $\mathscr{S}_0$) if there exist some nondecreasing functions σ_1 and σ_2 such that for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$,

$$|x(t)| \leqslant \max\{\sigma_1(|\xi|), \sigma_2(\|u\|)\} \quad \forall t \geqslant 0. \tag{11}$$

As shown in [13], if a system as in (4) is uBIBS w.r.t. $\mathscr{S}_0$, then it is forward complete w.r.t. $\mathscr{S}_0$.

The following was obtained in [13]:

Lemma 3.8. *Suppose system* (4) *is uBIBS w.r.t.* $\mathscr{S}(\Gamma)$.

1. *The system is uIOS w.r.t.* $\mathscr{S}(\Gamma)$ *if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

- *for some* $\alpha_1 \in \mathscr{K}_\infty, \alpha_2 \in \mathscr{K}_\infty$,

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n; \tag{12}$$

- *for some* $\chi \in \mathscr{K}_\infty$ *and* $\alpha_3 \in \mathscr{KL}$, *the following holds for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$ *and all* $v \in F_\gamma(\xi, \mu)$:

$$V(\xi) \geqslant \chi(|\xi|) \Rightarrow DV(\xi)v \leqslant -\alpha_3(V(\xi), |\xi|). \tag{13}$$

2. *The system is uOLIOS w.r.t.* $\mathscr{S}(\Gamma)$ *if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

- *for some* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$,

$$\alpha_1(|h(\xi)|) \leqslant V(\xi) \leqslant \alpha_2(|h(\xi)|) \quad \forall \xi \in \mathbb{R}^n, \tag{14}$$

- *for some* $\chi \in \mathscr{K}_\infty$ *and some* $\alpha_3 \in \mathscr{KL}$, (13) *holds for all* $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$ *and all* $v \in F_\gamma(\xi, \mu)$.

3. *The system is uSIIOS w.r.t* $\mathscr{S}(\Gamma)$ *if and only if there exists a smooth function* $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *such that*

- *for some* $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$, (14) *holds*; *and*
- *there exist* $\alpha_3 \in \mathscr{K}_\infty$ *and* $\chi \in \mathscr{K}_\infty$ *such that*

$$V(\xi) \geqslant \chi(|\mu|) \Rightarrow DV(\xi)v \leqslant -\alpha_3(|\xi|) \tag{15}$$

for all $\xi \in \mathbb{R}^n$, *all* $\mu \in \mathbb{R}^m$, *all* $\gamma \in \Gamma$ *and all* $v \in F_\gamma(\xi, \mu)$.

The function V as in Lemma 3.8 is called a common uIOS- uOLIOS- and uSIIOS-Lyapunov function w.r.t. $\mathscr{S}(\Gamma)$, respectively.

4. Systems with switchings governed by a digraph

In this section we consider switched systems whose switchings are governed by a given digraph $H^* \subseteq \Gamma \times \Gamma$. More precisely:

Definition 4.1. We say that a switching signal σ is an *admissible switching signal* if and only if $(\sigma(t_i), \sigma(t_{i+1})) \in H^*$ for all $0 \leqslant i < N_\sigma$.

We denote by $\mathscr{S}^{\mathrm{adm}}$ the class of all admissible switching signals.

Remark 4.2. Observe that if we consider the set-valued map $H : \Gamma \to 2^\Gamma$, defined by $H(\gamma) = \{\gamma' : (\gamma, \gamma') \in H^*\}$, then we have that $\sigma \in \mathscr{S}^{\mathrm{adm}}$ if and only if $\sigma(t_{i+1}) \in H(\sigma(t_i))$ for all $0 \leqslant i < N_\sigma$.

Our main concern is to characterize under suitable hypotheses the uIOSS, uIOS, uOLIOS and uSIIOS properties of system (4) with switching signals constrained to $\mathscr{S}^{\mathrm{adm}}$. Some terminology and results from graph theory will be used (see [6] for details).

Given $\gamma, \gamma' \in \Gamma$, we say that γ' is *accessible* from γ if there exists a finite sequence $\gamma = \gamma_0, \ldots, \gamma_k = \gamma'$ with $k \in \mathbb{N}_0$ such that $(\gamma_i, \gamma_{i+1}) \in H^*$ for all $0 \leqslant i < k$. In other words, $\gamma_{i+1} \in H(\gamma_i)$ for all $0 \leqslant i < k$. Then, if we consider in Γ the relation defined by $\gamma \sim \gamma'$ if and only if γ is accessible from γ' and γ' is accessible from γ, it follows that this is an equivalence relation and that its equivalence classes are the so-called strongly connected components of the digraph H^*. Throughout this work, we assume the following:

- **C3**: The digraph H^* has a finite number of strongly connected components $\Gamma_1, \ldots, \Gamma_N$.

Theorem 1. *Suppose that Assumptions* **C1–C3** *hold for system* (4). *Then the following statements are equivalent*:

1. *System* (4) *is uIOSS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$.
2. *System* (4) *is uIOSS w.r.t.* $\mathscr{S}(\Gamma_i)$ *for each* $i \in \{1, \ldots, N\}$.

For the input/output stability properties, we first have the following:

Lemma 4.3. *Suppose Assumptions* **C1–C3** *hold for system* (4). *Then the system is uBIBS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if and only if it is uBIBS w.r.t.* $\mathscr{S}(\Gamma_i)$ *for each* $i = 1, 2, \ldots, N$.

The proof of Lemma 4.3 will be given in Section 6.2.

Theorem 2. *Suppose that Assumptions* **C1–C3** *hold for system* (4). *Assume further that the system satisfies the uBIBS property w.r.t.* $\mathscr{S}^{\mathrm{adm}}$. *Then the following statements are equivalent*:

1. *System* (4) *is uOLIOS* (*uSIIOS, respectively*) *w.r.t.* $\mathscr{S}^{\mathrm{adm}}$.
2. *System* (4) *is uOLIOS* (*uSIIOS, respectively*) *w.r.t.* $\mathscr{S}(\Gamma_i)$ *for each* $i \in \{1, \ldots, N\}$.

The significance of Theorems 1 and 2 is that they reduce the stability properties of a system as in (4) with switchings governed by a digraph to the corresponding stability properties of the system with switchings taking arbitrary values in some given subsets of Γ. This will allow one to apply the previous results for systems with switchings taking arbitrary values to obtain the Lyapunov characterizations for systems whose switchings are governed by a digraph.

The implications 1. $\Rightarrow$ 2. of Theorems 1 and 2 follow from the following approximation result:

Lemma 4.4. *Suppose that Assumptions* **C1–C2** *hold. Let Γ^* be a strongly connected component of the digraph H^* and let $x \in \mathcal{T}^s(\xi, u, \sigma)$ with $\xi \in \mathbb{R}^n$, $u \in \mathcal{U}$ and $\sigma \in \mathcal{S}(\Gamma^*)$. Then, given $0 < T < T_x$ and $\varepsilon > 0$ there exists a trajectory $x^* \in \mathcal{T}^s(\xi, u, \sigma^*)$ with $\sigma^* \in \mathcal{S}^{\mathrm{adm}}$ such that $T_{x^*} > T$ and*

$$|x(\tau) - x^*(\tau)| < \varepsilon \quad \forall \tau \in [0, T]. \tag{16}$$

The proof of Lemma 4.4 will be given in Section 6.1.

4.1. Some remarks about the uIOS case

It is not hard to prove under Assumptions **C1–C2**, by using Lemma 4.4, that if a system is uIOS w.r.t $\mathcal{S}^{\mathrm{adm}}$ then it is uIOS w.r.t. $\mathcal{S}(\Gamma^*)$ for any strongly connected component Γ^* of H^*.

Unfortunately, in contrast to the results on the uOLIOS and uSIIOS stated in Theorem 1, the uIOS property w.r.t. $\mathcal{S}(\Gamma_i)$ for $1 \leqslant i \leqslant N$ does not imply the uIOS property w.r.t. $\mathcal{S}^{\mathrm{adm}}$ even when the system satisfies the additional hypothesis of being uBIBS stable, as shown by the following counterexample.

Example 4.5. Consider the switched linear system without inputs

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad y = x_2(t), \tag{17}$$

where $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$, $\sigma(t) \in \Gamma = \{1, 2\}$ and

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Consider $H^* = \{(1, 2)\}$. Then the strongly connected components of H^* are $\Gamma_1 = \{1\}$ and $\Gamma_2 = \{2\}$. We observe that each subsystem $\dot{x} = A_i x$, $y = x_2$, $i = 1, 2$, is output stable (OS) (and in consequence IOS). We also observe that, from the fact that the derivative of the positive definite and radially unbounded function $W(x) = x_1^2 + x_2^2$ along the trajectories of (17) is semidefinite negative, it follows that the trajectories of (17) are bounded by a $\mathcal{K}$-function in the initial values (and in consequence the system is uBIBS).

We claim that (17) is not uOS w.r.t $\mathcal{S}^{\mathrm{adm}}$ (and in consequence it is not uIOS w.r.t $\mathcal{S}^{\mathrm{adm}}$).

Suppose, on the contrary, that system (17) is uOS w.r.t $\mathscr{S}^{\mathrm{adm}}$. Therefore, the following straightforward consequence of the uOS property holds.

($\star$) Given a sequence $\{\sigma_k,\ k \in \mathbb{N}\}$ of admissible switching signals and $\xi_0 \in \mathbb{R}^2$, if y_k is the output of (17) corresponding to the switching signal σ_k and the initial condition $x(0) = \xi_0$, then $y_k(t) \to 0$ as $t \to +\infty$ and the convergence is uniform with respect to $k \in \mathbb{N}$.

Now, consider the sequence of admissible switching signals $\{\sigma_k,\ k \in \mathbb{N}\}$, with $\sigma_k(t) = 1$ for $0 \leqslant t < k$ and $\sigma_k(t) = 2$ for $t \geqslant k$. Let y_k be the output of (17) corresponding to the switching signal σ_k and the initial condition $(x_1(0), x_2(0)) = (1, 0)$.

An easy computation shows that

$$y_k(t) = \begin{cases} 0, & 0 \leqslant t < k, \\ (t-k)\mathrm{e}^{-(t-k)}, & t \geqslant k. \end{cases}$$

Thus, $y_k(k+1) = \mathrm{e}^{-1}$ for all $k \in \mathbb{N}$ and the convergence of $y_k(t)$ to zero is not uniform with respect to k, which contradicts ($\star$). This fact shows that (17) is not uOS w.r.t. $\mathscr{S}^{\mathrm{adm}}$.

The following sufficient condition for the uIOS w.r.t. $\mathscr{S}^{\mathrm{adm}}$ of system (4) is a simple consequence of Theorem 1.

Proposition 4.6. *Suppose that Assumptions* **C1–C3** *hold for a uBIBS system* (4). *Assume that there exists a continuous function $h_0 : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ such that the following hold*:

1. *for some $\mathscr{K}_\infty$-function χ it holds that*

$$|h(\xi)| \leqslant \chi(h_0(\xi)) \quad \forall \xi \in \mathbb{R}^n;$$

2. *system* (4) *with output $y = h_0(x)$ is uOLIOS w.r.t. $\mathscr{S}(\Gamma_i)$ for each $i \in \{1, \ldots, N\}$.*

Then the system is uIOS w.r.t. Γ^{adm}.

Remark 4.7. We observe that hypotheses 1 and 2 of Proposition 4.6 establish that system (4) is *uOLIOS under output redefinition* w.r.t. $\mathscr{S}(\Gamma_i)$ for each i, (see [15]) and that the redefined output function h_0 is *the same* for all the switched subsystems.

As a matter of fact, by using the results of [13], one can show for each $i \in \{1, \ldots, N\}$ the existence of a continuous function h_i such that system (4) with output $y = h_i(x)$ is uOLIOS w.r.t. $\mathscr{S}(\Gamma_i)$. The extra assumption made in Proposition 4.6 is that there is a common output redefinition h_0 valid for all $1 \leqslant i \leqslant N$.

4.2. Lyapunov characterizations

In this subsection we characterize, in terms of Lyapunov functions, the uIOSS and the uniform output stability properties of switched systems with switching governed by a digraph H^*.

Theorem 3. *Suppose that Assumptions* **C1–C3** *for system* (4) *hold. Assume that* $F_\gamma(0,0)=\{0\}$ *for all* $\gamma \in \Gamma$. *Then the system is uIOSS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if and only if there exists a common uIOSS-Lyapunov function* V_i *w.r.t.* $\mathscr{S}(\Gamma_i)$ *for each* $i=1,\ldots,N$.

Theorem 4. *Suppose that Assumptions* **C1–C3** *for system* (4) *hold. Assume further that the system is uBIBS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$. *Then the system is uOLIOS* (*uSIIOS, respectively*) *w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if and only if there exists a common uOLIOS-Lyapunov function* (*uSIIOS-Lyapunov function, respectively*) V_i w.r.t. $\mathscr{S}(\Gamma_i)$ *for each* $i=1,\ldots,N$.

Theorem 3 is a straightforward application of Lemma 3.6 and Theorem 1. Theorem 4 is a consequence of Lemma 3.8, Theorem 2 and Lemma 4.3.

For the uIOS case, we have the following as a consequence of Proposition 4.6, Lemmas 3.8 and 4.3.

Proposition 4.8. *Suppose that Assumptions* **C1–C2** *hold for system* (4). *System* (4) *is uIOS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if it is uBIBS w.r.t.* $\mathscr{S}(\Gamma_i)$ *for* $i=1,\ldots,N$ *and there exists a continuous function* $h_0:\mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ *that verifies the following*:

- *there exists a function* χ *of class* $\mathscr{K}_\infty$ *such that* $|h(\xi)| \leqslant \chi(h_0(\xi))$ *for all* $\xi \in \mathbb{R}^n$;
- *system* (4) *with output map* h_0 *admits a common OLIOS-Lyapunov function* V_i *w.r.t.* $\mathscr{S}(\Gamma_i)$ *for each* $i=1,\ldots,N$.

5. Input-measurement-to-error stability

To provide a unified proof for the implications 2. ⇒ 1. of Theorems 1 and 2, we consider a system as in (4) with two output maps

$$y(t)=h(x(t)), \quad e(t)=g(x(t)), \tag{18}$$

where $h:\mathbb{R}^n \to \mathbb{R}^p$ and $g:\mathbb{R}^n \to \mathbb{R}^q$ are continuous maps. Typically, y denotes the output variables that can be measured, and e denotes the output variables to be regulated.

Definition 5.1. Let a subclass of switched signals $\mathscr{S}_0 \subseteq \mathscr{S}$ and a system as in (4) with output maps (18) be given. We say that the system is *uniformly input-measurement-to-error stable w.r.t.* $\mathscr{S}_0$ (uIMES w.r.t. $\mathscr{S}_0$) if there exist $\beta \in \mathscr{KL}$, $\theta \in \mathscr{K}$ and $\alpha \in \mathscr{K}$ such that

$$|e(t)| \leqslant \beta(|\xi|,t)+\theta(\|y\|_{[0,t]})+\alpha(\|u\|_{[0,t]}) \quad \forall t \in [0,T_x) \tag{19}$$

holds for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi,u,\sigma)$.

The uIMES notion is a generalization of the uIOSS and the output stability properties. It was introduced in [5] for systems without switchings, where some primary work was developed for the Lyapunov characterizations for the special case when there is no input u acting on the system. In order to complete the proofs of Theorems 1 and 2, we will consider the following stronger variation of the uIMES notion.

Definition 5.2. Let a subclass of switched signals $\mathscr{S}_0 \subseteq \mathscr{S}$ and a system as in (4) with output maps (18) be given. We say that the system is *strongly uniformly input-measurement-to-error stable w.r.t.* $\mathscr{S}_0$ (strongly uIMES w.r.t. $\mathscr{S}_0$) if there exist $\beta \in \mathscr{K}\mathscr{L}$, $\theta \in \mathscr{K}$, $\alpha \in \mathscr{K}$ and a continuous, nondecreasing and nonnegative definite function $\kappa : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that

$$|e(t)| \leqslant \beta\left(|g(\xi)|, \frac{t}{1+\kappa(|\xi|)}\right) + \theta(\|y\|_{[0,t]}) + \alpha(\|u\|_{[0,t]}) \tag{20}$$

holds for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$.

As in the uIMES case, the strongly uIMES property requires the magnitude of the error output to be eventually small if the norms of the input and the measurement output are small. On the other hand, strongly uIMES is stronger than uIMES in the sense that it requires the "overshoot" of $e(t)$ to depend only on $|e(0)|$, while in the uIMES case, the overshoot is merely required to be dominated by $|\xi|$.

Remark 5.3. The strongly uIMES notion will enable one to consider the uIOSS and the input/output stability properties in a unified approach: uIOSS is the special case when $\kappa(s) \equiv 0$ and when $g(\xi)=\xi$ (i.e., when the output e represents the full set of state variables); and for a forward complete system, uOLIOS is the special case when $y \equiv 0$, which further results in uSIIOS when $\kappa(s) \equiv 0$.

The following is a robust version of the input–output-to-state boundedness property introduced in [1]:

Definition 5.4. Let a subclass of switched signals $\mathscr{S}_0 \subseteq \mathscr{S}$ and a system as in (4) with output maps (18) be given. We say that the system is *uniformly input–output-to-state bounded w.r.t. y* (uIO-BND) for $\mathscr{S}_0$ if there exist $\sigma_0 \in \mathscr{K}$, $\sigma_y \in \mathscr{K}$ and $\sigma_u \in \mathscr{K}$ such that

$$|x(t)| \leqslant \max\{\sigma_0(|\xi|),\ \sigma_y(\|y\|_{[0,t]}),\ \sigma_u(\|u\|_{[0,t]})\} \quad \forall t \in [0, T_x) \tag{21}$$

holds for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}_0$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$.

Lemma 5.5. *Suppose that Assumptions* **C1–C3** *hold for system* (4) *with the output maps given in* (18). *Then the system is uIO-BND w.r.t. y for* $\mathscr{S}^{\mathrm{adm}}$ *if and only if it is uIO-BND w.r.t. y for* $\mathscr{S}(\Gamma_i)$ *for each* $1 \leqslant i \leqslant N$.

The proof of Lemma 5.5 will be given in Section 6.2.

Theorem 5. *Suppose that Assumptions* **C1–C3** *hold for system* (4) *with the output maps given in* (18). *Assume that the system is uIO-BND w.r.t. y for* $\mathscr{S}^{\mathrm{adm}}$. *Then the system is strongly uIMES w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if and only if it is strongly uIMES w.r.t.* $\mathscr{S}(\Gamma_i)$ *for each* $1 \leqslant i \leqslant N$.

To prove Theorem 5, we associate with a given switching signal $\sigma \in \mathscr{S}^{\text{adm}}$ the subsequence $\{\tau_l\}_{l=0}^{L_\sigma}$ of the switching sequence $\{t_k\}_{k=0}^{N_\sigma}$, whose elements verify the following:

- $\tau_0 = 0$;
- for each $l = 0, 1, \ldots,$ there exists a connected component Γ_{i_l} such that $\sigma(t_k) \in \Gamma_{i_l}$ for all $t_k \in [\tau_l, \tau_{l+1})$ and $\sigma(\tau_{l+1}) \notin \Gamma_{i_l}$.

That is, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are the switching moments when σ switches from a value in one connected component to a value in a different connected component.

Lemma 5.6. *Suppose that Assumption* **C3** *holds. Then for each* $\sigma \in \mathscr{S}^{\text{adm}}$, $L_\sigma \leqslant N - 1$.

Proof. For $N = 1$ the result is clear. Suppose that $N \geqslant 1$ and $L_\sigma \geqslant N$; then there exist two integers $0 \leqslant i^* < i^* + 1 < j^* \leqslant L_\sigma$ such that $\Gamma_{i_{i^*}} = \Gamma_{i_{j^*}}$. Thus $\sigma(\tau_{j^*})$ is accessible from $\sigma(\tau_{i^*+1})$ because σ is an admissible switching signal. On the other hand, since $\sigma(\tau_{i^*}) \in \Gamma_{i_{i^*}} = \Gamma_{i_{j^*}}$, it follows that $\sigma(\tau_{i^*})$ is accessible from $\sigma(\tau_{j^*})$, which implies further that $\sigma(\tau_{i^*+1})$ is accessible from $\sigma(\tau_{j^*})$. Consequently, $\sigma(\tau_{i^*+1}) \in \Gamma_{i_{i^*}}$,which contradicts the definition of the sequence $\{\tau_k\}$. $\square$

Proof of Theorem 5. The *necessity* part (the "only if" part) of the theorem is a direct consequence of Lemma 4.4. Below we show the sufficiency part.

Assume that the system is strongly uIMES w.r.t. $\mathscr{S}(\Gamma_i)$ for $1 \leqslant i \leqslant N$ and N is finite. Without loss of generality, we assume that there exist $\beta \in \mathscr{KL}$, $\theta \in \mathscr{K}_\infty$, $\alpha \in \mathscr{K}_\infty$ and a nondecreasing and nonnegative definite continuous function κ such that for all $1 \leqslant i \leqslant N$, $\xi \in \mathbb{R}^n$, $u \in \mathscr{U}$, $\sigma \in \mathscr{S}(\Gamma_i)$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$.

$$|e(t)| \leqslant \max\left\{\beta\left(|g(\xi)|, \frac{t}{1+\kappa(|\xi|)}\right), \theta(\|y\|_{[0,t]}), \alpha(\|u\|_{[0,t]})\right\} \quad \forall t \in [0, T_x). \tag{22}$$

Let $\beta_0(s) = \beta(s, 0)$. Without loss of generality, we assume that $\beta_0(s) \geqslant s$ for all $s \geqslant 0$.

Lemma 5.7. *Consider* $\xi \in \mathbb{R}^n$, $u \in \mathscr{U}$, $\sigma \in \mathscr{S}^{\text{adm}}$ *and* $x \in \mathscr{T}^s(\xi, u, \sigma)$. *Let* $0 < t < T_x$ *and let k be the greatest nonnegative integer such that* $\tau_k \leqslant t$. *Then*

$$|e(t)| \leqslant \max\{\beta_0^{k+1}(|g(\xi)|), \beta_0^k \circ \theta(\|y\|_{[0,t]}), \beta_0^k \circ \alpha(\|u\|_{[0,t]})\}, \tag{23}$$

where $\beta_0^0(s) = s$, *and* $\beta_0^{l+1} = \beta_0 \circ \beta_0^l$ *for* $l \geqslant 0$.

Proof. We will prove it by induction on k. The case of $k = 0$ follows directly from (22). Assume now that (23) holds for $k - 1$ with $k \geqslant 1$. This means that

$$|e(\tau)| \leqslant \max\{\beta_0^k(|g(\xi)|), \beta_0^{k-1} \circ \theta(\|y\|_{[0,\tau]}), \beta_0^{k-1} \circ \alpha(\|u\|_{[0,\tau]})\} \quad \forall \tau \in [\tau_{k-1}, \tau_k). \tag{24}$$

Let $t_1 = t - \tau_k$, $\xi_1 = x(\tau_k)$ and u_1, σ_1 and x_1 be defined by $u_1(\tau) = u(\tau + \tau_k)$, $\sigma_1(\tau) = \sigma(\tau + \tau_k)$ and $x_1(\tau) = x(\tau + \tau_k)$, $e_1 = g(x_1)$, and $y_1 = h(x_1)$. Then $x_1 \in \mathcal{T}^s(\xi_1, u_1, \sigma_1)$ with $T_{x_1} > t_1$ and $\sigma_1(\tau) \in \Gamma_{i_k}$ for all $\tau \in [0, t_1]$. Hence,

$$|e_1(\tau)| \leqslant \max\{\beta_0(|g(\xi_1)|), \theta(\|y_1\|_{[0,\tau]}), \alpha(\|u_1\|_{[0,\tau]})\} \quad \forall \tau \in [0, t_1]. \tag{25}$$

Then, it follows from (24) (by letting $\tau \to \tau_k$) that

$$\begin{aligned}\beta_0(|g(\xi_1)|) &= \beta_0(|e(\tau_k)|) \\ &\leqslant \max\{\beta_0^{k+1}(|g(\xi)|), \beta_0^k \circ \theta(\|y\|_{[0,\tau_k]}), \quad \beta_0^k \circ \alpha(\|u\|_{[0,\tau_k]})\}.\end{aligned}$$

Combining this with (25) together with the fact that $e(t) = e_1(t_1)$ (and noticing that $\beta_0^k \circ \varphi(s) \geqslant \varphi(s)$ for any $\varphi \in \mathcal{K}$), we get (23). $\square$

We now continue with the proof of Theorem 5. Let $\xi \in \mathbb{R}^n$, $u \in \mathcal{U}$, $\sigma \in \mathcal{S}^{\text{adm}}$ and $x \in \mathcal{T}^s(\xi, u, \sigma)$. Fix $t \in [0, T_x)$. Let $a = \|y\|_{[0,t]}$, $b = \|u\|_{[0,t]}$ and consider the partition of $[0, t]$ given by $I_0 = [\tau_0, \tau_1), \ldots, I_{k-1} = [\tau_{k-1}, \tau_k)$, $I_k = [\tau_k, t]$ with $\tau_k \leqslant t$. By Lemma 5.6, $k \leqslant N - 1$. Consequently, at least one of these intervals, say I_j, has length equal to or greater than t/N, and from Lemma 5.7,

$$\begin{aligned}|e(\tau_j)| &\leqslant \max\{\beta_0^{j+1}(|g(\xi)|), \beta_0^j \circ \theta(a), \beta_0^j \circ \alpha(b)\} \\ &\leqslant \max\{\beta_0^N(|g(\xi)|), \beta_0^{N-1} \circ \theta(a), \beta_0^{N-1} \circ \alpha(b)\}.\end{aligned}$$

Arguments similar to those used in the proof of Lemma 5.7 show that, in the case when $j < k$,

$$\begin{aligned}|e(\tau_{j+1})| &\leqslant \max\left\{\beta\left(|g(\xi_j)|, \frac{t}{N(1+\kappa(|\xi_j|))}\right), \theta(a), \alpha(b)\right\} \\ &\leqslant \max\left\{\beta\left(\beta_0^N(|g(\xi)|), \frac{t/N}{1+\kappa(|\xi_j|)}\right), \beta_0^N \circ \theta(a), \beta_0^N \circ \alpha(b)\right\},\end{aligned} \tag{26}$$

where we have let $\xi_j = x(\tau_j)$, and in the case when $j = k$,

$$\begin{aligned}|e(t)| &\leqslant \max\left\{\beta\left(|g(\xi_j)|, \frac{t}{N(1+\kappa(|\xi_j|))}\right), \theta(a), \alpha(b)\right\} \\ &\leqslant \max\left\{\beta\left(\beta_0^N(|g(\xi)|), \frac{t/N}{1+\kappa(|\xi_j|)}\right), \beta_0^N \circ \theta(a), \beta_0^N \circ \alpha(b)\right\}.\end{aligned}$$

Since from the uIO-BND property we have

$$\kappa(|\xi_j|) \leqslant \max\{\kappa \circ \sigma_0(|\xi|), \kappa \circ \sigma_y(a), \kappa \circ \sigma_u(b)\},$$

it follows that for any $r > 0$,

$$\begin{aligned}\beta\left(r, \frac{t/N}{1+\kappa(|\xi_j|)}\right) \leqslant \max\left\{\beta\left(r, \frac{t/N}{1+\kappa \circ \sigma_0(|\xi|)}\right), \beta\left(r, \frac{t/N}{1+\kappa \circ \sigma_y(a)}\right),\right. \\ \left.\beta\left(r, \frac{t/N}{1+\kappa \circ \sigma_u(b)}\right)\right\}.\end{aligned}$$

Considering the two cases when $r \geqslant s$ and $r < s$, we see that for any $\mathscr{K}\mathscr{L}$-function $\widehat{\beta}$,

$$\widehat{\beta}\left(r, \frac{t}{1+s}\right) \leqslant \max\left\{\widehat{\beta}\left(r, \frac{t}{1+r}\right), \widehat{\beta}(s,0)\right\}.$$

Hence, with $\widehat{\beta}(r,t) := \beta(\beta_0^N(r), t)$, we get

$$\widehat{\beta}\left(r, \frac{t/N}{1+\kappa(|\xi_j|)}\right) \leqslant \max\left\{\widehat{\beta}\left(r, \frac{t/N}{1+\kappa\circ\sigma_0(|\xi|)}\right), \widehat{\beta}\left(r, \frac{t/N}{1+r}\right), \widehat{\beta}_0\circ\kappa\circ\sigma_y(a), \widehat{\beta}_0\circ\kappa\circ\sigma_u(b)\right\}.$$

Combining this with (26), we get, for the case when $j < k$,

$$|e(\tau_{j+1})| \leqslant \max\left\{\widehat{\beta}\left(|g(\xi)|, \frac{t/N}{1+\kappa\circ\sigma_0(|\xi|)}\right), \widehat{\beta}\left(|g(\xi)|, \frac{t/N}{1+|g(\xi)|}\right), \widehat{\sigma}_y(a), \widehat{\sigma}_u(b)\right\}, \tag{27}$$

where $\widehat{\sigma}_y$ and $\widehat{\sigma}_u$ are functions of class $\mathscr{K}_\infty$ that verify $\widehat{\sigma}_y(s) \leqslant \max\{\widehat{\beta}_0\circ\kappa\circ\sigma_y(s), \beta_0^N\circ\theta(s)\}$ and $\widehat{\sigma}_u(s) \leqslant \max\{\widehat{\beta}_0\circ\kappa\circ\sigma_u(s), \beta_0^N\circ\alpha(s)\}$, respectively. Let

$$\widetilde{\beta}(r,s) = \max\left\{\widehat{\beta}(r, s/N), \widehat{\beta}\left(r, \frac{s/N}{1+r}\right)\right\}.$$

Then, (27) yields, in the case when $j < k$,

$$|e(\tau_{j+1})| \leqslant \max\left\{\widetilde{\beta}\left(|g(\xi)|, \frac{t}{1+\kappa\circ\sigma_0(|\xi|)}\right), \widehat{\sigma}_y(a), \widehat{\sigma}_u(b)\right\}. \tag{28}$$

Similarly, for $j = k$, we have

$$|e(t)| \leqslant \max\left\{\widetilde{\beta}\left(|g(\xi)|, \frac{t}{1+\kappa\circ\sigma_0(|\xi|)}\right), \widehat{\sigma}_y(a), \widehat{\sigma}_u(b)\right\}. \tag{29}$$

In the case that $j < k$, it follows from Lemma 5.7 that

$$\begin{aligned}|e(t)| &\leqslant \max\{\beta_0^{k-j}(|g(x(\tau_{j+1}))|), \beta_0^{k-j-1}\circ\theta(a), \beta_0^{k-j-1}\circ\alpha(b)\}\\ &\leqslant \max\left\{\beta^*\left(|g(\xi)|, \frac{t}{1+\kappa\circ\sigma_0(|\xi|)}\right), \theta^*(a), \alpha^*(b)\right\},\end{aligned}$$

where $\beta^*(r,t) = \beta_0^N(\widetilde{\beta}(r,t))$, and

$$\theta^*(r) = \max\{\beta_0^N\circ\widehat{\sigma}_y(r), \beta_0^N\circ\theta(r)\}, \quad \alpha^*(r) = \max\{\beta_0^N\circ\widehat{\sigma}_u(r), \beta_0^N\circ\alpha(r)\}.$$

Combining this with (29) for the case when $j = k$, and taking into account the definitions of a and b, we get

$$|e(t)| \leqslant \max\left\{\beta^*\left(|g(\xi)|, \frac{t}{1+\kappa\circ\sigma_0(|\xi|)}\right), \theta^*(\|y\|_{[0,t]}), \alpha^*(\|u\|_{[0,t]})\right\}.$$

We complete the proof by noting that $\beta^* \in \mathcal{KL}$, $\theta^* \in \mathcal{K}$, $\alpha^* \in \mathcal{K}$ and $\kappa\circ\sigma_0$ is continuous and nondecreasing. $\square$

Remark 5.8. It can be seen in the proof above that the uIO-BND condition is redundant when $\kappa(s) \equiv 0$ in (20), since in this case, estimate (20) can be reduced to

$$|e(t)| \leqslant \beta(|g(\xi)|, t) + \theta(\|y\|_{[0,t]}) + \alpha(\|u\|_{[0,t]}), \qquad 0 \leqslant t < T_x. \tag{30}$$

Accordingly, one can derive (29) from (26) without using the uIO-BND condition.

6. Proofs

In this section, we provide proofs of Theorems 1 and 2 and some lemmas used in the paper.

6.1. Proof of Lemma 4.4

Take $\sigma \in \Gamma^*$, and let $\{t_i\}_{i=0}^{N_\sigma}$ be the sequence of switching times of σ and let $\gamma_i = \sigma(t_i)$. Consider the greatest nonnegative integer k such that $t_k \leqslant T$. As $\gamma_0, \ldots, \gamma_k$ belong to Γ^* then, for each integer $0 < l \leqslant k$, there exists a sequence $\gamma_{l-1} = \gamma_l^0, \gamma_l^1, \ldots, \gamma_l^{j_l} = \gamma_l$ of elements of Γ^* such that $\gamma_l^{j+1} \in H(\gamma_l^j)$ for all $j = 0, \ldots, j_l - 1$.

Then, if we consider for $\varepsilon_n = 1/n$, with $n \in \mathbb{N}$ large enough, the switching signal σ_n defined by

$$\sigma_n(t) = \begin{cases} \gamma_{l-1} & \text{if } t \in [t_{l-1}, t_l - (j_l - 1)\varepsilon_n),\ l = 1, \ldots, k, \\ \gamma_l^j & \text{if } t \in [t_l - (j_l - j)\varepsilon_n, t_l - (j_l - j - 1)\varepsilon_n),\ l = 1, \ldots, k, \\ & j = 1, \ldots, j_l - 1, \\ \gamma_k & \text{if } t \geqslant t_k, \end{cases} \tag{31}$$

we have that $\sigma_n \in \mathcal{S}^{\mathrm{adm}}$.

Claim. *There exists a sequence of maximal solutions* $\{x_n, n \in \mathbb{N}\}$ *of* (4) *so that*:

(i) $x_n \in \mathcal{T}^s(\xi, u, \sigma_n)$ *and* $T_{x_n} > T$, *for n large enough*;
(ii) $\{x_n\}$ *converges uniformly to x on* $[0, T]$.

Proof. As Assumptions **C1–C2** hold, it follows from Lemma 2.5 and Remark 2.6 of [13] that there exist an injective function $\iota : \Gamma \to D$, with D a compact metric space, and a set-valued map $F : \mathbb{R}^n \times \mathbb{R}^m \times D \to \mathcal{K}(\mathbb{R}^n)$ such that any $\bar{x} \in \mathcal{T}^s(\xi, u, \sigma)$ is also a maximal solution of $\dot{\bar{x}} \in F(\bar{x}, u, d_\sigma)$ with initial condition ξ, corresponding to u and to $d_\sigma = \imath \circ \sigma \in \mathscr{M}(D)$.

If we consider now, for each $n \in \mathbb{N}$, $d_n = \imath \circ \sigma_n$, then the family $\{d_n, n \in \mathbb{N}\}$ is locally equibounded and, in addition, $\lim_{n\to\infty} d_n(\tau) = d_\sigma(\tau)$ a.e. on $[0, t_{k+1})$ if $k < N_\sigma$ or on $[0, +\infty)$ if $k = N_\sigma$.

Let us define, for any $\tau \geqslant 0$, any $\eta \in \mathbb{R}^n$ and any $v \in D$ the set-valued map $G(\tau, \eta, v) := F(\eta, u(\tau), v)$. It follows that G verifies the hypotheses of Lemma 3.7 of [13] and that x is a maximal solution of the forced differential inclusion

$$\dot{z} \in G(\tau, z, d), \tag{32}$$

corresponding to $d = d_\sigma$. Then according to Lemma 3.7 of [13], there exists a sequence $\{x_n : n \in \mathbb{N}\}$, with x_n a maximal trajectory of (32) corresponding to d_n that verifies $x_n(0) = \xi$, and $T_{x_n} > T$ for n large enough. In addition, $\{x_n\}$ converges uniformly to x in $[0, T]$. But, according to Remark 1.4 of [12] and the definition of G, $x_n \in \mathscr{T}^s(\xi, u, \sigma_n)$, and hence the claim follows. □

6.2. *Proof of Lemmas 4.3 and 5.5*

Lemma 4.3 is a corollary of Lemma 5.5 with $h(\xi) \equiv 0$. Below we prove Lemma 5.5.

The necessity of Lemma 5.5 follows from the approximation result Lemma 4.4. In what follows we prove that if system (4) is uIO-BND w.r.t. y for $\mathscr{S}(\Gamma_i)$ for each $i \in \{1, \ldots, N\}$, then it is uIO-BND w.r.t. y for $\mathscr{S}^{\text{adm}}$.

Due to Assumption **C3**, we may assume that there exists $\mu \in \mathscr{K}$ such that for all $1 \leqslant i \leqslant N$,

$$|x(t)| \leqslant \max\{\mu(|\xi|), \mu(\|y\|_{[0,t]}), \mu(\|u\|_{[0,t]})\}, \quad 0 \leqslant t < T_x, \tag{33}$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathscr{U}$, all $\sigma \in \mathscr{S}(\Gamma_i)$ and all $x \in \mathscr{T}^s(\xi, u, \sigma)$. Without loss of generality, we assume that $\mu(s) \geqslant s$ for $s \geqslant 0$. As a consequence, $\mu^n \geqslant \mu^j$ when $n \geqslant j$.

Consider $\xi \in \mathbb{R}^n$, $u \in \mathscr{U}$, $\sigma \in \mathscr{S}^{\text{adm}}$ and $x \in \mathscr{T}^s(\xi, u, \sigma)$. We will prove the lemma by showing the following:

$$|x(t)| \leqslant \max\{\mu^N(|\xi|), \mu^N(\|y\|_{[0,t]}), \mu^N(\|u\|_{[0,t]})\}, \quad \forall t \geqslant 0. \tag{34}$$

In order to prove this claim let, as in the proof of Theorem 5, k be the greatest nonnegative integer such that $\tau_k \leqslant t$. Then

$$|x(\tau)| \leqslant \max\{\mu^j(|\xi|), \mu^j(\|y\|_{[0,t]}), \mu^j(\|u\|_{[0,t]})\} \quad \forall \tau \in [\tau_{j-1}, \tau_j),\ 1 \leqslant j \leqslant k. \tag{35}$$

We prove (35) by induction on j; the case $j = 1$ is a straightforward consequence of (33).

Suppose that (35) holds for $j - 1$. Then, from (35) and the continuity of x, we have that

$$|x(\tau_j)| \leqslant \max\{\mu^{j-1}(|\xi|), \mu^{j-1}(\|y\|_{[0,t]}), \mu^{j-1}(\|u\|_{[0,t]})\}$$
$$\forall \tau \in [\tau_{j-1}, \tau_j), \quad 1 \leqslant j \leqslant k.$$

By the same argument used in the proof of Lemma 5.7, we have, for $\tau \in [0, \tau_{j+1} - \tau_j)$,

$$\begin{aligned} |x(\tau_j + \tau)| &\leqslant \max\{\mu(|\xi_j|), \mu(\|y_j\|_{[0,\tau]}), \mu(\|u_j\|_{[0,\tau]})\} \\ &\leqslant \max\{\mu^j(|\xi|), \mu^j(\|y\|_{[0,t]}), \mu^j(\|u\|_{[0,t]})\}, \end{aligned}$$

where we have let $\xi_j = x(\tau_j)$, $x_j(\tau) = x(\tau_j + \tau)$, $y_j = h(x_j)$ and $u_j(\tau) = u(\tau_j + \tau)$. The induction is thus completed. Applying the uIO-BND estimate (34) once more with the estimate (35) with $j = k$, we get

$$|x(t)| \leqslant \max\{\mu^{k+1}(|\xi|), \mu^{k+1}(\|y\|_{[0,t]}), \mu^{k+1}(\|u\|_{[0,t]})\}.$$

Estimate (34) is thus proved by noting that $k \leqslant N - 1$.

6.3. *Proof of Theorems 1 and 2*

It follows from the following facts that Theorems 1 and 2 are consequences of Theorem 5. Let $\mathscr{S}_0$ be a subclass of $\mathscr{S}$.

1. A system as in (4) with the output map $y = h(x)$ is uIOSS w.r.t. $\mathscr{S}_0$ if and only if it is strongly uIMES w.r.t. $\mathscr{S}_0$ as in Definition 5.2 with $g(\xi) = \xi$. Also, note that if this is the case, then the uIO-BND condition follows from the uIOSS property.
2. A system as in (4) is uBIBS w.r.t. $\mathscr{S}_0$ if and only if it is uIO-BND w.r.t. y for $\mathscr{S}_0$ as in Definition 5.4 with the output map $h(\xi) \equiv 0$.
3. A forward complete system as in (4) with the output map $e = g(x)$ is uOLIOS w.r.t. $\mathscr{S}_0$ if and only if it is strongly uIMES w.r.t. $\mathscr{S}_0$ as in Definition 5.2 with $h(\xi) \equiv 0$.
4. A forward complete system as in (4) with the output map $e = g(x)$ is uSIIOS w.r.t. $\mathscr{S}_0$ if and only if it is strongly uIMES w.r.t. $\mathscr{S}_0$ as in Definition 5.2 with $h(\xi) \equiv 0$ and $\kappa(s) \equiv 0$.

6.4. *A remark about the uSIIOS property*

As indicated in [13], the uBIBS condition in the uSIIOS-Lyapunov characterization (c.f. Lemma 3.8) can be replaced by the forward completeness condition. Remark 5.8 also indicates that the uBIBS condition is redundant in the statement of Theorem 2 about the uSIIOS property. In order to state the results about uSIIOS without assuming the uBIBS condition, we need to introduce the notion of Zeno switching signal.

We say that $\sigma : [0, +\infty) \to \Gamma$ is a *Zeno switching signal* if there exist a sequence of real numbers $\{t_k, k \in \mathbb{N}_0\}$ with $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to+\infty} t_k = T_\sigma < +\infty$, a sequence $\{\gamma_k \in \Gamma, k \in \mathbb{N}_0\}$ with $\gamma_k \neq \gamma_{k+1}$ for all $k \geqslant 0$ and a point $\gamma^* \in \Gamma$ such that $\sigma(t) = \gamma_k$ for all $t_k \leqslant t < t_{k+1}$ and $\sigma(t) = \gamma^*$ for all $t \geqslant T_\sigma$.

The definition of trajectory and maximal trajectory of (4) when σ is a Zeno switching signal is the same as in the case in which σ is a piecewise constant one (see [13] for details).

We denote by $\mathscr{S}_Z$ the set of all (piecewise constant) switchings and all Zeno switchings taking values in Γ and we say that system (4) is forward complete with respect to $\mathscr{S}_Z$ if every maximal trajectory x of (4) corresponding to any initial condition ξ, any input u and any $\sigma \in \mathscr{S}_Z$ is defined for all $t \geqslant 0$. We observe that any uBIBS switched system is, in particular, forward complete w.r.t. $\mathscr{S}_Z$.

Proposition 6.1. *Suppose that Assumptions* **C1–C3** *hold for system* (4). *Assume that the system is forward complete w.r.t.* $\mathscr{S}_Z$. *Then the following hold*:

1. *The system is uSIIOS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if and only if the system is uSIIOS w.r.t.* $\mathscr{S}(\Gamma_i)$ *for all* $1 \leqslant i \leqslant N$.
2. *The system is uSIIOS w.r.t.* $\mathscr{S}^{\mathrm{adm}}$ *if and only if there exists a common uSIIOS-Lyapunov function* V_i *w.r.t.* $\mathscr{S}(\Gamma_i)$ *for all* $1 \leqslant i \leqslant N$.

7. Conclusion

In this paper we have studied different types of uniform stability properties of switched systems defined by differential inclusions with inputs and outputs. For such a switched system whose switchings are governed by a digraph, we have shown that, under suitable hypotheses, the stability properties are equivalent to the corresponding stability properties for the system when the switching signals are restricted to take arbitrary values in the strongly connected components of the digraph. As a consequence, we obtained Lyapunov characterizations, under suitable hypotheses, for the stability properties for systems with switchings governed by a digraph.

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