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Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems

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Abstract

We provide an explicit $\mathcal{KL}$ stability or input-to-state stability (ISS) estimate for a sampled-data nonlinear system in terms of the $\mathcal{KL}$ estimate for the corresponding discrete-time system and a $\mathcal{K}$ function describing inter-sample growth. It is quite obvious that a uniform inter-sample growth condition, plus an ISS property for the exact discrete-time model of a closed-loop system, implies uniform ISS of the sampled-data nonlinear system. Our results serve to quantify these facts by means of comparison functions. Our results can be used as an alternative to prove and extend results in [1] or extend some results in [4] to a class of nonlinear systems. Finally, the formulas we establish can be used as a tool for some other problems which we indicate.

Keywords: Discrete-time; Input-to-state stability; $\mathcal{KL}$ functions; Sampled data; Stability

1. Introduction

There is a strong motivation for the investigation of sampled-data systems due to the prevalence of computer controlled systems (see [4–6]). Moreover, very often linear theory does not suffice and we need to deal with nonlinear sampled-data systems (see [3,9,10]). Although the topic is very old there does not appear to be a comprehensive theory even for analysis of properties of sampled-data nonlinear systems. For instance, L_p stability properties of linear sampled-data systems were completely characterized recently in [4], whereas such a characterization is still lacking for nonlinear sampled-data systems.

The ideas of “sampling”, which are related to Poincare maps, can be used to analyze properties of continuous-time systems (see [1,2,16,24,18,21]). Recently, a Lyapunov-type theorem was proved in [1] to show uniform local asymptotic stability of time-varying nonlinear systems by using a Lyapunov function whose value decreases along the solutions only at sampling instants. This result was used in [2,16] to prove several new results on averaging of

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nonlinear systems. A generalization of the stability result from [1] was presented in [24] where it was shown that global asymptotic stability of the averaged system implies semi-global-practical stability of the original nonlinear system.

An important method in stability and ISS analysis of continuous-time systems is based on the use class-$\mathcal{KL}$ and class-$\mathcal{K}$ functions (for classical results on $\mathcal{KL}$ functions see [11, pp. 135–139] or [8, pp. 7–8 and 95–101]; for new results on $\mathcal{KL}$ functions see [20]; for ISS see [19]). We give precise definitions of these functions in the preliminaries section. By using this method, definitions and often some proofs are simplified and more obvious (for instance, see [12, Section 5.3] or [8,18,19]). However, the theory that would allow the use of class-$\mathcal{KL}$ and class-$\mathcal{K}$ functions in the context of sampled-data nonlinear systems seems to be lacking.

It is the purpose of this paper to relate discrete-time and sampled-data $\mathcal{KL}$ stability or ISS estimates. The explicit formulas that we present are a tool which allows us to prove new results as well as provide alternative proofs for some old results. A consequence of the established bounds is that stability or ISS of the exact discrete-time model of the system implies the same (equivalent) property of the sampled-data model, under a uniform inter-sample growth condition. Some of our results generalize the result on L_∞ stability of linear systems in [4] to a class of nonlinear systems. It is possible to use our method as an alternative to prove results of [1]. Moreover, in Section 4 we generalize the main result of [1] to cover the ISS property. The formulas we establish provide the last technical step in the proof of the main result in [15] where we presented conditions which guarantee that the controller that globally stabilizes an approximate discrete-time model of the plant also semi-globally practically stabilizes the sampled-data system. Also, the results in [13,17] can be alternatively proved using the results of our paper and the approach in [15]. We introduce a new property of class-$\mathcal{KL}$ functions (the UIB property, defined below), that is very useful when relating discrete-time and sampled-data estimates. Finally, we prove a comparison theorem for discrete-time systems based on the use of an auxiliary scalar differential equation. We emphasize that our results hold for a large class of nonlinear systems and for arbitrary fixed sampling periods (this is not a fast sampling result).

As an example of the relationship we obtain, suppose that there exist $r^s, r^b > 0$, $\beta \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$, such that: (1) the discrete-time model satisfies the estimate

$$|x(k_0)| \leqslant r^s \Rightarrow |x(k)| \leqslant \beta(|x(k_0)|, k - k_0), \quad k \geqslant k_0 \geqslant 0,$$

where $k, k_0 \in \mathbb{N}$; and (2) the inter-sample behavior is characterized in the following way:

$$\forall t_0 \geqslant 0, \quad |x(t_0)| \leqslant r^b \Rightarrow |x(t)| \leqslant \tilde{\gamma}(|x(t_0)|), \qquad t \in [t_0, t_0 + T],$$

where $T > 0$ is the sampling period. Then the sampled-data model satisfies the bound

$$|x(t_0)| \leqslant r_x \Rightarrow |x(t)| \leqslant \bar{\beta}(|x(t_0)|, t - t_0), \quad t \geqslant t_0 \geqslant 0,$$

where $r_x := \min\{\tilde{\gamma}^{-1}(r^s),\ \tilde{\gamma}^{-1} \circ \beta_0^{-1}(r^b)\}$, $\beta_0(s) = \beta(s, 0)$ (it is safe to assume that $\beta_0 \in \mathcal{K}_\infty$, see Remark 1 below) and $\bar{\beta}$ is as follows: If $\tilde{\gamma}(\beta(\tilde{\gamma}(s), \tau)) \in \mathcal{KL}$ satisfies a property that we precisely define later (uniform incremental boundedness), then

$$\bar{\beta}(s, \tau) := \max\left\{\tilde{\gamma}(s)\mathrm{e}^{T-\tau}, P\tilde{\gamma}\left(\beta\left(\tilde{\gamma}(s), \frac{\tau}{T}\right)\right)\right\},$$

where $P > 0$. Otherwise, we show that in general $\bar{\beta}(s, \tau) \in \mathcal{KL}$ can be constructed as:

$$\bar{\beta}(s, \tau) := \max\left\{\tilde{\gamma}(s)\mathrm{e}^{T-\tau},\ 4 \max_{\eta \in [0,\tau]} 2^{-\eta}\, \tilde{\gamma}\left(\beta\left(\tilde{\gamma}(s), \frac{\tau - \eta}{T}\right)\right)\right\}.$$

Similar formulas are derived for ISS and also under stronger hypotheses we obtain global estimates.

The paper is organized as follows. In Section 2 we introduce the class of systems we consider and present definitions and notation. In Sections 3 and 4 we present, respectively, main results and applications of main results. A summary is given in the last section. An important technical lemma is proved in the appendix.

2. Preliminaries

We concentrate on the class of nonlinear sampled-data systems (see, for example [9]). The model given below represents a continuous-time plant (S_{ct}) controlled in a closed-loop by a digital controller (S_{dt}), the two being interconnected via the sampler (S) and the zero-order hold (H). The system is described by the equations

$$S_{\mathrm{ct}}: \quad \dot{x}_1(t) = f_1(t, x_1(t), \tilde{y}_2(t), d_1(t)), \quad t \geqslant 0, \quad t \in \mathbb{R},$$
$$y_1(t) = h_1(x_1(t)),$$

$$S: \quad y_1(k) = y_1(kT), \quad k \geqslant 0, \; k \in \mathbb{N},$$

$$S_{\rm dt}: \quad x_2(k+1) = f_2(k, y_1(k), x_2(k), d_2(k)),$$

$$y_2(k) = h_2(x_2(k), y_1(k)),$$

$$H: \quad \tilde{y}_2(t) = y_2(k), \quad t \in [kT, (k+1)T[. \tag{1}$$

where $T > 0$ is a fixed sampling period. The following notation is used: $x_1 \in \mathbb{R}^{n_1}$, $y_1 \in \mathbb{R}^{p_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $y_2 \in \mathbb{R}^{p_2}$ represent, respectively, the states and outputs of the plant and controller; $d_1 \in \mathbb{R}^{s_1}$, $d_2 \in \mathbb{R}^{s_2}$ are exogenous disturbances. The vector disturbance is denoted as $d(t) = (d_1^{\rm T}(t)\; d_2^{\rm T}(k))^{\rm T}$, $t \in [kT, (k+1)T[$, where d_1 is measurable and essentially bounded and d_2 is bounded, and its infinity norm $\|d\|_\infty = \operatorname{ess\,sup}_{t \geqslant 0} |d_1(t)| + \sup_{k \geqslant 0} |d_2(k)|$. It is assumed that $h_1(0) = 0, h_2(0,0) = 0$ and all of the functions are continuous.

A function $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class-$\mathscr{G}$ ($\gamma \in \mathscr{G}$) if it is continuous, zero at zero and nondecreasing. It is of class-$\mathscr{K}$ if it is of class-$\mathscr{G}$ and strictly increasing. It is of class-$\mathscr{K}_\infty$ if it is of class-$\mathscr{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class-$\mathscr{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathscr{K}$ for each $\tau \geqslant 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s > 0$ (not all papers assume monotonicity as a property of $\mathscr{KL}$ functions). A class-$\mathscr{KL}$ function $\beta(s,\tau)$ is called exponential if $\beta(s,\tau) \leqslant Ks{\rm e}^{-c\tau}$, $K > 0$, $c > 0$. We introduce a new property which plays an important role in relating discrete-time and sampled-data stability estimates: A class-$\mathscr{KL}$ function $\beta(s,\tau)$ is called uniformly incrementally bounded (UIB) if there exists a number $P > 0$ such that $\beta(s,\tau) \leqslant P\beta(s, \tau+1)$, $\forall s \geqslant 0$, $\forall \tau \in \mathbb{N}$. Note that since $\beta \in \mathscr{KL}$ we actually have $P > 1$. For instance, given positive numbers B, α and $\beta(s,\tau) = Bs{\rm e}^{-\alpha\tau}$, then $\beta \in \mathscr{KL}$ is UIB and $P = {\rm e}^{\alpha}$. The following lemma, whose proof is given in the appendix, and the corollary are two of the most important properties of UIB class-$\mathscr{KL}$ functions that we use in the sequel:

Lemma 1. *Given an arbitrary class-$\mathscr{KL}$ function $\beta(s,\tau)$, there exists a class-$\mathscr{KL}$ function $\tilde{\beta}(s,\tau)$, which is UIB, such that $\beta(s,\tau) \leqslant \tilde{\beta}(s,\tau)$, $\forall s \geqslant 0$, $\forall \tau \geqslant 0$. More precisely, we can always take*

$$\tilde{\beta}(s,\tau) := \max_{\eta \in [0,\tau]} 2^{-\eta} \beta(s, \tau - \eta) \tag{2}$$

and $P = 2$.

The proof of the corollary below follows immediately by induction:

Corollary 1. *Given an arbitrary UIB function $\beta(s,\tau) \in \mathscr{KL}$ and any integer $l \geqslant 0$, we have that*

$$\beta(s,\tau) \leqslant P^l \beta(s, \tau + l), \quad \forall \tau \in \mathbb{N}, \; s \geqslant 0.$$

System (1) is considered on the time interval $t \geqslant 0$ for continuous-time part of the system and $k \geqslant 0$ for discrete-time part of the system. Let $x_1(0)$, $x_2(0)$ be specified. We choose the state of the sampled-data system $x_{\rm sd}(t)$ at time t to be (see also [4]):

$$x_{\rm sd}(t) := (x_1^{\rm T}(t) \;\; h_1(x_1(kT)) \;\; x_2^{\rm T}(k))^{\rm T},$$
$$t \in [kT, (k+1)T[, \;\; k \in \mathbb{N}. \tag{3}$$

Our choice is motivated by the fact that having $x_{\rm sd}(t_0)$ we can compute all signals in the system forward in time. Indeed, suppose that for some $t_1 \in [kT, (k+1)T[$ we know $x_{\rm sd}(t_1)$, $d[t_1, \infty)$. Then we can solve the following equations in order:

$$y_1(k) = h_1(x_1(kT)),$$

$$y_2(k) = h_2(x_2(k), y_1(k)),$$

$$\tilde{y}_2(t) = y_2(k), \quad t \in [t_1, (k+1)T],$$

$$\dot{x}_1(t) = f_1(t, x_1(t), \tilde{y}_2(t), d_1(t)),$$

$$x_1(t_1), \quad d_1[t_1, (k+1)T],$$

$$x_2(k+1) = f_2(k, h_1(x_1(kT)), x_2(k), d_2(k)),$$

$$x_2(k), \quad d_2(k), \tag{4}$$

to obtain the value of $x_{\rm sd}(t)$, $t \in [t_1, (k+1)T]$, assuming no finite escape time within this interval. By repeating similar calculations over successive sampling intervals we can find $x_{\rm sd}(t)$, $\forall t \geqslant t_1$, which shows that $x_{\rm sd}(t)$ is an appropriate choice for the state of the system. Moreover, $x_{\rm sd}(t)$ is, in a sense, a minimal choice since dropping either $x_1(t_1)$, $h_1(x_1(kT))$ or $x_2(k)$ from the definition of $x_{\rm sd}(t_1)$ proves insufficient to compute all variables for $t \geqslant t_1$.

The sampled-data system (1) is time-varying even without explicit dependence of f_1 and f_2 on t and k, respectively (since trajectories starting at $x(t_0) = x^*$, $t_0 = kT$ do not coincide in general with trajectories starting at $x(t_1) = x^*$, $t_1 \neq kT$) and we investigate the following properties:

Definition 1. System (1) is called

(1) uniformly locally asymptotically stable (ULAS) (uniformly globally asymptotically stable (UGAS)) if there exist $\beta \in \mathscr{KL}$, and $r_x > 0$ (there

exists $\beta \in \mathcal{KL}$) such that for any $t_0 \geqslant 0$ the following holds:

$$|x_{sd}(t_0)| \leqslant r_x \quad (\forall x_{sd}(t_0) \in \mathbb{R}^n) \Rightarrow$$
$$|x_{sd}(t)| \leqslant \beta(|x_{sd}(t_0)|, t - t_0), \quad \forall t \geqslant t_0, \tag{5}$$

(2) uniformly locally exponentially stable (ULES) if it is ULAS with an exponential estimate $\beta(s,\tau) = Bs\mathrm{e}^{-\alpha\tau}, \alpha,\ B > 0$; uniformly globally exponentially stable (UGES) if it is UGAS with an exponential estimate β,

(3) uniformly locally input-to-state stable (ULISS) (uniformly input-to-state stable (UISS)) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $r_x, r_d > 0$ (there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$) such that for any $t_0 \geqslant 0$ the following holds:

$$|x_{sd}(t_0)| \leqslant r_x,$$
$$\| d \|_\infty \leqslant r_d \ (\forall x_{sd}(t_0) \in \mathbb{R}^n,\ \| d \|_\infty < \infty) \Rightarrow$$
$$|x_{sd}(t)| \leqslant \beta(|x_{sd}(t_0)|, t - t_0) + \gamma(\| d \|_\infty),\ \forall t \geqslant t_0. \tag{6}$$

Remark 1. We note that $\beta_0(s) := \beta(s,0)$ can be assumed to belong to class-$\mathcal{K}_\infty$ since necessarily $\beta(s,0) \geqslant s,\ \forall s \leqslant r_x$ (just take $t = t_0$ and $\| d \|_\infty = 0$).

We will also consider the behavior of (1) at sampling instants only $x_{sd}(kT) = x_{sd}(k)$. If there are no disturbances, the exact discrete-time model of (1) is obtained by integrating the plant equations over one sampling interval. If there are some disturbances it is standard to assume that they are constant over sampling intervals. We do not necessarily take this approach in the sequel and we refer to the exact discrete-time model for (1) meaning the discrete-time model whose solutions coincide at sampling instants kT with solutions of (1) for the same initial states and inputs. We remark that if the plant and controller equations in (1) are time invariant, the exact discrete-time model is time invariant. For the stability properties of the exact discrete-time model of (1) we use Definition 1 where t and t_0 are, respectively, replaced by k and k_0, where $k, k_0 \in \mathbb{N}$.

Before we present main results we note that in [9] the "state" vector $x^r_{sd}(t) = (x_1^{\mathrm{T}}(t)\ x_2^{\mathrm{T}}(k))^{\mathrm{T}}$, $t \in [kT, (k+1)T[$ was used to prove a stability result for (1). We show below that the discrepancy with (3) is not important for the stability or ISS analysis. Denote $x_{sd}(kT) = x_{sd}(k)$ and $x^r_{sd}(kT) = x^r_{sd}(k)$.

Lemma 2. *Consider system* (1). *The following statements are equivalent*:

(1) *There exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *such that*

$$\forall x_{sd}(0) \in \mathbb{R}^{n_1 \times p_1 \times n_2},$$
$$\| d \|_\infty < \infty \Rightarrow$$
$$|x_{sd}(k)| \leqslant \beta(|x_{sd}(k_0)|, k - k_0) + \gamma(\| d \|_\infty),$$
$$\forall k \geqslant k_0 \geqslant 0.$$

(2) *There exist* $\beta_r \in \mathcal{KL}$ *and* $\gamma_r \in \mathcal{K}_\infty$ *such that*

$$\forall x^r_{sd}(0) \in \mathbb{R}^{n_1 \times n_2},$$
$$\| d \|_\infty < \infty \Rightarrow$$
$$|x^r_{sd}(k)| \leqslant \beta_r(|x^r_{sd}(k_0)|, k) + \gamma_r(\| d \|_\infty),$$
$$\forall k \geqslant k_0 \geqslant 0.$$

Proof. The norm of a vector $x = (x_1\ \ldots\ x_n)^{\mathrm{T}}$ in this proof is taken as $|x| = \sum_i |x_i|$. By assumption h_1 is continuous and zero at zero and hence it is $\mathcal{K}$-bounded; that is, there exists $\gamma_{h_1} \in \mathcal{K}_\infty$ such that $|h_1(x_1)| \leqslant \gamma_{h_1}(|x_1|)$. Both cases follow by direct computations and the relationship between the estimates is:

$1 \Rightarrow 2$: Given β and γ we compute $\beta_r(s,\tau) = \beta(s + \gamma_{h_1}(s), \tau)$, $\gamma_r(s) = \gamma(s)$.

$2 \Rightarrow 1$: Given β_r and γ_r we compute $\beta(s,\tau) = \beta_r(s,\tau) + \gamma_{h_1}(2\beta_r(s,\tau))$, $\gamma(s) = \gamma_r(s) + \gamma_{h_1}(2\gamma_r(s))$.

By equivalence of norms, we can write the result for any other norm. □

From Lemma 2 it follows that to conclude discrete-time stability or ISS of the whole system, we can use only part of the state vector. A straightforward consequence of the proof of Lemma 2 is

Corollary 2. *Consider system* (1) *and suppose that there exists* $K_{h_1} > 0$ *such that*

$$|h_1(x_1)| \leqslant K_{h_1} |x_1|, \quad \forall x_1. \tag{7}$$

Then if one of the conditions in Lemma 2 *holds with an exponential class-$\mathcal{KL}$ function, then the other condition also holds with an exponential class-$\mathcal{KL}$ function.*

Local versions of Lemma 2 and Corollary 2 can be easily formulated and details are omitted. Note that besides exponential convergence of x^r_{sd} at sampling instants we need also condition (7) to guarantee exponential convergence of x_{sd} at sampling instants.

Remark 2. In the sequel we use the discrete-time $\mathcal{KL}$ estimates on x_{sd} to state main results. However, by exploiting respectively Lemma 2 and Corollary 2 we can restate the results on asymptotic stability and exponential stability using the discrete-time estimates on the vector x^r_{sd}. These statements are omitted for space reasons.

The following definition is used in statements of our results.

Definition 2. Given a sampling period $T>0$, we say that the solutions of (1) are uniformly bounded over T (UBT) if there exist numbers $r^b>0$ and $r_1^b>0$ and class-$\mathscr{K}_\infty$ functions $\tilde{\gamma}$, $\tilde{\gamma}_1$ such that given any $t_0\geqslant 0$, $|x_{sd}(t_0)|\leqslant r^b$ and disturbance $d(t)$ such that $\|d\|_\infty\leqslant r_1^b$, the solution of system (1) exists on $[t_0,t_0+T]$ and satisfies the bound

$$|x_{sd}(t)|\leqslant\tilde{\gamma}(|x_{sd}(t_0)|)+\tilde{\gamma}_1(\|d\|_\infty),\quad t\in[t_0,t_0+T]. \tag{8}$$

If the given bounds hold for all $x(t_0)$ (and for all disturbances such that $\|d\|_\infty<\infty$) then we say that the solutions are uniformly globally bounded over T (UGBT). Also, in the second case if the function $\tilde{\gamma}$ above can be over bounded by a linear function, we say that the solutions are linearly uniformly bounded over T (LUBT) or linearly uniformly globally bounded over T (LUGBT).

When there are no disturbances we understand that $\tilde{\gamma}_1$, r_1 and $\|d\|_\infty$ are omitted from the above definition. This form of Definition 2 is used when we state stability results. The following two sufficient conditions for UBT and UGBT follow directly from continuity of solutions and can be easily proved using the same proof technique as in Theorems 2.5 and 2.6 in [11] (see also Lemma 1 in [1]):

Lemma 3. *Suppose that there exist k_1, $R>0$ and $\gamma_i\in\mathscr{K}_\infty$, $i=1,\dots,5$ such that if*

$$R\geqslant\max\{|x_1|,|x_2|,|\tilde{y}_2|,|h_1(x_1)|,|d_1|,|d|_2\},$$

then f_1 and f_2 in (1) *satisfy*

$$\begin{aligned}&|f_1(t,x_1,\tilde{y}_2,d_1)|\\&\quad\leqslant k_1|x_1|+\gamma_1(|\tilde{y}_2|)+\gamma_2(|d_1|),\quad\forall t\geqslant 0,\\&|f_2(k,h_1(x_1),x_2,d_2)|\\&\quad\leqslant\gamma_3(|x_2|)+\gamma_4(|h_1(x_1)|)+\gamma_5(|d_2|),\quad\forall k\geqslant 0.\end{aligned} \tag{9}$$

Then given any $T>0$ the solutions of (1) *are UBT.*

Lemma 4. *Suppose that f_1 and f_2 in* (1) *satisfy* (9) *for all x_1,x_2 and all d_1,d_2. Then given any $T>0$ the solutions of* (1) *are UGBT.*

3. Main results

We present below conditions that guarantee ULAS, UGAS, ULES, UGES, ULISS and UISS property for the sampled-data system. Moreover, we give the explicit formulas for computing the sampled-data $\mathscr{KL}$ estimates using such estimates for the discrete-time system, and the class-$\mathscr{K}_\infty$ functions given in Definition 2.

Theorem 1 (sampled-data ULAS $\Leftrightarrow$ discrete-time ULAS + UBT)**.** *The sampled-data system* (1) *is ULAS if and only if the following conditions hold*:

(1) *the discrete-time model is ULAS*,

(2) *the solutions are UBT.*

In particular, if there exist $r^s,r^b>0$, $\beta\in\mathscr{KL}$ and $\tilde{\gamma}\in\mathscr{K}_\infty$, such that

$$\begin{aligned}&(DT\text{-}ULAS)\quad |x_{sd}(k_0)|\leqslant r^s\Rightarrow\\&\qquad|x_{sd}(k)|\leqslant\beta(|x_{sd}(k_0)|,k-k_0),\\&\qquad k\geqslant k_0\geqslant 0,\\&(UBT)\quad\forall t_0\geqslant 0,\ |x_{sd}(t_0)|\leqslant r^b\Rightarrow\\&\qquad|x_{sd}(t)|\leqslant\tilde{\gamma}(|x_{sd}(t_0)|),\quad t\in[t_0,t_0+T],\end{aligned} \tag{10}$$

then

$$\begin{aligned}&(SD\text{-}ULAS)\quad |x_{sd}(t_0)|\leqslant r_x\Rightarrow\\&\qquad|x_{sd}(t)|\leqslant\bar{\beta}(|x_{sd}(t_0)|,t-t_0),\\&\qquad\forall t\geqslant t_0\geqslant 0,\end{aligned} \tag{11}$$

where $\bar{\beta}\in\mathscr{KL}$ is given by

(1) *when $\hat{\beta}(s,\tau):=\tilde{\gamma}(\beta(\tilde{\gamma}(s),\tau))$ is UIB with $P>1$, we can take*

$$\bar{\beta}(s,\tau)=\max\left\{\tilde{\gamma}(s)\mathrm{e}^{T-\tau},P^2\hat{\beta}\left(s,\frac{\tau}{T}\right)\right\}, \tag{12}$$

(2) *in general we can take*

$$\bar{\beta}(s,\tau)=\max\left\{\tilde{\gamma}(s)\mathrm{e}^{T-\tau},\ 4\max_{\eta\in[0,\tau]}2^{-\eta}\hat{\beta}\left(s,\frac{\tau-\eta}{T}\right)\right\} \tag{13}$$

and $r_x=\min\{\tilde{\gamma}^{-1}(r^s),\tilde{\gamma}^{-1}\circ\beta_0^{-1}(r^b)\}$, $\beta_0(s)=\beta(s,0)$, $\beta_0\in\mathscr{K}_\infty$.

Proof. Necessity is obvious and we address only sufficiency.

The proof is constructive and we show that (10) implies (11) with (12) or (13). Let $r_x = \min\{\tilde{\gamma}^{-1}(r^s), \tilde{\gamma}^{-1} \circ \beta_0^{-1}(r^b)\}$, where $\beta_0(s) = \beta(s,0)$ is a class-$\mathcal{K}_\infty$ function. Note that $\tilde{\gamma}(s) \geqslant s$, $\beta_0(s) \geqslant s$ so $r_x \leqslant \min\{r^s, \beta_0^{-1}(r^b)\} \leqslant \min\{\beta_0(r^s), r^b\}$.

With this choice for $r_x > 0$, the following holds: given any $t_0 \geqslant 0$ and $N \in \mathbb{N}$ such that $t_0 \in [NT, (N+1)T[$, we have

$$|x_{sd}(t_0)| \leqslant r_x \Rightarrow |x_{sd}(t)| \leqslant \tilde{\gamma}(|x_{sd}(t_0)|), \quad \forall t \in [t_0, (N+1)T]. \tag{14}$$

From (14) and the first condition in (10) we have

$$\begin{aligned} |x_{sd}(t_0)| \leqslant r_x &\Rightarrow |x_{sd}(N+1)| \leqslant \min\{r^s, \beta_0^{-1}(r^b)\} \\ &\Rightarrow |x_{sd}(k)| \leqslant \min\{\beta_0(r^s), r^b\}, \\ &\quad \forall k \geqslant N+1. \end{aligned} \tag{15}$$

The UBT property implies then that if $|x_{sd}(t_0)| \leqslant r_x$ then $x_{sd}(t)$ exists and is bounded for all $t \geqslant t_0$. We consider below only initial states such that $|x_{sd}(t_0)| \leqslant r_x$.

Also, since $e^{T-\tau} \geqslant 1$, $\forall \tau \leqslant T$, we can write using (14)

$$\begin{aligned} |x_{sd}(t)| &\leqslant \tilde{\gamma}(|x_{sd}(t_0)|) \\ &\leqslant \tilde{\gamma}(|x_{sd}(t_0)|) e^{T-(t-t_0)}, \\ &=: \beta_1(s, t-t_0), \quad \forall t \in [t_0, (N+1)T]. \end{aligned} \tag{16}$$

On the other hand, we can also write

$$|x_{sd}(t)| \leqslant \tilde{\gamma}(|x_{sd}(k+N+1)|), \quad t \in [(N+1+k)T, (N+2+k)T], \quad k \geqslant 0.$$

Then given any $t_0 \geqslant 0$ and $N \in \mathbb{N}$ such that $t_0 \in [NT, (N+1)T[$, from ULAS of the discrete-time model we have that for any $k \geqslant 0$ and $t \in [(N+1+k)T, (N+2+k)T[$ the following holds:

$$\begin{aligned} |x_{sd}(t)| &\leqslant \tilde{\gamma}(\beta(|x_{sd}(N+1)|, k)) \\ &\leqslant \tilde{\gamma}(\beta(\tilde{\gamma}(|x_{sd}(t_0)|), k)) \\ &=: \hat{\beta}(|x_{sd}(t_0)|, k). \end{aligned} \tag{17}$$

If $\hat{\beta}$ is UIB class-$\mathcal{KL}$ function (see Corollary 1), we can write

$$\begin{aligned} |x_{sd}(t)| &\leqslant \hat{\beta}(|x_{sd}(t_0)|, k) \\ &\leqslant P^2 \hat{\beta}(|x_{sd}(t_0)|, k+2), \\ &\quad t \in [(N+1+k)T, (N+2+k)T[, \ k \geqslant 0. \end{aligned} \tag{18}$$

Now note that with the above defined t_0, N, t we have $(t - t_0)/T < ((N+2+k)T - NT)/T = k+2$, $\forall k \geqslant 0$ and since $\hat{\beta}$ is class-$\mathcal{KL}$, we can rewrite (18)

$$\begin{aligned} |x_{sd}(t)| &\leqslant P^2 \hat{\beta}(|x_{sd}(t_0)|, k+2) \\ &< P^2 \hat{\beta}\left(|x_{sd}(t_0)|, \frac{t-t_0}{T}\right) \\ &=: \beta_2(|x_{sd}(t_0)|, t-t_0), \quad t \geqslant (N+1)T. \end{aligned} \tag{19}$$

Introduce a new class-$\mathcal{KL}$ function $\bar{\beta}$:

$$\bar{\beta}(s,\tau) := \max\{\beta_1(s,\tau), \beta_2(s,\tau)\}$$

and from (16) and (19) we can write

$$|x_{sd}(t_0)| \leqslant r_x \Rightarrow |x_{sd}(t)| \leqslant \bar{\beta}(|x_{sd}(t_0)|, t-t_0), \quad t \geqslant t_0,$$

where $t_0 \geqslant 0$ is arbitrary.

If $\hat{\beta}$ is not UIB, we majorize it using Lemma 1 with a UIB class-$\mathcal{KL}$ function $\tilde{\beta}$ ($P = 2$ in this case):

$$\tilde{\beta} = \max_{\eta \in [0,\tau]} 2^{-\eta} \tilde{\gamma}\left(\beta\left(\tilde{\gamma}(s), \frac{\tau-\eta}{T}\right)\right),$$

and repeat all the calculations, which completes the proof. □

Remark 3. We note that the above proof may be carried out without resorting to the UIB property. Indeed, what we really need in the proof is that given an arbitrary class-$\mathcal{KL}$ function $\beta(s,\tau)$, we can find another class-$\mathcal{KL}$ function $\beta_1(s,\tau)$ such that

$$\beta(s,\tau) \leqslant \beta_1(s, \tau+2), \quad \forall s, \ \tau \geqslant 0.$$

Another way to see that we can find such β_1 in addition to using Lemma 1 and Corollary 1 is to use Lemma 8 of [18] which states that given any $\beta \in \mathcal{KL}$, we can always find $u, v \in \mathcal{K}_\infty$ such that

$$\beta(s,\tau) \leqslant u(s) v(e^{-\tau}), \quad \forall s \geqslant 0, \ \tau \geqslant 0.$$

Also, using Corollary 10 in [18] we can always find $v_1, v_2 \in \mathcal{K}_\infty$ such that $v(cd) \leqslant v_1(c) v_2(d)$, $\forall c, d \geqslant 0$, which implies

$$v(e^{-\tau}) = v(e^2 e^{-\tau-2}) \leqslant v_1(e^2) v_2(e^{-\tau-2}).$$

We define $\beta_1(s,\tau) := u(s) v_1(e^2) v_2(e^{-\tau})$ and it follows that $\beta(s,\tau) \leqslant \beta_1(s, \tau+2)$, $\forall s, \tau \geqslant 0$.

We presented the UIB property for the following reasons: it allowed us to obtain more explicit formulas which relate the original β with its UIB over bound β_1; the construction we just showed may lead to a more conservative β_1 when compared to the approach we took in the paper; UIB is a new property of class-$\mathcal{KL}$ functions which seems to be of interest in its own right.

Note that formula (13) in Theorem 1 holds in general. In the results to follow, instead of stating separately formulas for $\mathcal{KL}$ functions depending on whether $\hat{\beta}$ is UIB or not, we only state the formulas that hold in general. The changes in statements when $\hat{\beta}$ is UIB are obvious. The following three results are proved in a similar manner and proofs are omitted.

Theorem 2 (sampled-data UGAS $\Leftrightarrow$ discrete-time UGAS + UGBT)**.** *The sampled-data system* (1) *is UGAS if and only if*:

(1) *the discrete-time model is UGAS*,

(2) *the solutions of* (1) *are UGBT.*

In particular, if there exist $\beta \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$, *such that*

$$(DT\text{-}UGAS) \quad |x_{sd}(k)| \leqslant \beta(|x_{sd}(k_0)|, k-k_0), \quad k \geqslant k_0 \geqslant 0, \quad \forall x_{sd}(k_0),$$

$$(UGBT) \quad |x_{sd}(t)| \leqslant \tilde{\gamma}(|x_{sd}(t_0)|), \quad \forall t_0 \geqslant 0, \quad t \in [t_0, t_0+T],\ \forall x_{sd}(t_0), \tag{20}$$

then, $\forall x_{sd}(t_0)$

$$(SD\text{-}UGAS) \quad |x_{sd}(t)| \leqslant \bar{\beta}(|x_{sd}(t_0)|, t-t_0), \quad \forall t \geqslant t_0 \geqslant 0, \tag{21}$$

where $\bar{\beta} \in \mathcal{KL}$ *is given by* (13).

Theorem 3 (discrete-time ULES + LUBT $\Leftrightarrow$ sampled-data ULES)**.** *The sampled-data system is ULES if and only if*

(1) *the discrete-time model is ULES*,

(2) *the solutions of* (1) *are LUBT.*

In particular, if there exist $r^s, r^b > 0,\ B > 0,\ \alpha > 0$ *such that*

$$(DT\text{-}ULES) \quad |x_{sd}(k_0)| \leqslant r^s \Rightarrow |x_{sd}(k)| \leqslant B e^{-\alpha(k-k_0)} |x_{sd}(k_0)|, \quad k \geqslant k_0 \geqslant 0,$$

$$(LUBT) \quad \forall t_0 \geqslant 0,\ |x_{sd}(t_0)| \leqslant r^b \Rightarrow |x_{sd}(t)| \leqslant \tilde{K} |x_{sd}(t_0)|, \quad t \in [t_0, t_0+T], \tag{22}$$

then

$$(SD\text{-}ULES) \quad |x_{sd}(t_0)| \leqslant r_x \Rightarrow |x_{sd}(t)| \leqslant K e^{(-\alpha(t-t_0))/T} |x_{sd}(t_0)|, \quad \forall t \geqslant t_0 \geqslant 0,$$

where $K := \tilde{K}^2 e^{2\alpha} B,\ r_x := \min\{r^s/\tilde{K}, r^b/\tilde{K}B\}$.

Theorem 4 (discrete-time UGES + LUGBT $\Leftrightarrow$ sampled-data UGES)**.** *The sampled-data system is UGES if and only if*

(1) *the discrete-time model is UGES*,

(2) *the solutions of* (1) *are UGBT.*

In particular, if there exist $B > 0,\ \alpha > 0$ *such that*

$$(DT\text{-}UGES) \quad |x_{sd}(k)| \leqslant B e^{-\alpha(k-k_0)} |x_{sd}(k_0)|, \quad k \geqslant k_0 \geqslant 0,\ \forall x_{sd}(k_0),$$

$$(LUGBT) \quad |x_{sd}(t)| \leqslant \tilde{K} |x_{sd}(t_0)|, \quad t \in [t_0, t_0+T], \quad \forall t_0 \geqslant 0,\ \forall x_{sd}(t_0), \tag{23}$$

then, $\forall x_{sd}(t_0)$

$$(SD\text{-}UGES) \quad |x_{sd}(t)| \leqslant K e^{(-\alpha(t-t_0))/T} |x_{sd}(t_0)|, \quad \forall t \geqslant t_0 \geqslant 0,$$

where $K := \tilde{K}^2 e^{2\alpha} B$.

Remark 4. Under the conditions of Lemma 3 (respectively, Lemma 4), namely a sector bound on f, it can be shown that if $|h_2(x_2, y_1)| \leqslant \gamma_{h_2}(|x|)$ then the solutions of (1) are UBT (respectively, GUBT) with $\tilde{\gamma}(s) \propto \gamma_{h_2}(s)$. Hence, if γ_{h_2} is locally (or globally) linearly bounded, $\tilde{\gamma}$ is also locally (globally) linearly bounded and exponential convergence of x_{sd} at sampling instants implies ULES (UGES) of the sampled-data system. Without the liner bound on γ_{h_2}, and therefore on $\tilde{\gamma}$, it is not clear whether ULES (UGES) of the discrete-time model implies ULES (UGES) of the sampled-data model.

Using a proof similar to that of Theorem 1, we can prove the following two results:

Theorem 5 (sampled-data ULISS $\Leftrightarrow$ discrete-time ULISS + UBT)**.** *The sampled-data system* (1) *is ULISS if and only if the following conditions hold*:

(1) *the discrete-time model is ULISS, and*

(2) *the solutions are UBT.*

In particular, if there exist $r^s, r_1^s, r^b, r_1^b > 0,\ \beta \in \mathcal{KL}$ *and* $\tilde{\gamma}, \tilde{\gamma}_1, \gamma_1 \in \mathcal{K}_\infty$, *such that*

$$(DT\text{-}ULISS) \quad |x_{sd}(k_0)| \leqslant r^s\ \|d\|_\infty \leqslant r_1^s \Rightarrow |x_{sd}(k)| \leqslant \beta(|x_{sd}(k_0)|, k-k_0) + \gamma_1(\|d\|_\infty), \quad k \geqslant k_0 \geqslant 0,$$

$$(UBT) \quad \forall t_0 \geqslant 0,\ |x_{sd}(t_0)| \leqslant r^b\ \|d\|_\infty \leqslant r_1^b \Rightarrow |x_{sd}(t)| \leqslant \tilde{\gamma}(|x_{sd}(t_0)|) + \tilde{\gamma}_1(\|d\|_\infty), \quad t \in [t_0, t_0+T], \tag{24}$$

then

$$(SD\text{-}ULISS)\quad |x_{sd}(t_0)|\leqslant r_x,\ \|d\|_\infty\leqslant r_d \Rightarrow |x_{sd}(t)|\leqslant\bar{\beta}(|x_{sd}(t_0)|,t-t_0)+\gamma_d(\|d\|_\infty),\quad \forall t\geqslant t_0\geqslant 0, \tag{25}$$

where we can take

$$\bar{\beta}(s,\tau):=\max\left\{\tilde{\gamma}(s)e^{T-\tau},\ 4\max_{\eta\in[0,\tau]}2^{-\eta}\tilde{\gamma}\times\left(4\beta\left(2\tilde{\gamma}(s),\frac{\tau-\eta}{T}\right)\right)\right\},$$

$$\gamma_d(s):=\tilde{\gamma}(4\beta(2\tilde{\gamma}_1(s),0))+\tilde{\gamma}(2\gamma_1(s))+\tilde{\gamma}_1(s),$$

$$r_x:=\max\left\{\tilde{\gamma}^{-1}\left(\frac{(1-\varepsilon)r^s}{2}\right),\ \tilde{\gamma}^{-1}\left(\beta_0^{-1}\left(\frac{(1-\varepsilon)r^b}{2}\right)\right)\right\},$$

$$r_d:=\min\left\{\tilde{\gamma}_1^{-1}\left(\frac{\varepsilon r^s}{2}\right),\gamma_1^{-1}\left(\frac{\varepsilon r^b}{2}\right),r_1^b,r_1^s\right\} \tag{26}$$

and ε *is an arbitrary number* $0<\varepsilon<1$.

We note that there is some freedom in choosing r_x and r_d in (26) since we can choose the number ε. Hence, we can decrease (increase) r_x while increasing (decreasing) r_d.

Theorem 6 (sampled-data UISS $\Leftrightarrow$ discrete-time UISS + UGBT)**.** *The sampled-data system* (1) *is UISS if and only if the following conditions hold*:

(1) *the discrete-time model is UISS*, *and*

(2) *the solutions are UGBT.*

In particular, *if there exist* $\beta\in\mathcal{KL}$ *and* $\tilde{\gamma},\tilde{\gamma}_1,\gamma_1\in\mathcal{K}_\infty$, *such that for* $\|d\|_\infty<\infty$ *we have*

$$(DT\text{-}UISS)\quad |x_{sd}(k)|\leqslant\beta(|x_{sd}(k_0)|,\ k-k_0)+\gamma_1(\|d\|_\infty),\ k\geqslant k_0\geqslant 0,\ \forall x_{sd}(k_0),$$

$$(UGBT)\quad |x_{sd}(t)|\leqslant\tilde{\gamma}(|x_{sd}(t_0)|)+\tilde{\gamma}_1(\|d\|_\infty),\ t\in[t_0,t_0+T],\ \forall t_0\geqslant 0,\ \forall x_{sd}(t_0), \tag{27}$$

then, $\forall x_{sd}(t_0)$, $\|d\|_\infty<\infty$ *we have*

$$(SD\text{-}UISS)\quad |x_{sd}(t)|\leqslant\bar{\beta}(|x_{sd}(t_0)|,\ t-t_0)+\gamma_d(\|d\|_\infty),\quad \forall t\geqslant t_0\geqslant 0, \tag{28}$$

where $\bar{\beta}\in\mathcal{KL}$ *and* $\gamma_d\in\mathcal{K}_\infty$ *are given in* (26).

4. Applications of main results

In this section we show how our results can be applied to some problems. We indicate some results from the literature that either become corollaries of our results or for which our method provides an alternative proof.

4.1. Relations between discrete-time stability and sampled-data ISS

As a first illustration of the utility of our results we consider the relationship between the stability and ISS properties of the sampled-data system. The classical approach to this problem is to use an appropriate converse stability theorem and some assumptions which are then exploited to show the robustness, that is an appropriate ISS property (see Theorem 6.1 in [11]). We are not aware of any converse stability theorems for sampled-data systems. However, such theorems are available for discrete-time systems, see [14,7]. Our results allow us to exploit the following implications in proving UISS or ULISS of sampled-data systems:

discrete-time stability + assumptions

$\Rightarrow$ discrete-time ULISS (UISS) + UBT

$\Rightarrow$ sampled-data ULISS(UISS),

the last implication being proved in the previous section. In this way, classical results on total stability (ULISS) of discrete-time systems, together with our results, give total stability results for the sampled-data systems.

As a simple illustration of this approach we provide conditions that guarantee that discrete-time UGES implies sampled-data UISS. The result is based on the converse Lyapunov theorem for UGES of time-invariant discrete-time systems in [14]. A converse theorem for ULAS of time-varying discrete-time systems in [7] can be used in a similar way to show ULISS of sampled-data systems. For another result on discrete-time ULISS (total stability) see Theorem 2.3.7 in [22, pp. 98].

Suppose that the exact discrete-time model of the system (1) is time invariant:

$$x(k+1)=F_e(x(k),d[k]) \tag{29}$$

where $F_e(0,0)=0$ and $d[k]:=\{d(t),\ t\in[kT,(k+1)T]\}$. We take the norm of $d[k]$ to be the supremum of $d(t)$ over $[kT,(k+1)T]$. If $d[k]=\{d(k)\},d(k)=$

const. then we obtain the more familiar discrete-time model

$$x(k+1) = F_e^*(x(k), d(k)).$$

Lemma 5. *Suppose that F_e is globally Lipschitz for $d[k] \equiv 0$ and there exists $K_d > 0$ such that*

$$|F_e(x, d[k]) - F_e(x, 0)| \leqslant K_d |d[k]|,$$

$$\forall x, \ |d[k]| < \infty, \ \forall k \geqslant 0.$$

Under these conditions, if (29) *with $d[k] \equiv 0$, $\forall k \geqslant 0$ is GES, then the discrete-time model* (29) *is ISS.*

Proof. From the converse Lyapunov theorem in [14] for global exponential stability, we know that if

$$x(k+1) = F_e(x(k), 0)$$

is GES, then we can find a Lyapunov function $V(x)$, such that

$$V(0) = 0,$$

$$V(x) \geqslant |x|, \quad \forall x \in \mathbb{R}^n,$$

$$\exists L_1 > 0, \ |V(x) - V(x_1)| \leqslant L_1 |x - x_1|, \quad \forall x, x_1 \in \mathbb{R}^n,$$

$$V(F_e(x(k), 0)) - V(x(k)) \leqslant (\lambda - 1) V(x(k)), \quad \forall x(k) \in \mathbb{R}^n, \ \lambda \in]0, 1[. \quad (30)$$

We add and subtract $V(F_e(x(k), d[k]))$ to the last condition and can write

$$V(F_e(x(k), d[k])) - V(x(k)) \leqslant (\lambda - 1) V(x(k)) + V(F_e(x(k), d[k])) - V(F_e(x(k), 0)), \quad \forall x(k) \in \mathbb{R}^n, \ |d[k]| < \infty$$

and using the second condition we obtain

$$V(F_e(x(k), d[k])) - V(x(k)) \leqslant (\lambda - 1) V(x(k)) + L_1 |F_e(x(k), d[k]) - F_e(x(k), 0)|, \quad \forall x(k) \in \mathbb{R}^n, \ |d[k]| < \infty.$$

Finally, define $V_{k+1} := V(F_e(x(k), d[k]), V_k := V(x(k))$ and from the conditions of theorem we have

$$V_{k+1} \leqslant \lambda V_k + L_1 K_d \| d \|_\infty, \quad \forall x \in \mathbb{R}^n, \ \| d \|_\infty < \infty.$$

Using the discrete Gronwall lemma [23, pp. 9] we obtain:

$$|x(k)| \leqslant L_1 \lambda^k |x(0)| + \frac{1 - \lambda^k}{1 - \lambda} L_1 K_d \| d \|_\infty \leqslant L_1 \lambda^k |x(0)| + \frac{L_1 K_d}{1 - \lambda} \| d \|_\infty, \ \forall x(0), \ \forall d \in L_\infty$$

which establishes the UISS of the discrete-time system (29). □

A simple consequence of Theorem 6 is:

Corollary 3. *If the conditions of Lemma* 5 *are satisfied and the solutions are UGBT, then the sampled-data system* (1) *is UISS.*

In the case of linear systems controlled with linear controllers, we can easily see that the conditions of Corollary 3 are always satisfied and we recover Theorem 5 in [6] on L_∞ stability of sampled-data linear systems.

4.2. A test for ISS of time-varying systems

As a second application of our results we note that they can be used as an alternative to prove Theorem 1 in [1] (see also [24]), where a new Lyapunov-type theorem was presented to test ULAS of time-varying nonlinear systems given by

$$\dot{x}(t) = f(t, x(t)) \quad (31)$$

by using an appropriate "discrete-time" condition. By using our approach we can prove the following generalized result on ULISS (UISS):

Theorem 7. *Consider the system*

$$\dot{x}(t) = f(t, x(t), d(t)) \quad (32)$$

and suppose that there exists $T_d > 0$ such that the following conditions hold:

(1) *There exist $\tilde{\gamma}, \tilde{\gamma}_1 \in \mathscr{K}_\infty$ and $r^b > 0, r_1^b > 0$ such that for any $t_0 \in \mathbb{R}, |x(t_0)| \leqslant r^b$, $\| d \|_\infty \leqslant r_1^b$ the solutions of* (32) *exist on $[t_0, t_0 + T_d]$ and satisfy*

$$|x(t)| \leqslant \tilde{\gamma}(|x(t_0)|) + \tilde{\gamma}_1(\| d \|_\infty), \quad t \in [t_0, t_0 + T_d]. \quad (33)$$

(2) *There exist $r^s, r_1^s > 0, \beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}_\infty$ and a positive number $T_d > 0$ such that for any $(x(t_0), t_0, d(t))$ such that $|x(t_0)| \leqslant r^s$, $\| d \|_\infty \leqslant r_1^s$ there exists an increasing sequence $\{t_i\}_{i=0}^\infty$ with $t_i \to +\infty$ as $i \to \infty$ and $t_{i+1} - t_i \leqslant T_d, \forall i \geqslant 0$ such that*

$$|x(t_i)| \leqslant \beta(|x(t_0)|, t_i - t_0) + \gamma(\| d \|_\infty), \quad i \geqslant 0. \quad (34)$$

Then the time-varying system (32) *is ULISS. Moreover, if all the assumptions hold globally, system* (32) *is UISS.*

Proof. We show that there exist $r>0, r_1>0$ and $\beta^*\in\mathscr{KL}, \gamma^*\in\mathscr{K}_\infty$ such that for arbitrary $t_0\geqslant 0$ and $|x(t_0)|\leqslant r$, $\|d\|_\infty\leqslant r_1$ we have

$$\forall t_0\geqslant 0,\ |x(t_0)|\leqslant r,\ \|d\|_\infty\leqslant r_1 \Rightarrow$$
$$|x(t_0+kT_d)|\leqslant \beta^*(|x(t_0)|,k)+\gamma^*(\|d\|_\infty),\quad k\geqslant 0 \tag{35}$$

and once this is established, the proof follows from Theorem 5.

Introduce

$$r:=\max\left\{\tilde{\gamma}^{-1}\left(\frac{(1-\varepsilon)r^s}{2}\right),\ \tilde{\gamma}^{-1}\left(\beta_0^{-1}\left(\frac{(1-\varepsilon)r^b}{2}\right)\right)\right\},$$

$$r_1:=\min\left\{\tilde{\gamma}_1^{-1}\left(\frac{\varepsilon r^s}{2}\right),\gamma_1^{-1}\left(\frac{\varepsilon r^b}{2}\right),r_1^b,r_1^s\right\},$$

where ε is an arbitrary number $0<\varepsilon<1$. Using the same argument as in Theorem 5 it can be shown that $|x(t_0)|\leqslant r$, $\|d\|_\infty\leqslant r_1$ imply $|x(t_i)|\leqslant\min\{\beta_0(r^s),r^b\}$, $\forall i$, and using (33), we conclude that the solutions exist and satisfy the inequalities (33) and (34), respectively, for all $t_0\geqslant 0$ and $\forall i\geqslant 0$.

Consider an arbitrary $t_0\geqslant 0$, $|x(t_0)|\leqslant r$ and $\|d\|_\infty\leqslant r_1$. Consider the corresponding sequence $\{t_i\}_{i=0}^\infty$ and introduce a subsequence $\{t_{i_k}\}_{k=0}^\infty$ such that $t_{i_0}=t_0$ and $t_{i_k}:=\max\{t_i\colon t_0+(k-1)T_d\leqslant t_i\leqslant t_0+kT_d\}, k\geqslant 1$, for which we can write for all $k\geqslant 1$:

$$\begin{aligned}|x(t_0+kT_d)|&\leqslant\tilde{\gamma}(|x(t_{i_k})|)+\tilde{\gamma}_1(\|d\|_\infty)\\&\leqslant\tilde{\gamma}(\beta(|x(t_0)|,t_{i_k}-t_0)+\gamma(\|d\|_\infty))+\tilde{\gamma}_1(\|d\|_\infty)\\&\leqslant\tilde{\gamma}(2\beta(|x(t_0)|,t_{i_k}-t_0))\\&\quad+\tilde{\gamma}(2\gamma(\|d\|_\infty))+\tilde{\gamma}_1(\|d\|_\infty).\end{aligned}\tag{36}$$

Since $t_{i_k}-t_0\leqslant(k-1)T_d, \forall k\geqslant 0$ and $\beta\in\mathscr{KL}$, we have that

$$\beta(|x(t_0)|,t_{i_k}-t_0)\leqslant\beta(|x(t_0)|,(k-1)T_d),\quad\forall k\geqslant 0.\tag{37}$$

We introduce $\beta_1(s,\tau):=\beta(s,T_d\tau)$ using Lemma 1 we find $\tilde{\beta}_1\in\mathscr{KL}$ such that $\beta_1(s,\tau)\leqslant\tilde{\beta}_1(s,\tau)\leqslant 2\tilde{\beta}(s,\tau+1)$. Hence, we can write that

$$\begin{aligned}\tilde{\gamma}(2\beta_1(|x(t_0)|,k-1))&\leqslant\tilde{\gamma}(2\tilde{\beta}_1(|x(t_0)|,k-1))\\&\leqslant\tilde{\gamma}(4\tilde{\beta}_1(|x(t_0)|,k)),\end{aligned}$$

which, together with (36) and (37), proves (35) with $\beta^*(s,\tau):=\tilde{\gamma}(4\tilde{\beta}_1(s,\tau)), \gamma^*(s):=\tilde{\gamma}(2\gamma(s))+\tilde{\gamma}_1(s)$. □

Theorem 1 in [1] is a corollary of Theorem 7 when $d(t)\equiv 0$. Indeed, Lemma 1 in [1] follows from Lemmas 3 and 4 with $d(t)\equiv 0$. Moreover, the existence of a positive definite, decrescent Lyapunov function which decreases at sampling instants along the solutions of (31) (see Theorem 1 in [1]) implies the existence of a class-$\mathscr{KL}$ function satisfying the second condition of Theorem 7 (more precisely (35) holds with $\|d\|_\infty=0$) as we show below. The result is interesting since we obtain a discrete-time $\mathscr{KL}$ estimate by using a $\mathscr{KL}$ function obtained from an auxiliary continuous-time differential equation.

Theorem 8. *Let $V:\mathbb{R}^n\to\mathbb{R}_{\geqslant 0}$ and suppose that*

$$V(k+1)-V(k)\leqslant-\alpha(V(k)),\quad V(k_0)=V_0,\tag{38}$$

where $\alpha\in\mathscr{K}$ is defined on $[0,b[, b>0$. Then there exists $\beta\in\mathscr{KL}$ such that for any $0\leqslant V_0<b$ we have

$$V(k)\leqslant\beta(V_0,k-k_0),\quad\forall k\geqslant k_0.\tag{39}$$

More specifically, the solution of the auxiliary scalar differential equation [4]

$$\dot{y}=-\alpha(y),\quad y(t_0)=y_0\tag{40}$$

is class-$\mathscr{KL}$ in initial condition and elapsed time, i.e. we can write $0\leqslant y_0<b\Rightarrow y(t)=\beta_1(y_0,t-t_0)$, where $\beta_1\in\mathscr{KL}$ and we can take $\beta(s,\tau):=\beta_1(s,\tau)$.

Proof. We introduce a variable $t\in\mathbb{R}$ and define $y(t):=V(k)+(t-k)(V(k+1)-V(k)), t\in[k,k+1], k\geqslant 0$. Note that $0\leqslant y(k)=V(k), k\geqslant 0$ and $y(t)$ is a continuous function of "time" t. Moreover, it is absolutely continuous in t (in fact, piecewise linear) and we can write for almost all t:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}y(t)&=V(k+1)-V(k),\quad t\in[k,k+1],\ k\geqslant 0\\&\leqslant-\alpha(V(k)),\quad t\in[k,k+1],\ k\geqslant 0\\&\leqslant-\alpha(y(t)).\end{aligned}\tag{41}$$

Let $v(t)=\beta(v_0,t)$ be the (unique) solution of $\dot{v}=-\alpha(v), v(t_0)=v_0$. It is shown in Lemma 6.1 in [19] that $\beta\in\mathscr{KL}$. By standard comparison theorems (see

[4] It can be assumed without loss of generality that α is locally Lipschitz since if it is not we can always find a locally Lipschitz class-$\mathscr{K}$ function α_1 defined on $[0,b[$ such that $-\alpha(s)\leqslant-\alpha_1(s), s\in[0,b[$. Hence, we can assume uniqueness of solutions of the scalar differential equation (40).

for instance [11, Theorem 1.10.2]) we have for $y_0=v_0$ that

$$y(t)\leqslant v(t)=\beta(y_0,t-t_0), \quad \forall t\geqslant t_0,$$

which implies using $V(k)=y(k)$ with $t=k$, $t_0=k_0, y_0=V_0$ that

$$V(k)\leqslant\beta(V_0,k-k_0), \quad k\geqslant k_0$$

and this completes the proof. □

4.3. On practical stability

As we indicated, our results require the knowledge of stability or ISS properties of the exact discrete-time model. The exact discrete-time model of (1) is difficult to obtain in general and especially when $d_1(t)\neq 0$. However, it is often possible to conclude about stability properties of the exact model by using an approximate discrete-time model, such as Euler approximation. For instance, the results of [15] draw conclusions about stability for a family of exact discrete-time control systems based on stability and other properties assumed for a family of approximate discrete-time control systems. The family of exact discrete-time control systems is shown to be practically stable. Thus, the stability bounds involve positive offsets like

$$|x(k_0)|\leqslant r \Rightarrow |x(k)|\leqslant\beta(|x(k_0)|,k-k_0)+R, \quad k\geqslant k_0\geqslant 0. \tag{42}$$

While we have not explicitly stated results with offsets here, they can be handled in the same way that disturbances are handled. In particular, under the condition (42) and UBT (in fact, an offset is now allowed in the UBT definition) and if R is sufficiently small we can show that there exists $\bar\beta\in\mathcal{KL}$ and $r_x,\bar R>0$ such that the solutions of the sampled-data system satisfy

$$|x(t_0)|\leqslant r_x \Rightarrow |x(t)|\leqslant\beta(|x(t_0)|,t-t_0)+\bar R, \quad t\geqslant t_0\geqslant 0.$$

(The proof of this result is the same as that of Theorem 5.) Hence, our results provide the last technical step in proving (practical) stability of a sampled-data system in the following way: approximate discrete-time model stability + assumptions ⇒ exact discrete-time model (practical) stability ⇒ sampled-data model (practical) stability.

5. Summary

We presented formulas that relate $\mathcal{KL}$ stability and ISS estimates between the discrete-time and sampled-data models for a large class of systems. The estimates are very important in the analysis of sampled-data nonlinear systems and they allowed us to recover or generalize some results from the literature. We showed that ULISS (total stability) and UISS results for the sampled-data system can be deduced from the corresponding results for the discrete-time model. A new result on ULISS and UISS for time-varying nonlinear systems also follows from our approach. A new property of $\mathcal{KL}$ functions was presented and used to prove the results.

Appendix

Proof of Lemma 1. Let $\tilde\beta(s,\tau)$ be given by (2). This function satisfies the UIB property with $P=2$:

$$\begin{aligned}\tilde\beta(s,\tau) &= \max_{\eta\in[0,\tau]} 2^{-\eta}\beta(s,\tau-\eta)\\ &= \max_{R\in[1,\tau+1]} 2^{-(R-1)}\beta(s,\tau-(R-1))\\ &= 2\max_{R\in[1,\tau+1]} 2^{-R}\beta(s,\tau+1-R)\\ &\leqslant 2\max_{\eta\in[0,\tau+1]} 2^{-\eta}\beta(s,\tau+1-\eta)\\ &= 2\tilde\beta(s,\tau+1).\end{aligned}$$

We also have

$$\tilde\beta(s,\tau)\geqslant 2^{-\eta}\beta(s,\tau-\eta)\Big|_{\eta=0}=\beta(s,\tau). \tag{A.1}$$

Now we show that $\tilde\beta$ is a class-$\mathcal{KL}$ function.

$\mathcal{K}$ property. Since $\beta\in\mathcal{KL}$, for arbitrary $\tau\geqslant 0$, $\tilde\beta(\cdot,\tau)$ is continuous and we have $\tilde\beta(0,\tau)=0$. For arbitrary fixed τ, consider $s_2>s_1\geqslant 0$. Since $\beta\in\mathcal{KL}$, for arbitrary τ and $\eta\in[0,\tau]$ we have that $2^{-\eta}\beta(s_2,\tau-\eta)>2^{-\eta}\beta(s_1,\tau-\eta)$ which implies $\tilde\beta(s_2,\tau)>\tilde\beta(s_1,\tau)$.

$\mathcal{L}$ property. Given arbitrary fixed $s>0$, consider $\tau_2>\tau_1$. Introduce $\delta:=\tau_2-\tau_1>0$. Then we can write

$$\begin{aligned}\tilde\beta(s,\tau_2) &= \max_{\eta\in[0,\tau_2]} 2^{-\eta}\beta(s,\tau_2-\eta)\\ &= \max\left\{\max_{\eta\in[0,\delta]} 2^{-\eta}\beta(s,\tau_2-\eta),\right.\end{aligned}$$

$$\left.\max_{\eta\in[\delta,\tau_1+\delta]} 2^{-\delta}2^{-(\eta-\delta)}\beta(s,\tau_1-(\eta-\delta))\right\}$$

$$= \max\left\{\max_{\eta\in[0,\delta]} 2^{-\eta}\beta(s,\tau_2-\eta),\right.$$

$$\left.\max_{h\in[0,\tau_1]} 2^{-\delta}2^{-h}\beta(s,\tau_1-h)\right\}$$

$$=: \max\{\beta_1(s,\tau_2), 2^{-\delta}\tilde{\beta}(s,\tau_1)\}. \quad \text{(A.2)}$$

Obviously we have that $2^{-\delta}\tilde{\beta}(s,\tau_1) < \tilde{\beta}(s,\tau_1)$. Moreover, by considering cases $0 \leqslant \eta \leqslant \delta/2$ and $\delta/2 < \eta \leqslant \delta$ we can write

$$\beta_1(s,\tau_2) = \max\left\{\max_{\eta\in[0,\delta/2]} 2^{-\eta}\beta(s,\tau_2-\eta),\right.$$

$$\left.\max_{\eta\in[\delta/2,\delta]} 2^{-\eta}\beta(s,\tau_2-\eta)\right\}$$

$$\leqslant \max\{\beta(s,\tau_1+\delta/2), 2^{-\delta/2}\beta(s,\tau_1)\} \quad \text{(A.3)}$$

and using (A.1) and the fact that $\beta \in \mathscr{KL}$ we can write $\beta(s,\tau_1+\delta/2) < \beta(s,\tau_1) \leqslant \tilde{\beta}(s,\tau_1)$ and $2^{-\delta/2}\beta(s,\tau_1) < \beta(s,\tau_1) \leqslant \tilde{\beta}(s,\tau_1)$, which shows that $\tilde{\beta}(s,\tau_2) < \tilde{\beta}(s,\tau_1)$.

Finally we show that for arbitrary $s \geqslant 0$ we have that $\lim_{\tau\to\infty} \tilde{\beta}(s,\tau) = 0$. If we take the two cases $\eta \leqslant \tau/2$ and $\eta > \tau/2$, we have

$$\tilde{\beta}(s,\tau) \leqslant \max\left\{\beta\left(s,\frac{\tau}{2}\right), 2^{-\tau/2}\beta(s,0)\right\}$$

and the conclusion follows by letting $\tau \to \infty$. This shows that $\tilde{\beta} \in \mathscr{KL}$, which completes the proof. □

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