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Characterizations of detectability notions in terms of discontinuous dissipation functions

MIKHAIL KRICHMAN†* and EDUARDO D. SONTAG‡

We consider a new Lyapunov-type characterization of detectability for non-linear systems without controls, in terms of lower-semicontinuous (not necessarily smooth, or even continuous) dissipation functions, and prove its equivalence to the GASMO (global asymptotic stability modulo outputs) and UOSS (uniform output-to-state stability) properties studied in previous work. The result is then extended to provide a construction of a discontinuous dissipation function characterization of the IOSS (input-to-state stability) property for systems with controls. This paper complements a recent result on smooth Lyapunov characterizations of IOSS. The utility of non–smooth Lyapunov characterizations is illustrated by application to a well-known transistor network example.

1. Introduction

Stabilizing a control system is one of the most important goals of control theory. If we are lucky enough so that at each instant of time all the relevant information ('the full state of the system') is available, the problem reduces to designing a *static* or *state* feedback law. In practical applications a perfect measurement of the state is not available, and it is desirable to come up with a control law based on an 'output'—a smooth function of the state, which may or may not give adequate information on how far the true state is from a target set. Obviously, a feedback based on these output measurements will not perform very well in the parts of the state space where the measurements are small whereas the states are nevertheless far from the target. Thus, it is desired that the system be itself stable in the 'unobservable' parts of the state space. In other words, when the distance from the state to the target is large without the output seeing it, the trajectory must tend to the target by itself. This is the intuitive idea behind the non-linear detectability notion of GASMO (global asymptotic stability modulo outputs). Before introducing this property, it will be helpful to consider a few important particular cases.

Suppose we are given an autonomous system on $\mathbb{X} = \mathbb{R}^n$ with outputs in $\mathbb{Y} = \mathbb{R}^m$

$$\dot{x} = f(x); \qquad y = h(x)$$

with an equilibrium at $x = 0$. If the measurement does not provide any information about the state (i.e. $h(\cdot) \equiv 0$), then the GASMO property would simply mean global asymptotic stability, that is, $|x(t)| \leq$

$\beta(|x(0)|, t)$ for some function β of class $\mathcal{KL}$.† This case justifies the use of the word 'stability' in the definition of GASMO. To convince ourselves that GASMO is not quite a 'stability' notion, we can consider, on the other extreme, the case when the output provides full information on the state (for example, $h(x) = x$). Then, according to our intuitive definition, the system will be GASMO no matter what the dynamics are. In between the two extremes, in a most usual situation, the output provides satisfactory information in one part of the state space (say, for example, on $\{x: |x| < \rho(|x|)\}$, for some $\rho \in \mathcal{K}_\infty$) and unsatisfactory information in another part $\{x: |x| \geq \rho(|x|)\}$.

To get a better feel about what the right definition of GASMO, and more generally of detectability, should be for the non–linear systems without controls and what should be the property corresponding to GASMO in the most general case, we will first see what happens when the system of interest is linear (and with inputs).

A *linear*, time-invariant system Σ_{lin} with outputs is one for which f and h are linear, that is

$$\left.\begin{array}{l} \dot{x} = \mathbf{A}x + \mathbf{B}u \\ y = \mathbf{C}x \end{array}\right\} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{C} \in \mathbb{R}^{p \times n}$.

A well-known property of linear systems with outputs, consistent with our intuitive definition of GASMO, is the notion of *zero-detectability*, requiring that any trajectory, producing zero output under zero control, tends to zero. It is a well known fact (see, for example, Sontag 1998 b) that if a system (1) is detectable, then there exists a matrix $\mathbf{L} \in \mathbb{R}^{n \times p}$, such that the matrix $\mathbf{A} + \mathbf{LC}$ is Hurwitz. Then, rewriting (1) in form

† As usual, we say that a function $\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing, and is *of class* $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *of class* $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$.

$$\dot{x} = (\mathbf{A} + \mathbf{LC})x + \mathbf{B}u - \mathbf{L}y$$

we can write down the solution

$$x(t) = e^{t(\mathbf{A}+\mathbf{LC})}x(0)$$
$$+ \int_0^t e^{s(\mathbf{A}+\mathbf{LC})}[\mathbf{B}u(t-s) - \mathbf{L}y(t-s)]\,\mathrm{d}s \quad (2)$$

By routine manipulations it is possible to find positive constants K and δ such that for any initial state $x(0)$ and control $\mathbf{u}$ we have a bound

$$|x(t)| \le K e^{-\delta t}|x(0)| + K\frac{\|\mathbf{B}\|}{\delta}\||\mathbf{u}\|_{[0,t]}\| + K\frac{\|\mathbf{L}\|}{\delta}\||y|_{[0,t]}\|$$

(see Krichman *et al.* (2001) for the thorough treatment of the example). In other words, for a zero-detectable linear system, the magnitude of the state is bounded by a decaying 'overshoot' term accounting for the magnitude of the initial state, and by the effects ('*gains*') of inputs and outputs. One legitimate way of extending the notion of zero-detectability to non–linear systems lies in taking the above property, appropriately generalized as below, as a *definition* of zero-detectability.

Namely, replacing the exponential decay term $K e^{-\delta t}|x(0)|$ by an arbitrary function $\beta(|x(0)|, t)$ of class $\mathcal{KL}$, and replacing $K(\|\mathbf{B}\|/\delta)\||\mathbf{u}\|_{[0,t]}\|$ and $K(\|\mathbf{L}\|/\delta)\||y|_{[0,t]}\|$ by the arbitrary $\mathcal{K}$-gains $\gamma_u(\||\mathbf{u}\|_{[0,t]}\|)$ and $\gamma_y(\||y|_{[0,t]}\|)$, we obtain the following bound

$$|x(t)| \le \beta(|x(0)|, t) + \gamma_u(\||\mathbf{u}\|_{[0,t]}\|) + \gamma_y(\||y|_{[0,t]}\|) \quad (3)$$

This property is called IOSS (input–output to state stability) or just OSS when no inputs are present, and originates from the notion of ISS (input-to-state stability).

The ISS property was first introduced in the late 1980s by one of the authors and has proved to be a useful paradigm for characterizing stability notions of non-linear systems. Various applications and equivalent characterizations of this property have been extensively discussed in the literature since then. In particular, much attention has been given to the Lyapunov-type characterizations of the ISS-related properties (Lyapunov characterization of ISS has been established in Sontag and Wang 1995). Lyapunov techniques are well known to provide a powerful tool for gaining insights into the behaviour of a system. For example, smooth Lyapunov functions can be used in control design (see for example Krstić *et al.* 1995); but sometimes finding a non-smooth, or even a discontinous function satisfying a Lyapunov-type dissipation inequality can be sufficient to establish a corresponding stability-like property of a system.

The first attempt to approach the question of detectability of non-linear systems from the ISS-like viewpoint was made in Sontag and Wang (1996), where the OSS (output-to-state stability) property

$$|x(t)| \le \beta(|x(0)|, t) + \gamma_y(\||y|_{[0,t]}\|) \quad (4)$$

(where β and σ_y are as in (3)) was introduced for non-linear differential equations with outputs and no inputs, and the basic characterizations of this property were obtained. In particular, it was shown (though not explicitly stated) that OSS implies that there exist a class $\mathcal{K}_\infty$-function ρ such that any trajectory of the system, lying entirely outside the set $\mathcal{D} := \{x \in \mathbb{X}: |x| \le \rho(|h(x)|)\}$, will tend to the equilibrium of interest, in a uniform manner (an estimate such as $|x(t, \xi)| \le \lambda(|\xi|, t)$ holds on such trajectories). This property was later called GASMO. The GASMO property, in turn, was shown to imply the existence of a lower-semicontinuous Lyapunov-like function, satisfying the dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \le -\alpha(|x(t)|) + \sigma_2(|h(x(t))|) \quad (5)$$

thus decreasing along trajectories whenever the state is not dominated in magnitude by the output gain $(|x| > \alpha^{-1}(\sigma_2(|h(x)|)))$. Besides its intrinsic value as a notion of relative stability, the GASMO property played a key role in the characterization of OSS.

The notion of IOSS, described by (3), is a way to characterize a detectability property when both outputs and inputs come into the picture. It first appeared in Sontag (1989 b) stated in input/output terms, and in Jiang *et al.* (1994), called 'detectability' and 'strong unboundedness observability' respectively there, and the term IOSS was first introduced in Sontag and Wang (1997), where the natural Lyapunov characterization was conjectured. This conjecture was proved in Krichman *et al.* (2001), thus showing the equivalence of all the non-linear zero-detectability notions mentioned above. The first step of the proof was extending the converse Lyapunov result from Sontag and Wang (1996) to systems driven by disturbances, confined to a compact set, whose trajectories satisfy the bound (5) uniformly with respect to disturbances (hence the name UOSS—uniform output-to-state stability—for the corresponding property), and also showing that the resulting Lyapunov function may be chosen to be *smooth*. Next, a small-gain argument was used to generalize this result to non-linear systems driven with both controls and disturbances and satisfying (3) uniformly with respect to disturbances. The latter property was called UIOSS (uniform IOSS).

In this paper, we will explore characterizations of the UIOSS and GASMO properties in terms of *non-smooth* Lyapunov-like functions. It is well-known in differential equation theory that non–differentiable Lyapunov functions are often more natural for applications (an example based on a result in Sandberg (1969) is provided in the last section, see also Leonessa *et al.* (2001). Thus,

sufficiency statements in the Lyapunov characterizations can be made stronger if one introduces a weaker notion of UIOSS-Lyapunov function. This weaker concept does not require smoothness (or even continuity), and it replaces the *pointwise* dissipation inequality as in (5) with an *integral dissipation inequality* which is only required to hold when states are large in comparison to inputs. On the other hand, it was shown in Krichman *et al.* (2001) that a UIOSS system of type (13) will always admit a smooth UIOSS-Lyapunov function. In that sense, the *necessary* part of the result presented in this paper is weaker as stated. However, the main difficulty associated with building a smooth Lyapunov function as in Krichman *et al.* (2001) for a system with disturbances lay in finding a *continuous* quantity, decreasing along the trajectories of the system, contained in an 'unobservable' subset $\mathcal{D}$ of the state space. As we will discuss below, mimicking the argument from Sontag and Wang (1996) in presence of disturbances will yield a Lyapunov-like function not even continuous along the trajectories of the system, in which case the resulting function cannot be made smooth. This difficulty is avoided if only discontinuous dissipation functions are of interest. However, extending this discontinuous quantity (a so-called GASMO-Lyapunov function of a related control-free system, obtained from the original one by scaling controls appropriately) to $\mathbb{X} \setminus \mathcal{D}$ requires more elaborate techniques. The main result of the paper, Proposition 1, provides a universal method for extending *any* lower-semicontinuous GASMO-Lyapunov function to the rest of the state space. This result may become useful in proving converse Lyapunov theorems for different types of systems which may not admit continuous Lyapunov functions.

Among other references on IOSS and related notions, one can mention for instance the detectability studies in Lu (1995), Morse (1995) and Krener (1999), where Lyapunov-like definitions are used, as well as the recent work (Liberzon *et al.* 2002) on an ISS-like notion of minimum-phase systems and asymptotic gain characterizations in Angeli *et al.* (2001). A preliminary version of this paper appeared in Krichman and Sontag (1998).

2. Main definitions and statements of results

We start by considering a class of systems not subject to external inputs. Later, we extend to a more general class of systems.

The systems to be considered first are of the type

$$\dot{x}(t) = f(x(t), \mathbf{d}(t)), \qquad y(t) = h(x(t)) \qquad (6)$$

where $\Omega \subseteq \mathbb{R}^m$ is compact and $\mathbb{X} = \mathbb{R}^n$ for some positive integer n, $f: \mathbb{X} \times \Omega \to \mathbb{X}$ is locally Lipschitz in x uniformly on d and jointly continuous in x and d, and $f(0, d) = 0$ for every $d \in \Omega$. We think of measurable

functions $\mathbf{d}: \mathcal{I} \to \Omega$ as *disturbances* acting on the system, and denote by $\mathcal{M}_\Omega$ the collection of all such functions.

We will use the notation $t_{\max}$, or more precisely $t_{\max}(\xi, \mathbf{d})$, for the supremum (possibly $+\infty$) of the times $t > 0$ such that the solution $x(t, \xi, \mathbf{d})$ of $\dot{x} = f(x, \mathbf{d})$, $x(0) = \xi$ exists, and denote $y(t, \xi, \mathbf{d}) := h(x(t, \xi, \mathbf{d}))$ for each such t. (When ξ and $\mathbf{d}$ are clear from the context, we often just write $x(t)$ or $y(t)$.) For any measurable function $z(\cdot)$, the L_∞ (essential supremum) norm of the restriction of z to the interval $[t_1, t_2]$ is denoted by $\|z|_{[t_1, t_2]}\|$.

Definition 1: A system (6) is *uniformly output-to-state stable (UOSS)* if there exist some $\beta \in \mathcal{KL}$ and $\gamma_y \in \mathcal{K}$ such that

$$|x(t, \xi, \mathbf{d})| \leq \max \{\beta(|\xi|, t), \gamma_y(\|y|_{[0,t]}\|)\} \qquad (7)$$

for any disturbance $\mathbf{d}$, any initial state $\xi \in \mathbb{X}$, and every $t \in [0, t_{\max})$.

Definition 2: A *UOSS-Lyapunov-like function* for system (6) is a lower semicontinuous function

$$V: \mathbb{X} \to \mathbb{R}_{\geq 0}$$

such that the following properties hold:

(1) there exist $\mathcal{K}_\infty$-functions α_1 and α_2 such that

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \qquad (8)$$

for all ξ in $\mathbb{X}$, and

(2) there exist $\mathcal{K}_\infty$-functions α and γ such that the dissipation inequality

$$V(x(\tau, \xi, \mathbf{d})) - V(x(\sigma, \xi, \mathbf{d}))$$

$$\leq \int_\sigma^\tau (-\alpha(|x(s, \xi, \mathbf{d})|) + \gamma(|y(s, \xi, \mathbf{d})|)) \, ds \qquad (9)$$

is satisfied for any initial state $\xi \in \mathbb{X}$, any disturbance $\mathbf{d}$, and any $[\sigma, \tau] \subset [0, t_{\max}(\xi, \mathbf{d}))$.

One of our results will be as follows.

Theorem 1: *A system* (6) *is UOSS if and only if it admits a UOSS-Lyapunov-like function.*

In Krichman *et al.* (2001), we established a smooth version of this result, namely, we showed that a system is UOSS if and only if there is an infinitely differentiable UOSS-Lyapunov-like function V. (For smooth V, the integral dissipation inequality can be recast in the equivalent infinitesimal form $\nabla V(\xi) \cdot f(\xi, w) \leq -\alpha(|\xi|) + \sigma_2(|h(\xi)|)$.) Obviously, the necessity part of this theorem is a corollary of the stronger result shown in that paper, while the sufficiency part is stronger here. It is well-known that Lyapunov functions are sometimes easier to find, and hence stability properties are easier to verify, if smoothness (or even continuity) is not required.

Thus, it is of interest to have a proof of sufficiency when only a general lower semicontinuous V is available.

Although redundant in view of Krichman *et al.* (2001), we also give a self-contained proof of necesssity. The reason for including this proof is two-fold. First of all, the first step of the proof of existence of a merely lower semicontinuous V, which consists of the construction of a 'relative stability' Lyapunov function, is substantially simpler than the argument (involving a relaxed problem) needed to prove the stronger result, so it seems worth having this simpler proof documented in the literature. Second, and more importantly, the next step in the proof, consisting of showing that UOSS-Lyapunov-like functions can be always obtained by extension of a relative stability Lyapunov function (in the sense that we make precise next) is far less trivial than in the smooth case, and is of considerable independent interest.

Given a system (6) and any function η of class $\mathcal{K}_\infty$, let us denote by $\mathcal{F}(\eta)$ (or simply $\mathcal{F}$, when η is clear from the context) the set

$$\mathcal{F}(\eta) := \{x \in \mathbb{X} : |x| > \eta(|h(x)|)\}$$

We may think of $\mathcal{F}$ as the set of states that are large compared to the current observations. The following notion of stability, originally considered in Sontag and Wang (1996), is a natural generalization of global asymptotic stability of an equilibrium, to the case when stability is needed only for the 'unobservable' motions of a system with outputs.

Definition 3: A system of type (6) satisfies the GAS-MO (*global asymptotic stability modulo output*) property if there exist a function ρ of class $\mathcal{K}_\infty$ and a function λ of class $\mathcal{KL}$, such that, for all $\xi \in \mathbb{X}$, $\mathbf{d} \in \mathcal{M}_\Omega$, and any $T < t_{\max}(\xi, \mathbf{d})$, if

$$x(t, \xi, \mathbf{d}) \in \mathcal{F}(\rho)$$

for all $0 \leq t \leq T$, then also the estimate

$$|x(t, \xi, \mathbf{d})| \leq \lambda(|\xi|, t) \qquad (10)$$

holds for all $0 \leq t \leq T$.

The next lemma, originally proved in Sontag and Wang (1996) for the disturbance-free case and in Krichman *et al.* (2001) for the present setting, makes it possible to use the GAS-MO property as the main tool for constructing a UOSS Lyapunov-like functions. We will not repeat the proof in this paper.

Lemma 1: *Any UOSS system* (6) *satisfies the GAS-MO property.*

Definition 4: A *GASMO-Lyapunov-like function* for a system (6) is a lower-semicontinuous function

$$V_0 : \mathcal{F}(\eta) \to \mathbb{R}_{\geq 0}$$

for some η of class $\mathcal{K}_\infty$, such that the following two properties hold:

(1) there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that

$$\underline{\alpha}(|\xi|) \leq V_0(\xi) \leq \overline{\alpha}(|\xi|) \qquad \forall \, \xi \in \mathcal{F}(\eta) \qquad (11)$$

(2) there exists a class-$\mathcal{K}_\infty$ function φ so that, along all trajectories of (6) which are entirely contained in $\mathcal{F}(\eta)$

$$V_0(x(t_2)) - V_0(x(t_1)) \leq - \int_{t_1}^{t_2} \varphi(|x(t)|) \, \mathrm{d}t \qquad (12)$$

Suppose now that we are given a UOSS-Lyapunov-like function V for (6). Let $\eta(r) := \alpha^{-1}(2\gamma(r))$. Then

$$|x| \geq \eta(|h(x)|) \quad \Rightarrow \quad -\alpha(|x|) + \gamma(|h(x)|) \leq -\gamma(|h(x)|)$$

for all x. This means that the restriction V_0 of V to $\mathcal{F}$ is a GASMO-Lyapunov-like function (use $\varphi = \gamma$). The second of our main results is as follows; it states that, conversely, any GASMO-Lyapunov-like function can be extended to an UOSS-Lyapunov-like function, provided that a rescaling and a small 'thickening' of $\mathcal{F}$ are allowed.

Proposition 1: *Suppose that* $V_0 : \mathcal{F}(\eta) \to \mathbb{R}_{\geq 0}$ *is a GASMO-Lyapunov-like function for* (6)*. Then there exist a UOSS-Lyapunov-like function* V_2 *for* Σ*, and a class* $\mathcal{K}_\infty$ *function* Φ*, such that*

$$V_2 = \Phi \circ V_0 \qquad \forall \, x \in \mathcal{F}(2\eta)$$

Moreover, if V_0 *is continuous, then* V_2 *can be chosen continuous as well.*

2.1. *Systems with inputs and disturbances*

Our main results, such as the extension theorem for GASMO-Lyapunov-like functions, are for systems (6), but it is easy to state and prove corollaries for the more general class of systems treated in Krichman *et al.* (2001). These are systems whose dynamics depend on two types of inputs, which we call respectively *controls* and *disturbances*

$$\dot{x} = f(x(t), \mathbf{u}(t), \mathbf{w}(t)), \qquad y(t) = h(x(t)) \qquad (13)$$

Here states evolve in $\mathbb{X} = \mathbb{R}^n$, controls are measurable, essentially bounded functions $\mathbf{u}$ on $\mathcal{I} = \mathbb{R}_{\geq 0}$ with values in $\mathbb{U} := \mathbb{R}^{m_u}$, and disturbances, as earlier, are measurable functions $\mathbf{w} : \mathcal{I} \to \Gamma$, where Γ is a compact subset of $\mathbb{R}^{m_w}$.

In those cases when a different interval $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ of definition for a control $\mathbf{u}$ is specified, we always apply the definitions to the extension of $\mathbf{u}$ to $\mathbb{R}_{\geq 0}$, using $\mathbf{u} \equiv 0$ on $\mathbb{R}_{\geq 0} \setminus \mathcal{I}$. The function $f : \mathbb{X} \times \mathbb{U} \times \Gamma \to \mathbb{X}$ is locally Lipschitz in (x, u) uniformly on w, jointly continuous in x, u and w, and such that $f(0, 0, w) = 0$ for any $w \in \Gamma$; and $h : \mathbb{X} \to \mathbb{Y} := \mathbb{R}^p$ is smooth and vanishes at 0.

Extending the previous notations, given a state $\xi \in \mathbb{X}$, for each pair $(\mathbf{u}, \mathbf{w})$ we denote by $x(t, \xi, \mathbf{u}, \mathbf{w})$ the unique maximal solution of the system (13), which is

defined on some maximal interval $[0, t_{\max}(\xi, \mathbf{u}, \mathbf{w}))$, and again we use the notation $y(t, \xi, \mathbf{u}, \mathbf{w}) := h(x(t, \xi, \mathbf{u}, \mathbf{w}))$, writing $x(t)$ and $y(t)$ when $\xi, \mathbf{u}, \mathbf{w}$ are clear from the context.

The UOSS property and its associated Lyapunov-like notion generalize as follows.

Definition 5: A system of type (13) is said to be *uniformly input–output-to-state stable (UIOSS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma_u, \gamma_y \in \mathcal{K}$ such that the estimate

$$|x(t, \xi, \mathbf{u}, \mathbf{w})| \leq \max\{\beta(|\xi|, t), \gamma_u(\|\mathbf{u}\|_{[0,t]}\|), \gamma_y(\|y\|_{[0,t]}\|)\}$$
$$(14)$$

holds for any initial state $\xi \in \mathbb{X}$, control $\mathbf{u}$, disturbance $\mathbf{w}$ and time $t \in [0, t_{\max}(\xi, \mathbf{u}, \mathbf{w}))$.

Definition 6: A *UIOSS-Lyapunov-like function* for system (13) is a lower semicontinuous function $V: \mathbb{X} \to \mathbb{R}_{\geq 0}$ satisfying the following conditions:

- there exist $\mathcal{K}_\infty$-functions α_1, α_2 such that (8) holds for all ξ in $\mathbb{X}$, and
- there exist α_3, γ, and $\chi \in \mathcal{K}_\infty$ such that, for each $\xi \in \mathbb{X}$, $\mathbf{u}: \mathbb{R}_{\geq 0} \to \mathbb{U}$, disturbance $w \in \mathcal{M}_\Gamma$, and over any interval $[\sigma, \tau] \subset [0, t_{\max}(\xi, \mathbf{u}, \mathbf{w}))$ in which

$$|x(s, \xi, \mathbf{u}, \mathbf{w})| \geq \chi(|\mathbf{u}(s)|) \qquad \text{for almost all } s \in [\sigma, \tau]$$
$$(15)$$

the inequality

$$V(x(\tau, \xi, \mathbf{u}, \mathbf{w})) - V(x(\sigma, \xi, \mathbf{u}, \mathbf{w}))$$
$$\leq \int_\sigma^\tau (-\alpha_3|x(s, \xi, \mathbf{u}, \mathbf{w})| + \gamma|y(s, \xi, \mathbf{u}, \mathbf{w})|) \, \mathrm{d}s \quad (16)$$

holds.   □

Note that, when there are no control inputs u, the condition (15) is satisfied (we may think of $u \equiv 0$), so this notion specializes to the previous one when there are no u's. Theorem 1 generalizes like this:

Theorem 2: *A system* (13) *is UIOSS if and only if it admits a UIOSS-Lyapunov-like function.*

Remark 1: Actually, a much weaker condition than existence of a UIOSS-Lyapunov-like function can be proved sufficient for UIOSS. This condition generalizes the one given for GASMO-Lyapunov-like functions, and is as follows. We will show in the proof of Theorem 2 that for a system to be UIOSS it is enough to have a lower-semicontinuous function V_0, defined on a set $\mathcal{F}(\eta)$, for some η of class $\mathcal{K}_\infty$, and satisfying (11), such that there exist some fixed $\mathcal{K}_\infty$-functions φ and χ so that the estimate (12) holds along any piece of trajectory

$$x(t) := x(t, \mathbf{u}, \mathbf{w}), \qquad t \in [t_1, t_2]$$

of (13) lying entirely in $\mathcal{F}(\eta)$, as long as $|x(t)| \geq \chi(|\mathbf{u}(t)|)$ for almost all $t \in [t_1, t_2]$. It is easy to see that any UIOSS-Lyapunov-like function satisfies (12), with $\varphi = \alpha_3/2$ and $\eta(\cdot) = \alpha_3^{-1}(2\gamma(\cdot))$.   □

3. Proof of sufficiency in Theorems 1 and 2

To relax the regularity assumptions on the Lyapunov-like function V in the sufficiency statements of the Lyapunov-type theorems, it is useful to have a version of the comparison principle which underlies most of the sufficiency proofs in Lyapunov theory.

3.1. An integral comparison principle

Definition 7: A function $w: \mathcal{I} \to \mathbb{R}$ is said to satisfy the *no upward jump (NUJ)* condition on an interval $[a, b] \subset \mathrm{Dom}(w)$ if

$$\liminf_{s \to r} w(s) \geq w(r) \qquad \text{for any } r \in (a, b]$$

and

$$w(r) \geq \limsup_{s \to r^+} w(s) \qquad \text{for any } r \in [a, b)$$

It is easy to see that satisfying the inequality

$$w(\tau) - w(\sigma) \leq \int_\sigma^\tau \psi(t) \, \mathrm{d}t$$

$$\forall \, \sigma, \tau \text{ such that } a \leq \sigma \leq \tau \leq b$$

with some integrable function ψ would imply the NUJ property for w.

Lemma 2: *For each $\mathcal{K}$-function $\alpha: [0, \infty) \to \mathbb{R}_{\geq 0}$, there exists a $\mathcal{KL}$ function β_α with the following property: For any $T > 0$ and any lower semicontinuous function $w: [0, T] \to \mathbb{R}_{\geq 0}$, if, for almost all σ, τ such that $0 \leq \sigma < \tau \leq T$ the following inequality holds*

$$w(\tau) - w(\sigma) \leq -\int_\sigma^\tau \alpha(w(t)) \, \mathrm{d}t \qquad (17)$$

then $w(T) \leq \beta_\alpha(w(0), T)$.   □

Proof: The proof follows along the lines of the proof of the similar comparison principle (with pointwise dissipation inequality) in Sontag (1989 a).

Define $\eta(s) := -\int_1^s \mathrm{d}r/\alpha(r)$, and let $a := -\lim_{s \to +\infty} \eta(s)$ and $b := \lim_{s \to 0^+} \eta(s)$ (a and b may be infinite in case the improper integral in the definition of η diverges as the upper integration limit becomes 0 or $+\infty$). Note that η is strictly decreasing on its domain, so that its inverse function is well defined, and $\mathrm{Range}(\eta) = \mathrm{Dom}(\eta^{-1}) = (-a, b)$.

Define, for $s \neq 0$

$$\beta(s, t) := \begin{cases} 0, & \text{if } t + \eta(s) \geq b \\ \eta^{-1}(t + \eta(s)), & \text{if } t + \eta(s) < b \end{cases}$$

and let $\beta(0,t) \equiv 0$. Then β is continuous in each argument. Also, for each fixed s, $\beta(s,t)$ is non-increasing in t and strictly decreasing in t when $t < b - \eta(s)$. For each fixed t, $\beta(\cdot,t)$ is strictly increasing when positive. For any s, t such that $t + \eta(s) \neq b$

$$\frac{\partial}{\partial t}\beta(s,t) = -\alpha(\beta(s,t))$$

Hence $\beta \in \mathcal{KL}$ and

$$\beta(s,\tau) = \beta(s,\sigma) - \int_\sigma^\tau \alpha(\beta(s,t))\,\mathrm{d}t \qquad \forall\, 0 < \sigma < \tau$$

Note that $\beta(s,0) = s$. Now pick any measurable function w satisfying (17).

Claim: $w(T) \leq \beta(w(0),T)$.

Proof of claim: Suppose the contrary, that is, $w(T) > \beta(w(0),T)$. By the NUJ property of w, $\liminf_{t\to T^-} w(t) \geq w(T)$ so, by continuity of $\beta(w(0),\cdot)$, there exists a $\delta > 0$ such that $w(t) > \beta(w(0),t)$ for all $t \in [T-\delta, T]$. Let

$$\tau = \inf\{\tilde{\tau} \mid w(t) > \beta(w(0),t) \qquad \forall t \in [\tilde{\tau}, T]\}$$

We next show that

$$w(\tau) \leq \beta(w(0),\tau) \tag{18}$$

If $\tau = 0$ then $w(\tau) = w(0) = \beta(w(0),0) = \beta(w(0),\tau)$, because $\beta(s,0) = s$ for all s. If $\tau > 0$ then, because of the continuity of β and the NUJ property of w, the inequality $w(\tau) > \beta(w(0),\tau)$ would imply that there exists some $\tilde{\delta} > 0$ such that $w(t) > \beta(w(0),t)$ for all t in $[\tau - \tilde{\delta}, \tau]$, contradicting the definition of τ, so, (18) indeed holds. Hence

$$w(T) - \beta(w(0),T) \leq [w(\tau) - \beta(w(0),\tau)]$$
$$- \int_\tau^T [\alpha(w(t)) - \alpha(\beta(w(0),t))]\,\mathrm{d}t \tag{19}$$

The first term in the right-hand side of (19) is non-positive by (18). The second term is also non-positive, because of the choice of τ. Hence, the right-hand side of (19) is non-positive, contradicting our assumption. This proves the claim, which completes the proof of the lemma. $\qquad \square$

3.2. *Existence of a UIOSS-Lyapunov-like function implies UIOSS*

We now prove the sufficiency part of Theorem 2. Note that, as Theorem 1 is a particular case of Theorem 2, the sufficiency part of Theorem 1 will follow automatically.

Lemma 3: *Let q_1, q_2 be two positive numbers. Suppose that*

- $x: \mathcal{I} \to \mathbb{R}^n$ *is absolutely continuous,*
- $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is lower semicontinuous and satisfies* (8) *with some $\alpha_i \in \mathcal{K}_\infty$, and*
- $V \circ x$ *is non-increasing on any subinterval $[\sigma,\tau]$ of $\mathcal{I}$ such that $|x(t)| \geq q_1 > 0$ is satisfied for all $t \in [\sigma,\tau]$.*

Then if $\alpha_1(q_2) > \alpha_2(q_1)$ and $|x(\tilde{t})| \leq q_1$ for some $\tilde{t}$, then $|x(t)| < q_2$ for all $t > \tilde{t}$.

Proof: Suppose $|x(t_2)| \geq q_2$ for some $t_2 > \tilde{t}$. Then, since $x(t)$ is continuous, there is a $t_1 \geq \tilde{t}$ such that

$$t_1 = \max\{t < t_2 \mid |x(t)| = q_1\}$$

Then $|x(t)| \geq q_1$ for all $t \in [t_1, t_2]$. So, $V \circ x$ is non-increasing on $[t_1, t_2]$, hence

$$\alpha_1(q_2) \leq V(x(t_2)) \leq V(x(t_1)) \leq \alpha_2(q_1)$$

The obtained contradiction proves the lemma. $\qquad \square$

Now suppose a system (13) admits a Lyapunov-like function V_0 as in Remark 1. We extend V_0 to the state space $\mathbb{X}$, by the formula

$$V(x) = \begin{cases} V_0, & x \in \mathcal{F}(\eta) \\ \underline{\alpha}(|x|), & x \notin \mathcal{F}(\eta) \end{cases}$$

Notice that V is continuous at 0 because of (11) and lower semicontinuous on $\mathbb{X}$, because V_0 is lower-semicontinuous on $\mathcal{F}(\eta)$, $\underline{\alpha}$ is continuous, and $\underline{\alpha}(|\xi|) \leq \liminf_{x\to\xi, x\in\mathcal{F}(\eta)} V_0(x)$ for all $\xi \in \partial\mathcal{F}(\eta)$. We will show that (13) is UIOSS with $\gamma_u(\cdot) := \underline{\alpha}^{-1}(2\overline{\alpha}(2\chi(\cdot)))$, $\gamma_y(\cdot) := \underline{\alpha}^{-1}(2\overline{\alpha}(2\eta(\cdot)))$, and $\beta(s,t) := \underline{\alpha}^{-1}(\beta_{\hat{\varphi}}(\overline{\alpha}(s),t))$, where $\beta_{\hat{\varphi}}$ is provided by Lemma 2 for the $\mathcal{K}$-function $\hat{\varphi} = \varphi \circ \underline{\alpha}^{-1}$.

Pick an initial state ξ, a control $\mathbf{u}$, a disturbance $\mathbf{w}$, and a time $T < t_{\max}(\xi, \mathbf{u}, \mathbf{w})$, and write $x(t) = x(t, \xi, \mathbf{u}, \mathbf{w})$ and $y(t) := h(x(t))$. Let $q_1 := \max\{2\chi(\|\mathbf{u}|_{[0,T]}\|), 2\eta(\|y|_{[0,T]}\|)\}$. By definition

$$q_1 \geq \eta(|y(t)|) \qquad \text{for all } t \in [0,T]$$

with equality only in case $q_1 = 0$. Therefore, if, for some $s \in [0,T]$, it holds that

$$|x(s)| \geq q_1 > 0 \quad \text{or} \quad |x(s)| > q_1 = 0 \tag{20}$$

then $x(s) \in \mathcal{F}(\eta)$. Hence, if one of the inequalities in (20) holds on some subinterval $[\sigma,\tau]$ of $[0,T]$, then (12) holds on $[\sigma,\tau]$, and, in particular, $V \circ x$ is non-increasing on $[\sigma,\tau]$.

Suppose first that $q_1 > 0$. If

$$|x(t)| \geq q_1 \tag{21}$$

holds for all $t \in [0, T]$, then $x(t) \in \mathcal{F}(\eta)$ for all $t \in [0, T]$, so that (12) must hold on any subinterval $[\sigma, \tau]$ of $[0, T]$, therefore

$$
\begin{aligned}
V(x(\tau)) - V(x(\sigma)) &\leq \int_\sigma^\tau -\varphi(|x(s)|) \, \mathrm{d}s \\
&\leq \int_\sigma^\tau -\varphi(\overline{\alpha}^{-1}(V(x(s)))) \, \mathrm{d}s \\
&= -\int_\sigma^\tau \hat{\varphi}(V(x(s))) \, \mathrm{d}s
\end{aligned}
$$

Thus, in this case the conditions of Lemma 2 are satisfied for the function $w(t) := V(x(t))$, so that

$$
V(x(T)) \leq \beta_{\hat{\varphi}}(V(x(0)), T)
$$

with some function $\beta_{\hat{\varphi}}$ of class $\mathcal{KL}$. So

$$
\begin{aligned}
|x(T)| &\leq \underline{\alpha}^{-1}(V(x(T))) \\
&\leq \underline{\alpha}^{-1}(\beta_{\hat{\varphi}}(V(x(0)), T)) \\
&\leq \underline{\alpha}^{-1}(\beta_{\hat{\varphi}}(\overline{\alpha}(|x(0)|), T)) = \underline{\alpha}^{-1}(\beta_{\hat{\varphi}}(\overline{\alpha}(|\xi|), T))
\end{aligned}
\tag{22}
$$

If instead (21) fails for some $\tilde{t} < T$, then

$$
|x(\tilde{t})| < \max \{2\chi(\|\mathbf{u}\|_{[0,T]}), 2\eta(\|y\|_{[0,T]})\}
$$

so, by Lemma 3 applied for q_1 and

$$
q_2 = \underline{\alpha}^{-1}(2\overline{\alpha}(q_1))
$$

we have

$$
|x(T)| < \max \{\gamma_u(\|\mathbf{u}\|_{[0,T]}), \gamma_y(\|y\|_{[0,T]})\}
\tag{23}
$$

Combining (22) and (23), we obtain the estimate (14).

Suppose now that $q_1 = 0$. If $x(t) \neq 0$ for all $t \in [0, T]$, then the trajectory lies entirely in $\mathcal{F}(\eta)$, so that (22) holds. Otherwise, let $\bar{t} := \min \{t \in [0, T]: x(t) = 0\}$. We will show that $x(t) = 0$ for all $t \in [\bar{t}, T]$. Indeed, the set

$$
C := \{t \in [\bar{t}, T]: x(t) \neq 0\}
$$

is open in the topology of $[\bar{t}, T]$, and thus can be written as a disjoint union of open intervals (σ_i, τ_i) and perhaps $[\bar{t}, \tau_0)$ and $(\sigma_0, T]$. Fix any i. Take any *closed* subinterval $[\sigma, \tau]$ of (σ_i, τ_i). The piece of trajectory over $[\sigma, \tau]$ lies entirely in $\mathcal{F}(\eta)$, so that the inequality (12) holds, implying

$$
|x(\tau)| \leq \underline{\alpha}^{-1}(\beta_{\hat{\varphi}}(\overline{\alpha}(|x(\sigma)|), \tau - \sigma))
\tag{24}
$$

Since $x(\cdot)$ is continuous and $x(\sigma_i) = 0$, we can choose σ close enough to σ_i so that $|x(\sigma)|$ is as small as desired, thus, showing by (24) that $|x(\tau)|$ is infinitely small for any $\tau \in [\sigma, \tau_i)$. This proves that $x(\cdot) \equiv 0$ on (σ_i, τ_i) and on all of C, so, $C = \varnothing$ and the conclusion follows.

4. Proof of necessity in theorem 1

Suppose a system Σ of type (6) is UOSS. By Lemma 1, Σ satisfies the GASMO property with some $\mathcal{K}_\infty$ function ρ. Majorizing ρ with another $\mathcal{K}_\infty$ function if necessary, we may assume ρ to be smooth when restricted to $\mathbb{R}_{>0}$ and satisfy $\rho(s) > s$ for all positive s. Introduce the following notation:

- $\mathcal{D} := \{\xi \in \mathbb{X}: |\xi| \leq \rho(|h(\xi)|)\} = \mathbb{X} \setminus \mathcal{F}(\rho)$,
- $\mathcal{E} := \mathcal{F}(\rho)$.

If $\mathcal{D} = \mathbb{X}$, then any proper, smooth and positive definite function $V: \mathbb{X} \to \mathbb{R}$ is a UOSS-Lyapunov function for (6). This fact is proved in Krichman *et al.* (2001), but is relatively easy to convince oneself of.

Suppose now that $\mathcal{D} \neq \mathbb{X}$. For each $\mathbf{d} \in \mathcal{M}_\Omega$ and $\xi \in \mathcal{E}$, define

$$
\lambda_{\xi, \mathbf{d}} = \inf \{t \in [0, t_{\max}): x(t, \xi, \mathbf{d}) \in \mathcal{D}\}
\tag{25}
$$

with the convention $\lambda_{\xi, \mathbf{d}} = t_{\max}(\xi, \mathbf{d})$ if the trajectory never enters $\mathcal{D}$.

The GASMO property then implies

$$
|x(t, \xi, \mathbf{d})| \leq \lambda(|\xi|, t)
$$
$$
\forall \xi \in \mathcal{E}, \quad \forall d \in \mathcal{M}_\Omega, \quad \forall t \in [0, \lambda_{\xi, \mathbf{d}})
\tag{26}
$$

for some $\lambda \in \mathcal{KL}$.

Note that, because of property (26), the system cannot have any equilibria in $\mathcal{E}$, that is

$$
f(\xi, d) \neq 0
$$

for every $\xi \in \mathcal{E}$ and every $d \in \Omega$. Moreover, replacing $\rho(s)$ by $c\rho(s)$ for some $c > 1$ if necessary, one may also assume that $f(\xi, d) \neq 0$ for all $\xi \in \partial \mathcal{D} \setminus \{0\}$, all $d \in \Omega$.

We introduce an auxiliary system $\hat{\Sigma}$ which slows down the motions of the original one

$$
\dot{z} = \hat{f}(z, d) = \frac{1}{1 + |f(z, d)|^2 + \kappa(z)} f(z, d)
\tag{27}
$$

where κ is any smooth function $\mathbb{X} \to [0, \infty)$ with the property that

$$
\kappa(\xi) \geq 2 \max_{d \in \Omega} |\nabla(\rho \circ h)(\xi) \cdot f(\xi, d)|
\tag{28}
$$

whenever $|h(\xi)| \geq 1$. For each disturbance $\hat{\mathbf{d}}$ (defined on $\mathbb{R}_{\geq 0}$), denote by

$$
z(s, \xi, \hat{\mathbf{d}})
$$

the value at time s of the solution of the equation $\dot{z} = \hat{f}(z, \hat{\mathbf{d}})$ with initial state ξ. Observe that, as $\hat{f}$ is bounded, this solution exists for all non-negative s. The next two claims, summarizing relevant properties of $\hat{\Sigma}$, were proved in Krichman *et al.* (2001).

Claim 1: For each ξ and each $\mathbf{d}$

$$x(t, \xi, \mathbf{d}) = z(\sigma_{\xi, \mathbf{d}}(t), \xi, \mathbf{d} \circ \sigma_{\xi, \mathbf{d}}^{-1}) \qquad \forall\, t \in [0, t_{\max}(\xi, \mathbf{d}))$$

$$(29)$$

where $\sigma_{\xi, \mathbf{d}} \colon [0, t_{\max}(\xi, \mathbf{d})) \to \mathbb{R}_{\geq 0}$ is defined by

$$\sigma_{\xi, \mathbf{d}}(t) = \int_0^t [1 + |f(x(s, \xi, \mathbf{d}), \mathbf{d}(s))|^2 + \kappa(x(s, \xi, \mathbf{d}))]\, \mathrm{d}s$$

Moreover, $\sigma_{\xi, \mathbf{d}}(t) \to \infty$ as $t \to t_{\max}(\xi, \mathbf{d})$, so, we can define $\sigma_{\xi, \mathbf{d}}(t_{\max}(\xi, \mathbf{d})) := +\infty$ for convenience.

Claim 2: System $\hat{\Sigma}$ satisfies the GASMO property.

For each initial state ξ and each disturbance function $\mathbf{d}$, define

$$\theta_{\xi, \mathbf{d}} = \inf \{ t \geq 0 \colon z(t, \xi, \mathbf{d}) \in \mathcal{D} \} \qquad (30)$$

where $\theta_{\xi, \mathbf{d}} = \infty$ if $z(t, \xi, \mathbf{d}) \notin \mathcal{D}$ for all $t \geq 0$. Note that $\theta_{\xi, \mathbf{d}} > 0$ for all $\mathbf{d} \in \mathcal{M}_\Omega$ and all $\xi \in \mathcal{E}$, because $\mathcal{E}$ is open. Observe also that if $\mathbf{d}_1 = \mathbf{d} \circ \sigma_{\xi, \mathbf{d}}$

$$\theta_{\xi, \mathbf{d}} = \sigma_{\xi, \mathbf{d}_1}(\lambda_{\xi, \mathbf{d}_1})$$

Lemma 4: *For each fixed disturbance $\mathbf{d} \in \mathcal{M}_\Omega$, $\theta_{\xi, \mathbf{d}}$ is a lower-semicontinuous function of ξ on $\mathcal{E}$.*

Proof: Let $\xi_0 \in \mathbb{X}$. Pick a disturbance $\mathbf{d} \in \mathcal{M}_\Omega$ and a sequence $\{\xi_k\}$ that converges to ξ_0. Let $\theta_k = \theta_{\xi_k, \mathbf{d}}$ and $\theta_0 = \theta_{\xi_0, \mathbf{d}}$. We need to show that

$$\theta_0 \leq \hat{\theta} = \liminf_{k \to \infty} \theta_k$$

If $\hat{\theta} = \infty$, there is nothing to prove, so we may assume without loss of generality that $\lim_{k \to \infty} \theta_k = \hat{\theta} < \infty$. It follows by continuity on initial conditions and the fact that $\mathcal{D}$ is a closed set, that $z(\hat{\theta}, \xi_0, \mathbf{d}) \in \mathcal{D}$, and hence, $\theta_0 \leq \hat{\theta}$, as required. $\qquad \square$

4.1. *Defining a discontinuous GASMO-Lyapunov-like function on $\mathcal{F}(2\rho)$*

Let the function λ of class $\mathcal{KL}$ be as in Definition 3 for the system $\hat{\Sigma}$, that is, the following estimate holds for system (27)

$$|z(t, \xi, \mathbf{d})| \leq \lambda(|\xi|, t) \qquad \forall\, t \in [0, \theta_{\xi, \mathbf{d}}) \qquad (31)$$

for all $\xi \in \mathcal{E}$ and all $\mathbf{d} \in \mathcal{M}_\Omega$.

According to Proposition 7 in Sontag (1998 a), there exist $\mathcal{K}_\infty$-functions μ_1 and μ_2 such that

$$\lambda(r, t) \leq \mu_1(\mu_2(r) \mathrm{e}^{-t}) \qquad \forall\, r, \quad t \geq 0 \qquad (32)$$

Define

$$\varphi(s) := \mu_1^{-1}(s)$$

Recall that for all $\xi \in \mathcal{E}$ and any $\mathbf{d}$, the GASMO property gives the estimate $|z(t, \xi, \mathbf{d})| \leq \lambda(|\xi|, t)$ for all $t \in [0, \theta_{\xi, \mathbf{d}})$. Then

$$\int_0^{\theta_{\xi, \mathbf{d}}} \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t \leq \int_0^{\theta_{\xi, \mathbf{d}}} \varphi(\lambda(|\xi|, t))\, \mathrm{d}t$$

$$\leq \mu_2(|\xi|) \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}t = \mu_2(|\xi|) \qquad (33)$$

For $\xi \in \mathcal{E}$ and $\mathbf{d} \in \mathcal{M}_\Omega$, define

$$\hat{V}_0(\xi, \mathbf{d}) := \int_0^{\theta_{\xi, \mathbf{d}}} \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t \qquad (34)$$

Equation (33) implies that the integral in (34) converges, so that the definition makes sense for all $\xi \in \mathcal{E}$.

Lemma 5: *For any fixed disturbance $\mathbf{d} \in \mathcal{M}_\Omega$, $\hat{V}_0(\xi, \mathbf{d})$ is a lower semicontinuous function of ξ on $\mathcal{E}$.*

Proof: Fix a disturbance $\mathbf{d} \in \mathcal{M}_\Omega$ and a state $\xi \in \mathcal{E}$. Let $\varepsilon > 0$ be given. Since

$$\int_0^{\theta_{\xi, \mathbf{d}}} \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t < \infty$$

there exists some $0 < T < \theta_{\xi, \mathbf{d}}$ such that

$$\int_T^{\theta_{\xi, \mathbf{d}}} \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t < \frac{\varepsilon}{2}$$

By lower semicontinuity of $\theta_{\xi, \mathbf{d}}$, one can find a bounded neighbourhood U_1 of ξ contained in $\mathcal{E}$, such that $\theta_{\bar{\xi}, \mathbf{d}} \geq T$ for any $\bar{\xi}$ in U_1. Since $\varphi(|z(\cdot, \cdot, \mathbf{d})|)$ is uniformly continuous on $[0, T] \times U_1$, there is some neighbourhood U of ξ contained in U_1 such that

$$|\varphi(z(t, \xi, \mathbf{d})) - \varphi(z(t, \bar{\xi}, \mathbf{d}))| < \frac{\varepsilon}{2T}$$

for all $\bar{\xi} \in U$, all $t \in [0, T]$. Consequently, for all $\bar{\xi} \in U$

$$\hat{V}_0(\xi, \mathbf{d}) - \hat{V}_0(\bar{\xi}, \mathbf{d})$$

$$= \int_0^{\theta_{\xi, \mathbf{d}}} \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t - \int_0^{\theta_{\bar{\xi}, \mathbf{d}}} \varphi(|z(t, \bar{\xi}, \mathbf{d})|)\, \mathrm{d}t$$

$$\leq \int_0^T \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t + \int_T^{\theta_{\xi, \mathbf{d}}} \varphi(|z(t, \xi, \mathbf{d})|)\, \mathrm{d}t$$

$$- \int_0^T \varphi(|z(t, \bar{\xi}, \mathbf{d})|)\, \mathrm{d}t$$

$$\leq \int_0^T |\varphi(z(t, \xi, \mathbf{d})) - \varphi(z(t, \bar{\xi}, \mathbf{d}))|\, \mathrm{d}t + \varepsilon/2 \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\hat{V}_0(\xi) \leq \liminf_{\bar{\xi} \to \xi} \hat{V}_0(\bar{\xi})$. $\qquad \square$

For each $\xi \in \mathcal{E}$, define

$$V_0(\xi) = \sup_{\mathbf{d} \in \mathcal{M}_\Omega} \hat{V}_0(\xi, \mathbf{d})$$

Note that V_0 is positive on $\mathcal{E}$. Estimate (33) implies that

$$V_0(\xi) \leq \mu_2(|\xi|) \qquad (35)$$

for all $\xi \in \mathcal{E}$.

To show properness of V_0 we will need the following lemma, proved in Krichman *et al.* (2001) along the lines of the proof of the disturbance-free version of this result in Sontag and Wang (1996).

Lemma 6: *Suppose* $\Sigma: \dot{z} = f(z, d)$, $y = h(z)$ *is a system of type* (6), *and* $p(\cdot)$ *is a smooth function of class* $\mathcal{K}_\infty$, *such that the following conditions hold:*

- $|f(\xi, d)| \leq 1$ *for all* $\xi \in \mathbb{X}$ *and* $d \in \Omega$,
- $|\nabla(p \circ h)(\xi) \cdot f(\xi, d)| \leq 1$ *for all* $d \in \Omega$ *and all* ξ *with* $|h(\xi)| \geq 1$,
- $p(s) \geq s$ *for all* $s > 0$.

Pick any constant $a > 0$ *and define* $K_0 = p(1) + (2 + a)/a + 1$. *Then for each* $\xi \in \mathbb{X}$ *such that*

$$|\xi| \geq (1 + a)p(|h(\xi)|) \quad \text{and} \quad |\xi| \geq K_0$$

it holds that
$$|z(t, \xi, \mathbf{d})| > p(|h(z(t, \xi, \mathbf{d}))|)$$

for all $t \in [0, 1)$ *and any* $\mathbf{d} \in \mathcal{M}_\Omega$.

Lemma 7: *If* $\xi \in \mathcal{F}(2\rho)$ *and* $|\xi| > \rho(1) + 4$, *then* $V_0(\xi) > \varphi(|\xi| - 1)$.

Proof: Recall that we have assumed that $\rho(s) > s$ for any positive s. Lemma 6, applied with $f = \hat{f}$, $a = 1$, $p = \rho$, implies that $\theta_{\xi,\mathbf{d}} \geq 1$ for all $\xi \in \mathcal{F}(2\rho)$ with $|\xi| > \rho(1) + 4$ and all $\mathbf{d} \in \mathcal{M}_\Omega$. Note that $|\hat{f}(z, d)| < 1$, so, if $|\xi| > \rho(1) + 4$, then $|z(t, \xi, \mathbf{d})| > |\xi| - 1$ for all $t \in [0, 1]$. Therefore, since $\theta_{\xi,\mathbf{d}} \geq 1$ for any $\mathbf{d}$, we have

$$V_0(\xi) \geq \int_0^1 \varphi(|z(t, \xi, \mathbf{d})|) \, dt \geq 1 \cdot \varphi(|\xi| - 1)$$

proving the lemma. $\qquad \square$

Lemma 8: V_0 *is lower semicontinuous on* $\mathcal{E}$.

Proof: Suppose a sequence $\{\xi_k\}$ in $\mathcal{E}$ converges to $\xi \in \mathcal{E}$. Pick any disturbance $\mathbf{d} \in \mathcal{M}_\Omega$. For this particular $\mathbf{d}$ we have

$$\hat{V}_0(\xi, \mathbf{d}) \leq \liminf_{k \to \infty} \hat{V}_0(\xi_k, \mathbf{d}) \leq \liminf_{k \to \infty} V_0(\xi_k)$$

where the first inequality follows from the lower semicontinuity of $\hat{V}_0$. Since this estimate holds for arbitrary $\mathbf{d}$, we will also have

$$V_0(\xi) = \sup_{\mathbf{d} \in \mathcal{M}_\Omega} \hat{V}_0(\xi, \mathbf{d}) \leq \liminf_{k \to \infty} V_0(\xi_k)$$

proving the lemma. $\qquad \square$

The following is an immediate consequence of Lemmas 7 and 8 and inequality (35).

Corollary 1: *There exists a* $\mathcal{K}_\infty$-*function* α *such that*

$$\alpha(|\xi|) \leq V_0(\xi) \leq \mu_2(|\xi|) \tag{36}$$

for all $\xi \in \mathcal{F}(2\rho)$, *where* μ_2 *is as in inequality* (32). $\quad \square$

Proof: Since V_0 is lower semicontinuous, it attains its minimum on any compact set. For each positive l define

$$r_l := (\rho(1) + 4)/l$$
$$\text{and} \quad m_l = \inf \{V_0(z) : z \in \mathcal{F}(2\rho), r_l \leq |z| \leq r_1\}$$

Since the sequence $\{m_l\}$ is non–increasing and positive, and φ is of class $\mathcal{K}_\infty$, we can find a $\mathcal{K}_\infty$-function α such that

$$\alpha(s) < m_l \qquad \forall\, s \in [r_l, r_{l-1}], \qquad \forall\, l > 1$$

and

$$\alpha(s) < \varphi(s - 1) \qquad \forall\, s \geq \rho(1) + 4$$

By construction, α will be a lower bound for V_0 on $\mathcal{F}(2\rho)$. $\qquad \square$

Lemma 9: *For any* $\xi \in \mathcal{E}$ *and any* σ, τ *such that* $0 \leq \sigma < \tau < \theta_{\xi,\mathbf{d}}$, *the following inequality holds*

$$V_0(z(\tau, \xi, \mathbf{d})) - V_0(z(\sigma, \xi, \mathbf{d})) \leq -\int_\sigma^\tau \varphi(z(t, \xi, \mathbf{d})) \, dt \tag{37}$$

Proof: Fix $\xi \in \mathcal{E}$ and $d \in \mathcal{M}_\Omega$, and fix $0 \leq \sigma < \tau < \theta_{\xi,\mathbf{d}}$. Let $\varepsilon > 0$ be given. Find $\mathbf{d}_1$ such that

$$V_0(z(\tau, \xi, \mathbf{d})) - \hat{V}_0(z(\tau, \xi, \mathbf{d}), \mathbf{d}_1) < \varepsilon$$

Let $\tilde{\mathbf{d}}$ be the disturbance defined by

$$\tilde{\mathbf{d}}(t) = \begin{cases} \mathbf{d}(t + \sigma) & \text{if } 0 \leq t \leq \tau - \sigma \\ \mathbf{d}_1(t - (\tau - \sigma)) & \text{if } t > \tau - \sigma \end{cases}$$

Then $z(t, z(\tau, \xi, \mathbf{d}), \mathbf{d}_1) = z(t + (\tau - \sigma), z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}})$ for all $t \geq 0$. Let $\tilde{\theta} = \theta_{z(\tau, \xi, \mathbf{d}), \mathbf{d}_1}$, and $\bar{\theta} = \theta_{z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}}}$. Then $\bar{\theta} = \tilde{\theta} + (\tau - \sigma)$, because $z(t, \xi, \mathbf{d}) \notin \mathcal{D}$ for $t \leq \tau$, by choice of τ (if $\tilde{\theta} = +\infty$, then $\bar{\theta} = +\infty$ satisfies our definition). Consequently, one has

$$\hat{V}_0(z(\tau, \xi, \mathbf{d}), \mathbf{d}_1)$$
$$= \int_0^{\tilde{\theta}} \varphi(|z(s, z(\tau, \xi, \mathbf{d}), \mathbf{d}_1)|) \, ds$$
$$= \int_0^{\tilde{\theta}} \varphi(|z(s + (\tau - \sigma), z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}})|) \, ds$$
$$= \int_{\tau - \sigma}^{\bar{\theta}} \varphi(|z(s, z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}})|) \, ds$$
$$= \hat{V}_0(z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}}) - \int_0^{\tau - \sigma} \varphi(|z(s, z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}})|) \, ds$$
$$= \hat{V}_0(z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}}) - \int_0^{\tau - \sigma} \varphi(|z(s + \sigma, \xi, \mathbf{d})|) \, ds$$
$$= \hat{V}_0(z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}}) - \int_\sigma^\tau \varphi(|z(s, \xi, \mathbf{d})|) \, ds$$

Thus,

$$V_0(z(\tau, \xi, \mathbf{d})) \le \hat{V}_0(z(\tau, \xi, \mathbf{d}), \mathbf{d}_1) + \varepsilon$$

$$= \hat{V}_0(z(\sigma, \xi, \mathbf{d}), \tilde{\mathbf{d}}) - \int_\sigma^\tau \varphi(|z(s, \xi, \mathbf{d})|) \, \mathrm{d}s + \varepsilon$$

$$\le V_0(z(\sigma, \xi, \mathbf{d})) - \int_\sigma^\tau \varphi(|z(s, \xi, \mathbf{d})|) \, \mathrm{d}s + \varepsilon$$

Letting $\varepsilon \to 0$, we get the desired inequality. $\qquad \square$

To see that the same estimate holds along the original system, pick an initial state $\xi \in \mathcal{E}$, disturbance $\mathbf{d} \in \Omega$ and t_1 and t_2 such that $0 \le t_1 < t_2 < \lambda_{\xi, \mathbf{d}}$. Then

$$V_0(x(t_2, \xi, \mathbf{d})) - V_0(x(t_1, \xi, \mathbf{d}))$$

$$= V_0(z(\sigma_{\xi, \mathbf{d}}(t_2), \xi, \mathbf{d} \circ \sigma_{\xi, \mathbf{d}}^{-1})) - V_0(z(\sigma_{\xi, \mathbf{d}}(t_1), \xi, \mathbf{d} \circ \sigma_{\xi, \mathbf{d}}^{-1}))$$

$$\le - \int_{\sigma_{\xi, \mathbf{d}}(t_1)}^{\sigma_{\xi, \mathbf{d}}(t_2)} \varphi(|z(s, \xi, \mathbf{d} \circ \sigma_{\xi, \mathbf{d}}^{-1})|) \, \mathrm{d}s$$

$$= - \int_{t_1}^{t_2} \varphi(|z(\sigma_{\xi, \mathbf{d}}(t), \xi, d \circ \sigma_{\xi, \mathbf{d}}^{-1})|) \, \mathrm{d}\sigma_{\xi, \mathbf{d}}(t)$$

$$= - \int_{t_1}^{t_2} \varphi(|x(t, \xi, \mathbf{d})|) \frac{\mathrm{d}}{\mathrm{d}t} \sigma_{\xi, \mathbf{d}}(t) \, \mathrm{d}t$$

$$= - \int_{t_1}^{t_2} \varphi(|x(t, \xi, \mathbf{d})|)$$

$$\times [1 + |f(x(t, \xi, \mathbf{d}), \mathbf{d}(t))|^2 + \kappa(x(t, \xi, \mathbf{d}))] \, \mathrm{d}t$$

$$\le - \int_{t_1}^{t_2} \varphi(|x(t, \xi, \mathbf{d})|) \, \mathrm{d}t$$

Remark 2: Note that if the system we consider is disturbance-free, $V_0(\xi) = \int_0^{\theta_\xi} \varphi(|z(t, \xi)|) \, \mathrm{d}t$. For any positive t, we have $\theta_{z(t, \xi)} = \theta_\xi - t$ so that $V_0(z(t, \xi))$ is differentiable in t and

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0(z(t, \xi)) = -\varphi(|z(t, \xi)|)$$

One can easily see (by a discussion similar to that in case with disturbances, treated above), that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0(x(t, \xi)) \le -\varphi(|x(t, \xi)|)$$

Thus, we have a lower semicontinuous function V_0, defined on the set $\mathcal{E}$, where it satisfies the integral dissipation inequality (37). Moreover, the inequality (36) holds for all $x \in \mathcal{F}(2\rho)$.

4.1. *Proof of Proposition 1*

To construct a UOSS-Lyapunov-like function defined on the whole state-space $\mathbb{X}$, we must extend V_0. This is done next, and in the process we prove Proposition 1. (Letting $\eta := 2\rho$, the set $\mathcal{F}(2\rho)$ is the same as $\mathcal{F}(\eta)$. We will construct a Lyapunov-like func-

tion V_2 for Σ and a class $\mathcal{K}_\infty$ function Φ such that $V_2 = \Phi \circ V_0$ on $\mathcal{F}(3\rho)$. As this set contains $\mathcal{F}(2\eta)$, the proposition will follow.)

We will first make a few observations. The following two lemmas serve as discontinuous versions of the chain rule and the product rule respectively in the form relevant to our setting. Other versions of the chain rule for non–smooth functions were proven in the literature, (see for example Clarke *et al.* 1998, Theorem 9.1), but most of them require more restrictive regularity assumptions, such as Lipschitz continuity of both functions to be composed. Also, in many cases, the needed property of generalized gradients is proven to hold at some point in a given neighbourhood of a given point, rather than at a given point itself. Thus, to keep the presentation self-contained, it will be helpful to have the following integral versions on hand.

Lemma 10: *Let $\gamma(t)$, $t \in [\sigma, \tau]$, be an absolutely continuous curve in $\mathbb{X}$. Let W be a function defined on a subset of $\mathbb{X}$ containing γ, such that for any t_1, t_2 in $[\sigma, \tau]$ the function $W \circ \gamma$ is Riemann integrable and the following dissipation inequality holds*

$$W(\gamma(t_2)) - W(\gamma(t_1)) \le \int_{t_1}^{t_2} \alpha(t) \, \mathrm{d}t \qquad (38)$$

where the function α is Lebesgue integrable on $[\sigma, \tau]$. Assume that Φ is continuously differentiable with $\Phi'(r) \ge 0$ for all r. Then

$$\Phi(W(\gamma(\tau))) - \Phi(W(\gamma(\sigma))) \le \int_\sigma^\tau \Phi'(W(\gamma(t))) \alpha(t) \, \mathrm{d}t$$

Proof: Let $\mu(s) = \int_\sigma^s \alpha(t) \, \mathrm{d}t$. Observe first that for any t_1, t_2 in $[\sigma, \tau]$, we have

$$\Phi(W(\gamma(t_2))) - \Phi(W(\gamma(t_1)))$$

$$= \Phi'(\zeta)[W(\gamma(t_2)) - W(\gamma(t_1))]$$

where ζ is some number between $W(\gamma(t_2))$ and $W(\gamma(t_1))$ furnished by the Mean Value Theorem. By (38), we have

$$\Phi(W(\gamma(t_2))) - \Phi(W(\gamma(t_1)))$$

$$= \Phi'(\zeta)(W(\gamma(t_2)) - W(\gamma(t_1))) \le \Phi'(\zeta) \int_{t_1}^{t_2} \alpha(t) \, \mathrm{d}t$$

$$= \Phi'(\zeta)(\mu(t_2) - \mu(t_1))$$

Now let $\sigma = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = \tau$ be any partition of $[\sigma, \tau]$. Then

$$\Phi(W(\gamma(\tau))) - \Phi(W(\gamma(\sigma)))$$

$$= \sum_{i=1}^m [\Phi(W(\gamma(t_i))) - \Phi(W(\gamma(t_{i-1})))]$$

$$\le \sum_{i=1}^m \Phi'(\zeta_i)(\mu(t_i) - \mu(t_{i-1})) \qquad (39)$$

where ζ_i is between $W(\gamma(t_{i-1}))$ and $W(\gamma(t_i))$. Note that

$$\min_{[t_{i-1}, t_i]} \Phi'(W(\gamma(t))) \le \Phi'(\zeta_i) \le \sup_{[t_{i-1}, t_i]} \Phi'(W(\gamma(t)))$$

As we let the mesh of the partition tend to zero, the expression in the right-hand side of the inequality (39) will converge to the Riemann–Stiltjes integral (which exists and equal to the corresponding Lebesgue integral with respect to the measure, generated by μ, because $\Phi' \circ W \circ \gamma$ is Riemann integrable on $[\sigma, \tau]$, and μ is absolutely continuous). Finally, we obtain

$$\Phi(W(\gamma(\tau))) - \Phi(W(\gamma(\sigma))) \le \int_\sigma^\tau \Phi'(W(\gamma(t))) \, \mathrm{d}\mu(t)$$
$$= \int_\sigma^\tau \Phi'(W(\gamma(t)))\alpha(t) \, \mathrm{d}t$$

as stated in the lemma. $\qquad\square$

Thus, if Φ is continuously differentiable with $\Phi' \in \mathcal{K}$ and $\Phi(0) = 0$, and V_0 is as we constructed, i.e. satisfies (36) and (37) for all $\xi \in \mathcal{F}(2\rho)$, then $V_0(x(t, \xi, \mathbf{d}))$ is decreasing on $[\sigma, \tau]$ for any $0 \le \sigma < \tau < \lambda_{\xi, \mathbf{d}}$. Thus, for any $\xi \in \mathcal{F}(2\rho)$, $V_0(x(\cdot, \xi, \mathbf{d}))$ is Riemann integrable on $[\sigma, \tau]$ for any $0 \le \sigma < \tau < \lambda_{\xi, \mathbf{d}}$. Consequently, by Lemma 10 the function $V_1 := \Phi \circ V_0$ will satisfy the following dissipation inequality along any piece of trajectory contained in $\mathcal{F}(2\rho)$

$$V_1(x(\tau, \xi, \mathbf{d})) - V_1(x(\sigma, \xi, \mathbf{d}))$$
$$\le -\int_\sigma^\tau \Phi'(\alpha(|x(t, \xi, \mathbf{d})|))\varphi(|x(t, \xi, \mathbf{d})|) \, \mathrm{d}t \quad (40)$$

Since $[\Phi' \circ \alpha(\cdot)] \varphi(\cdot)$ is of class $\mathcal{K}_\infty$, V_1 may also serve as a UOSS Lyapunov function, if appropriately extended to the complement of $\mathcal{F}(2\rho)$.

Lemma 11: *Let γ be as in the previous lemma, and suppose that ϕ is a smooth function defined on a subset of $\mathbb{X}$ containing γ, with values in $[0, 1]$. Assume that W is a measurable function on $\mathbb{X}$. Then:*

(1) *If W is decreasing along γ, then*

$$\phi(\gamma(\tau))W(\gamma(\tau)) - \phi(\gamma(\sigma))W(\gamma(\sigma))$$
$$\le \int_\sigma^\tau W(\gamma(t)) \, \mathrm{d}\phi(\gamma(t))$$

(2) *If W is increasing along γ, then*

$$\phi(\gamma(\tau))W(\gamma(\tau)) - \phi(\gamma(\sigma))W(\gamma(\sigma))$$
$$\le \int_\sigma^\tau W(\gamma(t)) \, \mathrm{d}\phi(\gamma(t)) + W(\gamma(\tau)) - W(\gamma(\sigma))$$

Proof: Observe first that for any $\sigma \le t_1 < t_2 \le \tau$, it holds that

$$\phi(\gamma(t_2))W(\gamma(t_2)) - \phi(\gamma(t_1))W(\gamma(t_1))$$
$$= \phi(\gamma(t_2))[W(\gamma(t_2)) - W(\gamma(t_1))]$$
$$+ W(\gamma(t_1))[\phi(\gamma(t_2)) - \phi(\gamma(t_1))] \quad (41)$$

Assume $W(\gamma(t))$ is increasing. Since $0 \le \phi(\xi) \le 1$, it follows that the right-hand side of (41) is bounded by

$$W(\gamma(t_2)) - W(\gamma(t_1)) + W(\gamma(t_1))[\phi(\gamma(t_2)) - \phi(\gamma(t_1))]$$

If $W(\gamma(t))$ is decreasing, the right-hand side of

$$W(\gamma(t_1))(\phi(\gamma(t_2)) - \phi(\gamma(t_1)))$$

Now let $\sigma = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m = \tau$ be any partition of $[\sigma, \tau]$. Then, in the case when $W(\gamma(t))$ is increasing, we have

$$\phi(\gamma(\tau))W(\gamma(\tau)) - \phi(\gamma(\sigma))W(\gamma(\sigma))$$
$$= \sum_{i=1}^m \phi(\gamma(t_i))W(\gamma(t_i)) - \phi(\gamma(t_{i-1}))W(\gamma(t_{i-1}))$$
$$\le \sum_{i=1}^m \{W(\gamma(t_i)) - W(\gamma(t_{i-1}))$$
$$+ W(\gamma(t_{i-1}))(\phi(\gamma(t_i)) - \phi(\gamma(t_{i-1})))\}$$
$$= W(\gamma(\tau)) - W(\gamma(\sigma))$$
$$+ \sum_{i=1}^m W(\gamma(t_{i-1}))[\phi(\gamma(t_i)) - \phi(\gamma(t_{i-1}))]$$

The estimate

$$\phi(\gamma(\tau))W(\gamma(\tau)) - \phi(\gamma(\sigma))W(\gamma(\sigma))$$
$$\le \sum_{i=1}^m W(\gamma(t_{i-1}))[\phi(\gamma(t_i)) - \phi(\gamma(t_{i-1}))]$$

for the case of decreasing $W(\gamma(t))$ is obtained in a similar way.

As we let the mesh of the partition tend to zero, the sum in the right-hand side of the last inequalities will converge to the Riemann–Stiltjes integral (which exists, since $W \circ \gamma$ is monotone, and hence Riemann integrable, and the integrating function $\phi \circ \gamma$ is smooth), so, the result follows. $\qquad\square$

We will also need the following simple separation result from topology (it is a well-known fact, see, for example, Boothby 1986).

Lemma 12: *Let $\mathcal{M}$ be a smooth manifold, and suppose K_1 and K_2 are two closed subsets of $\mathcal{M}$ such that $K_1 \cap K_2 = \varnothing$. Then there exists a smooth function $\phi: \mathcal{M} \to [0, 1]$ such that*

$$\phi(x) = \begin{cases} 1 & x \in K_1 \\ 0 & x \in K_2 \end{cases}$$

$V_2(x) = \phi(V_0(x))$

$\mathcal{E}_2 = \{x : |x| > 3\rho(|h(x)|)\}$

$\mathcal{E} = \{x : |x| > 2\rho(|h(x)|)\}$

$V_2(x) = \text{"mix"}$

$D_2 = \{x : |x| < 4\rho(|h(x)|)\}$

$V_1(x) = \phi(|x|^2)$

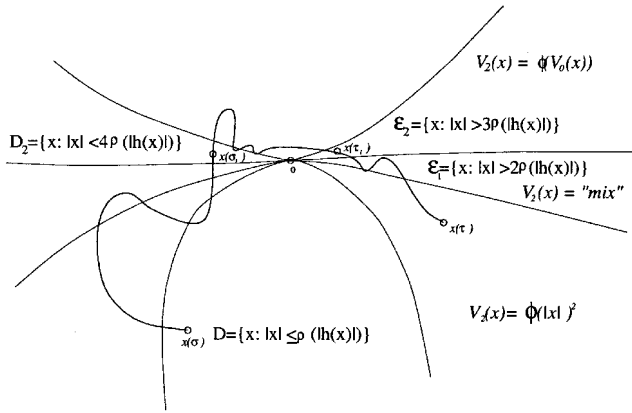$D = \{x : |x| \leq \rho(|h(x)|)\}$

Figure 1.   Partitioning of the trajectory.

Proof (of Proposition 1):   Let

$$\mathcal{E}_2 := \mathcal{F}(3\rho) = \{\xi : |\xi| > 3\rho(|h(\xi)|)\}$$

and

$$\mathcal{D}_2 := \text{int}\left(\mathbb{X} \setminus \mathcal{F}(4\rho)\right) = \{\xi : |\xi| < 4\rho(|h(\xi)|)\}$$

(see figure 1). Lemma 12 provides a smooth function $\phi : \mathbb{X} \setminus \{0\} \to [0, 1]$ with the property that

$$\phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \geq 3\rho(|h(\xi)|) \\ 0, & \text{if } |\xi| \leq 2\rho(|h(\xi)|) \end{cases}$$

and ϕ is non–zero elsewhere. It is easy to see that $|\nabla\phi(x)|$ is bounded above by a $\mathcal{K}$ function ν_2 of $|x|$ outside the unit ball centered at 0. One can also find a smooth, strictly increasing function $\nu_1 : [0, 1] \to \mathbb{R}_{\geq 0}$, such that $\nu_1(0) = 0$ and $|\nabla\phi(x)| \leq 1/\nu_1(|x|)$ for all x such that $0 < |x| \leq 1$.

Our next goal is to find a $\mathcal{K}$-function Φ whose derivative Φ' is also in $\mathcal{K}$ such that for the function

$$V_2 : \mathbb{X} \to \mathbb{R}_{\geq 0}$$

given by

$$V_2(\xi) = \phi(\xi)\Phi(V_0(\xi)) + (1 - \phi(\xi))\Phi(|\xi|^2) \qquad (42)$$

there exists some $\mathcal{K}$-function α_5 such that, for any $\xi \in \mathbb{X}$ and $\mathbf{d} \in \mathcal{M}_\Omega$, the estimate

$$V_2(x(\tau, \xi, \mathbf{d})) - V_2(x(\sigma, \xi, \mathbf{d})) \leq \int_\sigma^\tau \alpha_5(|x(t, \xi, \mathbf{d})|)\, dt \qquad (43)$$

holds on any interval $[\sigma, \tau] \subseteq [0, t_{\max})$. (Strictly speaking, the definition (42) makes no sense on $\mathcal{D}$, because V_0 is not defined there, however, since $\phi \equiv 0$ on $\mathcal{D}$, this does not matter).

Once this is accomplished, the inequality

$$V_2(x(\tau, \xi, \mathbf{d})) - V_2(x(\sigma, \xi, \mathbf{d}))$$

$$\leq \int_\sigma^\tau [-\Phi'(\alpha(|x(t, \xi, \mathbf{d})|))\varphi(|x(t, \xi, \mathbf{d})|)$$

$$+ \Phi'(\alpha(4\rho(|h(x(t))|)))\varphi(4\rho(|h(x(t))|))$$

$$+ \alpha_5(4\rho(|x(t)|))]\, dt \qquad (44)$$

will hold for any ξ and $\mathbf{d}$ on any subinterval $[\sigma, \tau] \subset [0, t_{\max})$, so that V_2 will be a UOSS-Lyapunov-like function for system (6), as in Definition 2, with

$$\alpha_3 = \Phi'(\alpha(\cdot))\varphi(\cdot) \qquad (45)$$

$$\gamma = \alpha_5(4\rho(\cdot)) + \Phi'(\alpha(4\rho(\cdot)))\varphi(4\rho(\cdot)) \qquad (46)$$

and

$$\left. \begin{array}{l} \alpha_1 = \min\{\Phi(|\cdot|^2), \Phi \circ \underline{\alpha}\} \\ \alpha_2 = \max\{\Phi(|\cdot|^2), \Phi \circ \overline{\alpha}\} \end{array} \right\} \qquad (47)$$

Indeed, pick $\xi \in \mathbb{X}$ and $\mathbf{d} \in \mathcal{M}_\Omega$ and let $x(\cdot) = x(\cdot, \xi, \mathbf{d})$. Since f is locally Lipschitz in x uniformly in d, $x(t) \neq 0$ for all $t \in [0, t_{\max})$. Consider some interval $[\sigma, \tau] \subset [0, t_{\max})$. The sets $\mathcal{E}_2$ and $\mathcal{D}_2$ are open and $\mathcal{E}_2 \cup \mathcal{D}_2 = \mathbb{X} \setminus \{0\}$, so, for every $t \in [\sigma, \tau]$, there will be an open (relative to $[\sigma, \tau]$) interval $\mathcal{I}_t$, containing t and such that the corresponding piece of trajectory is contained entirely in at least one of the sets $\mathcal{E}_2$ or $\mathcal{D}_2$. By compactness, we may pick from $\{\mathcal{I}_t\}$ a finite subcovering $\{[\sigma_0, \tau_0), (\sigma_k, \tau_k], (\sigma_i, \tau_i), i = 1, 2, \ldots, k-1\}$ of $[\sigma, \tau]$. By shrinking the intervals if necessary, we may assume that for every integer j such that $0 \leq j \leq k-2$, we have $\sigma_{j+1} < \tau_j < \sigma_{j+2}$, and for any j between 2 and k we have $\tau_{j-2} < \sigma_j < \tau_{j-1}$. Then the interval $[\sigma, \tau]$ is partitioned as

$$\sigma = \sigma_0 < \sigma_1 < \tau_0 < \sigma_2 < \tau_1 < \sigma_3 < \cdots$$

$$< \sigma_j < \tau_{j-1} < \sigma_{j+1} < \cdots < \sigma_k < \tau_{k-1} < \tau_k = \tau$$

For each subinterval $[t_j, t_{j+1}]$ in this partition, the corresponding piece of trajectory lies entirely in at least one of the sets $\mathcal{E}_2$ and $\mathcal{D}_2$. Split the trajectory into these pieces. By definition of V_2, $V_2 \equiv V_1 = \Phi \circ V_0$ on the closure of $\mathcal{E}_2$, so, on any piece of trajectory, which lies entirely in $\mathcal{E}_2$, the function V_2 satisfies the inequality (40), from which (44) trivially follows with σ, τ replaced by σ_j, τ_j respectively. On the other hand, if a piece of trajectory is contained in $\mathcal{D}_2$ on $[\sigma_j, \tau_j]$, then (43) implies

$$V_2(x(t_j)) - V_2(x(t_{j+1})) \leq \int_{t_j}^{t_{j+1}} \alpha_5(4\rho(|h(x(t))|))\, dt \quad (48)$$

which in turn implies (44) (with σ and τ replaced by t_j and t_{j+1}), because on $\mathcal{D}_2$

$$\Phi'(\alpha(4\rho(|h(x(t))|)))\varphi(4\rho(|h(x(t))|))$$

$$- \Phi'(\alpha(|x(t, \xi, \mathbf{d})|))\varphi(|x(t, \xi, \mathbf{d})|) \geq 0$$

Adding the estimates of type (44) we obtained for each of the subintervals, we conclude that (44) holds on the whole interval $[\sigma, \tau]$.

To construct the function Φ we need, take any ξ and $\mathbf{d}$, let $x(\cdot) = x(\cdot, \xi, \mathbf{d})$, and consider the difference

$$
\begin{aligned}
V_2(x(\tau)) - V_2(x(\sigma)) = {} & \phi(x(\tau))\Phi(V_0(x(\tau))) \\
& - \phi(x(\sigma))\Phi(V_0(x(\sigma))) \\
& + \Phi(|x(\tau)|^2) - \Phi(|x(\sigma)|^2) \\
& + \phi(x(\sigma))\Phi(|x(\sigma)|^2) \\
& - \phi(x(\tau))\Phi(|x(\tau)|^2) \quad (49)
\end{aligned}
$$

Next we get upper bounds for each of the three terms in the right side of (49). Recall that $\phi(\xi) = 0$ if $\xi \in \mathbb{X} \setminus \mathcal{F}(2\rho)$, and V_0 is decreasing along $x(t)$ when $x(t) \notin \mathcal{D}$ (and hence, so is $\Phi \circ V_0$ if $\Phi \in \mathcal{K}$). Arguing as before, we can partition the interval $[\sigma, \tau]$ into a disjoint union of subintervals $\cup_0^k [t_j, t_{j+1}]$ so that for each subinterval $[t_j, t_{j+1}]$ the corresponding piece of trajectory lies entirely in at least one of the sets $\mathbb{X} \setminus \mathcal{F}(2\rho)$ or $\mathcal{F}(\rho)$. If $x(t) \in \mathbb{X} \setminus \mathcal{F}(2\rho)$ for all $t \in [t_j, t_{j+1}]$, then

$$
\phi(x(t_{j+1}))\Phi(V_0(x(t_{j+1}))) - \phi(x(t_j))\Phi(V_0(x(t_j)))
$$
$$
= 0 = \int_{t_j}^{t_{j+1}} \Phi(V_0(x(t))) \, d\phi(x(t))
$$

On the other hand, if $x(t) \in \mathcal{F}(\rho)$ for all $t \in [t_j, t_{j+1}]$, then, by Lemma 11, we get

$$
\phi(x(t_{j+1}))\Phi(V_0(x(t_{j+1}))) - \phi(x(t_j))\Phi(V_0(x(t_j)))
$$
$$
\leq \int_{t_j}^{t_{j+1}} \Phi(V_0(x(t))) \, d\phi(x(t))
$$

Summing the estimates obtained on each subinterval, we get, on the whole interval $[\sigma, \tau]$,

$$
\phi(x(\tau))\Phi(V_0(x(\tau))) - \phi(x(\sigma))\Phi(V_0(x(\sigma)))
$$
$$
= \sum_{j=0}^{k} \phi(x(t_{j+1}))\Phi(V_0(x(t_{j+1}))) - \phi(x(t_j))\Phi(V_0(x(t_j)))
$$
$$
\leq \int_{\sigma}^{\tau} \Phi(V_0(x(t))) d\,\phi(x(t))
$$
$$
= \int_{\sigma}^{\tau} \Phi(V_0(x(t)))[\nabla_\phi(x(t)) \cdot \dot{x}(t)] \, dt
$$
$$
\leq \int_{\sigma}^{\tau} \Phi(\overline{\alpha}(|x(t)|))|\nabla_\phi(x(t))||f(x(t), \mathbf{d}(t))| \, dt \quad (50)
$$

For the second term, we have

$$
\Phi(|x(\tau)|^2) - \Phi(|x(\sigma)|^2)
$$
$$
= 2 \int_{\sigma}^{\tau} \Phi'(|x(t)|^2)[x(t) \cdot f(x(t), \mathbf{d}(t))] \, dt \quad (51)
$$

and for the last term, we have

$$
\phi(x(\sigma))\Phi(|x(\sigma)|^2) - \phi(x(\tau))\Phi(|x(\tau)|^2)
$$
$$
\leq \int_{\sigma}^{\tau} \Phi(|x(t)|^2) \, d\phi(x(t)) + \int_{\sigma}^{\tau} \Phi'(|x(t)|^2)\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \, \mathrm{d}t
$$
$$
\leq \int_{\sigma}^{\tau} \Phi(|x(t)|^2)|\nabla_\phi(x(t))||f(x(t), \mathbf{d}(t))| \, \mathrm{d}t
$$
$$
+ 2\int_{\sigma}^{\tau} \Phi'(|x(t)|^2)|x(t)||f(x(t), \mathbf{d}(t))| \, \mathrm{d}t \quad (52)
$$

Let ν_3 be a $\mathcal{K}$-function such that $\max_{d \in \Omega} |f(\xi, d)| \leq \nu_3(|\xi|)$. Take any smooth function $\pi_1 \colon [0, \overline{\alpha}(1)] \to \mathbb{R}_{\geq 0}$ with $\pi_1'(s) > 0$ for all $s \in (0, \overline{\alpha}(1))$, such that

$$
\frac{\pi_1(\overline{\alpha}(r))}{\nu_1(r)} < s(r), \quad \frac{\pi_1(r^2)}{\nu_1(r)} < s(r)
$$

for some $\mathcal{K}$-function s and all $0 < r \leq 1$. Let π_2 be any $\mathcal{K}$-function such that $\pi_2(r) \leq \pi_1'(r)$ for all non-negative $r \leq 1$.

Let $\Phi(r) = \int_0^r \pi_2(r_1) \, dr_1$. Then $\Phi(r) \leq \pi_1(r)$ for all $r \leq 1$, so that $\Phi(\overline{\alpha}(r))/\nu_1(r) < s(r)$ and $\Phi(r^2)/\nu_1(r) < s(r)$ for all $r \in (0, 1]$. Hence, the following $\mathcal{K}$ bounds can be obtained for the integrands in (50), (51) and (52) respectively

$$
\Phi(\overline{\alpha}(|\xi|))|\nabla_\phi(\xi)||f(\xi, d)|
$$
$$
\leq \max\{s(|\xi|), \Phi(\overline{\alpha}(|\xi|))\nu_2(|\xi|)\}\nu_3(|\xi|)
$$
$$
2\Phi'(|\xi|^2)[\xi \cdot f(\xi, d)] \leq 2\pi_2(|\xi|^2)|\xi|\nu_3(|\xi|)
$$

and

$$
\Phi(|\xi|^2)|\nabla_\phi(\xi)||f(\xi, d)| + 2\Phi'(|\xi|^2)|\xi||f(\xi, d)|
$$
$$
\leq \max\{s(|\xi|), \Phi(|\xi|^2)\nu_2(|\xi|)\}\nu_3(|\xi|) + 2\pi_2(|\xi|^2)|\xi|\nu_3(|\xi|)
$$

The functions in the right sides of the last three inequalities are all of class $\mathcal{K}$. Define

$$
\begin{aligned}
\alpha_5(\cdot) := {} & 4\pi_2(|\cdot|^2) \cdot |\nu_3(|\cdot|) \\
& + 2\max\{s(|\cdot|), \Phi(\overline{\alpha}(|\cdot|))\nu_2(|\cdot|), \Phi(|\cdot|^2)\nu_2(|\cdot|)\} \\
& \times \nu_3(|\cdot|)
\end{aligned}
$$

By the previous discussion, V_2 is a UOSS-Lyapunov-like function for the system (6). $\qquad\square$

5. Proof of necessity in Theorem 2

We now show how to reduce Theorem 2 to the particular case of systems with no controls. Since this was done in detail in Krichman *et al.* (2001), we will only sketch the main steps here. Let $\mathbb{U}_1$ denote the closed unit ball $\{u \in \mathbb{U}: |u| \leq 1\}$ in $\mathbb{U}$.

Definition 8: System (13) is said to be *robustly output to state stable (ROSS)* if there exists a locally

Lipschitz $\mathcal{K}_\infty$-function Δ, called a *stability margin*, such that the system

$$\dot{x}(t) = g(x(t), \mathbf{d}(t)) := f(x(t), \mathbf{d}_u(t)\Delta(|x(t)|), \mathbf{w}(t)) \quad (53)$$

with disturbances $d := [d_u, w] \in \mathbb{U}_1 \times \Gamma$ and outputs $y = h(x)$ is UOSS.

Note that the set $\mathbb{U}_1 \times \Gamma$ is a compact subset of $\mathbb{R}^{m_u + m_w}$. We will denote it by Ω in this section. Observe also that the dynamics g of system (53) are locally Lipschitz in x uniformly in d, and also $g(0, d) = 0$ for all $d \in \Omega$.

The following lemma, linking ROSS and UIOSS properties, was proved in Krichman *et al.* (2001) via a small gain argument.

Lemma 13 (see Lemma 3.2 in Krichman *et al.* 2001)**:** *If a system* (13) *is UIOSS, then it is ROSS.*

We show now how the necessity part of Theorem 2 follows from the necessity part Theorem 1.

Lemma 14 (see Lemma 3.5 in Krichman *et al.* 2001)**:** *Suppose a system* Σ *of type* (13) *is ROSS. Let* V *be a UOSS-Lyapunov-like function for the system* (53) *associated with* Σ. *Then* V *is an UIOSS-Lyapunov-like function for* Σ.

Proof: Let Δ be a stability margin for Σ. Since V is a UOSS-Lyapunov-like function for (53), inequalities (8) and (9) hold with some α_1, α_2, α_3 and γ. Pick an initial state ξ, control $\mathbf{u}$, and disturbance $\mathbf{w}$, and write $x(t) := x(t, \xi, \mathbf{u}, \mathbf{w})$. Take $\tau, \sigma \in [0, t_{\max}(\xi, \mathbf{u}, \mathbf{w}))$. If for all $t \in [\sigma, \tau]$ we have $|x(t)| \geq \Delta^{-1}(|\mathbf{u}(t)|)$, then $\mathbf{u}(t) = \Delta(|x(t)|)\mathbf{d}_u(t)$ for some $\mathbf{d}_u \in \mathcal{M}_{\mathbb{U}_1}$, that is, $x(t)$ is a trajectory of the UOSS system (53) corresponding to Σ, with the disturbance $\mathbf{d} = [\mathbf{d}_u, \mathbf{w}]$. Along this trajectory, the estimate (16) for the function V is the same as (9). So, V is a UIOSS-Lyapunov-like function for Σ, with $\chi = \Delta^{-1}$, and α_i and γ as before. $\qquad\square$

6. Example: transistor networks

In the following example, treated in detail in Rouche *et al.* (1977), a non–smooth (yet continuous) Lyapunov function is used to establish the global asymptotic stability of a transistorized network. The setup is sketched next, and then we will modify the example to illustrate how OSS and IOSS properties can be proved, assuming that some components of the state can be observed and/or some parameters can serve as inputs.

Consider a network of n transistors, plugged into a linear resistive port, as on the figure 2. The state $x = (x_1, x_2, ..., x_{2n-1}, x_{2n})$ of the system consists of voltages (x_{2i-1}, x_{2i}), $i = 1, \ldots, n$, on each of the transistors. The system is described by the equations

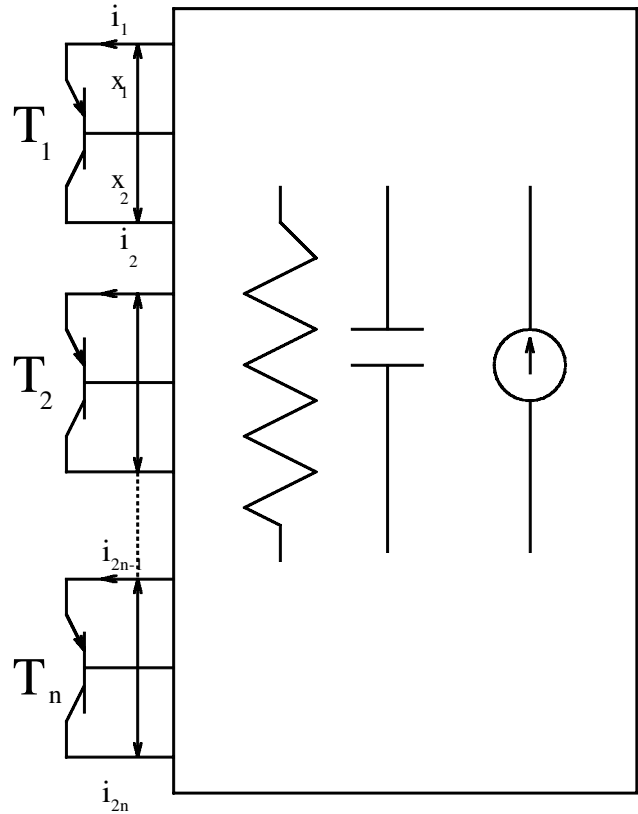$$C(x)\dot{x} = -TF(x) - Gx + b \quad (54)$$

Figure 2.   Transistor network.

with

- a block-diagonal matrix

$$T = \mathrm{diag}\left[\begin{pmatrix} p_1 & -r_1 \\ -r_1 & q_1 \end{pmatrix}, \ldots, \begin{pmatrix} p_n & -r_n \\ -r_n & q_n \end{pmatrix}\right]$$

where, for each integer $i = 1, \ldots, n$, the positive parameters p_i, q_i and r_i describe, correspondingly, resistors and voltage controlled current sources, constituting the ith transistor,

- a vector function $F(\cdot) = (f_1(\cdot), \ldots, f_{2n}(\cdot))^{\mathrm{T}}$, where $f_j \colon \mathbb{R} \to \mathbb{R}$ are strictly increasing continuous functions (such as exponentials, which are used in typical transistor models), each pair (f_{2i-1}, f_{2i}) of which describes the ith transistor,

- a diagonal matrix

$$C(x) = \mathrm{diag}\left[C_1(x_1), \ldots, C_{2n}(x_{2n})\right]$$

with continuous functions $C_j \colon \mathbb{R} \to \mathbb{R}$, such that there exists a positive ε satisfying

$$C_j(v) \geq \varepsilon$$

for all v and all positive integers $j \leq 2n$. For each positive integer $i \leq n$, the pair (C_{2i-1}, C_{2i}) describes the capacitors in the ith transistor (see figure 3).
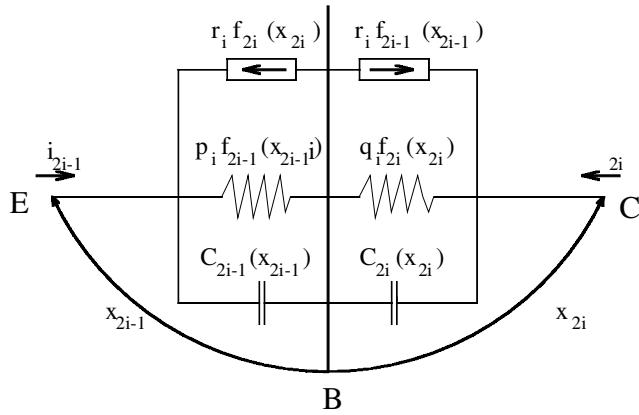
Figure 3.   The *i*th transistor.

- a $2n \times 2n$ constant matrix $G = \{g_{k}\}_{j,k=1}^{2n}$ and a $2n$-vector b, describing the resistive n-port.

Coordinatewise equations of the system will then read as follows for $1 \leq i \leq n$

$$\dot{x}_{2i-1} = \Lambda_{2i-1}(x) := \left(-p_i f_{2i-1}(x_{2i-1}) + r_i f_{2i}(x_{2i}) \right.$$
$$\left. - \sum_{j=1}^{2n} g_{j,2i-1} x_j + b_{2i-1} \right) \Big/ C_{2i-1}(x_{2i-1})$$

$$\dot{x}_{2i} = \Lambda_{2i}(x) := \left(r_i f_{2i-1}(x_{2i-1}) - q_i f_{2i}(x_{2i}) \right.$$
$$\left. - \sum_{j=1}^{2n} g_{j,2i} x_j + b_{2i} \right) \Big/ C_{2i}(x_{2i})$$

Suppose there exists an equilibrium point $\xi = (\xi_1, \xi_2, \ldots, \xi_{2n})$, i.e. a point satisfying

$$-TF(\xi) - G\xi + b = 0 \qquad (55)$$

(Generally speaking, such an equilibrium point is not guaranteed to exist.) We are interested in exploring the stability of the system with respect to ξ.

Let

$$V_j(x_j) = \left| \int_{\xi_j}^{x_j} C_j(v)\, dv \right|, \qquad 1 \leq j \leq 2n$$

and consider

$$V(x) = \sum_{j=1}^{2n} d_j V_j(x_j) \qquad (56)$$

where d_j are some arbitrary positive numbers (we will later decide how to choose them).

By routine computations it can be shown that

$$V(x) \geq \varepsilon \sum_{j=1}^{2n} d_j |x_i - \xi_i| \qquad (57)$$

so that V is indeed proper and positive definite. Notice also that V is not differentiable at any x such that $x_j = \xi_j$ for some $j \in \{1, 2, \ldots, 2n\}$, although (see Rouche *et al.* (1977) for an explicit calculation) the directional derivative along Λ does exist for all x and the following dissipation inequality holds for V

$$L_\Lambda V(x) \leq -\sum_{i=1}^{n} ((d_{2i-1} p_i - d_{2i} r_i)$$
$$\times |f_{2i-1}(x_{2i-1}) - f_{2i-1}(\xi_{2i-1})|$$
$$+ (-d_{2i-1} r_i + d_{2i} q_i)|f_{2i}(x_{2i}) - f_{2i}(\xi_{2i})|)$$
$$- \sum_{j=1}^{2n} \left(d_j g_{j,j} - \sum_{\substack{1 \leq k \leq 2n \\ k \neq j}} d_k |g_{k,j}| \right) |x_j - \xi_j| \quad (58)$$

Recall that a matrix $A \in \mathbb{R}^{n \times n}$ is said to be *column-sum dominant* if

$$a_{jj} - \sum_{i \neq j} |a_{ij}| > 0$$

for all $j = 1, 2, \ldots, n$. It is easy to see from (58) that in order to establish the negative-definiteness of $L_\Lambda V$ (and thus to prove the global asymptotic stability of the transistor network with respect to the equilibrium ξ), it is sufficient to find positive numbers d_j, $1 \leq j \leq 2n$, such that the matrices $\Delta^T := \text{diag}[d_1, \ldots, d_{2n}]T$ and $\Delta^G := \text{diag}[d_1, \ldots, d_{2n}]G$ are column sum dominant. This fact can be seen as a version of Sandberg's theorem (cf. Sandberg 1969).

Now suppose that, for some given T, F, and G, it is in fact impossible to find coefficients d_j to establish the desired global asymptotic stability property, that is, the matrices Δ^T and Δ^G cannot be made column sum dominant with any choice of D (this will happen, for example, if the matrix G is singular). However, assume that it is possible to find a set of $d_1, \ldots, d_{2n-2}$ so that Δ_{2n-2}^T and Δ_{2n-2}^G, the upper left $(2n-2) \times (2n-2)$ minors of Δ^T and Δ^G, are column sum dominant. Suppose also that the voltages x_{2n-2} and x_{2n} on the nth transistor can be observed, that is, in our notation, take

$$y_1 = h_1(x) = x_{2n-1} - \xi_{2n-1}$$
$$y_2 = h_2(x) = x_{2n} - \xi_{2n}$$

Since Δ_{2n-2}^G is column-sum dominant, we can find positive d_{2n-1} and d_{2n} small enough so that

$$d_j g_{j,j} - \sum_{\substack{1 \leq k \leq 2n \\ k \neq j}} d_k |g_{k,j}| > 0 \qquad \forall j \leq 2n - 2 \quad (59)$$

Let $\tilde{\gamma}$ be a $\mathcal{K}_\infty$-function, majorizing deviations of f_{2n-1} from its value at ξ_{2n-1} and deviations of f_{2n} from its value at ξ_{2n}

$$\tilde{\gamma}(|x_{2n-1} - \xi_{2n-1}|) \geq |f_{2n-1}(x_{2n-1}) - f_{2n-1}(\xi_{2n-1})|$$

and

$$\tilde{\gamma}(|x_{2n} - \xi_{2n}|) \geq |f_{2n}(x_{2n}) - f_{2n}(\xi_{2n})|$$

(such $\tilde{\gamma}$ is possible to find by the continuity of f_j). Now define V as in (56). Then V is proper because of (57), and, chopping off the last two terms in each of the summations in (58), we obtain the following bounds on the increments of V

$$L_A V(x)$$

$$\leq - \sum_{i=1}^{n-1}((d_{2i-1}p_i - d_{2i}r_i)|f_{2i-1}(x_{2i-1}) - f_{2i-1}(\xi_{2i-1})|$$

$$+ (-d_{2i-1}r_i + d_{2i}q_i)|f_{2i}(x_{2i}) - f_{2i}(\xi_{2i})|)$$

$$- \sum_{j=1}^{2n-2}\left(d_j g_{j,j} - \sum_{\substack{1 \leq k \leq 2n \\ k \neq j}} d_k|g_{k,j}|\right)|x_j - \xi_j|$$

$$- |x_{2n-1} - \xi_{2n-1}| - |x_{2n} - \xi_{2n}|$$

$$+ |x_{2n-1} - \xi_{2n-1}| + |x_{2n} - \xi_{2n}| \qquad (60)$$

$$+ (|d_{2n-1}p_n| + |d_{2n}r_n|)|f_{2n-1}(x_{2n-1}) - f_{2n-1}(\xi_{2n-1})|$$

$$\qquad\qquad (61)$$

$$+ (|d_{2n-1}r_n| + |d_{2n}q_n|)|f_{2n}(x_{2n}) - f_{2n}(\xi_{2n})|) \qquad (62)$$

$$+ \left(\sum_{k=1}^{2n} d_k|g_{k,2n-1}|\right)|x_{2n-1} - \xi_{2n-1}| \qquad (63)$$

$$+ \left(\sum_{k=1}^{2n} d_k|g_{k,2n}|\right)|x_{2n} - \xi_{2n}| \qquad (64)$$

The terms (60), (61), (62), (63), and (64), which account for the voltages on the n-th transistor, can be bounded above by

$$(|d_{2n-1}p_n| + |d_{2n}r_n| + |d_{2n-1}r_n| + |d_{2n}q_n|)\tilde{\gamma}(|y|)$$

$$+ 2\left(\sum_{k=1}^{2n} d_k|g_{k,2n-1}| + \sum_{k=1}^{2n} d_k|g_{k,2n}| + 1\right)|y|$$

Take the functions $\alpha_3(\cdot)$ and $\sigma_2(\cdot)$ such that

$$\alpha_3(s) \leq s\left(\min\left(1, \min_{1 \leq j \leq 2n-2}\left(d_j g_{j,j} - \sum_{\substack{1 \leq k \leq 2n \\ k \neq j}} d_k|g_{k,j}|\right)\right)\right)$$

$$\qquad\qquad (65)$$

$$\sigma_2(s) \geq (|d_{2n-1}p_n| + |d_{2n}r_n| + |d_{2n-1}r_n| + |d_{2n}q_n|)\tilde{\gamma}(s)$$

$$+ 2\left(\sum_{k=1}^{2n} d_k|g_{k,2n-1}| + \sum_{k=1}^{2n} d_k|g_{k,2n}| + 1\right)s \qquad (66)$$

so that V satisfies

$$L_A V(x) \leq -\alpha_3(|x - \xi|) + \sigma_2(|y|)$$

Thus, V is a non–differentiable OSS-Lyapunov-like function for the transistor network with the voltages on the last transistor serving as an output. Thus, we arrive at the following result (Sandberg's Theorem for the output-to-state stability of a transistor network).

Theorem 3: *Suppose a transistor network Σ, governed by equation* (54), *has an equilibrium at a point ξ. Suppose furthermore that measurements of the voltages on the last k transistors are available, that is, for all $i = 1, 2, \ldots, k$*

$$h_{2i-1}(x) = x_{2(n-k+i)-1} - \xi_{2(n-k+i)-1}$$

$$h_{2i}(x) = x_{2(n-k+i)} - \xi_{2(n-k+i)}$$

Then Σ will be OSS with respect to output h if there exist positive coefficients $d_1, d_2, \ldots, d_{2(n-k)}$ so that $\Delta_{2(n-k)}^T$ and $\Delta_{2(n-k)}^G$ are column-sum dominant.

Remark 3: The argument above proves this theorem for the case of 1 observed transistor. Similar arguments will work for any number k $(k < n)$ of observed tranistors.

Remark 4: If we further assume that the parameters r_n, p_n and q_n can serve as controls, we can use the function V to show that the system is in fact IOSS with respect to these controls. Indeed, let

$$u_1 := p_n$$
$$u_2 := r_n$$
$$u_3 := q_n$$

and find a $\mathcal{K}$-function σ_1 such that

$$(|d_{2n-1}||u_1| + |d_{2n}||u_2| + |d_{2n-1}||u_2| + |d_{2n}||u_3|)^2/2$$

$$\leq \sigma_1(|u|)$$

Then

$$(|d_{2n-1}p_n| + |d_{2n}r_n| + |d_{2n-1}r_n| + |d_{2n}q_n|)\tilde{\gamma}(s)$$

$$\leq \sigma_1(|u|) + \tilde{\gamma}(s)^2/2$$

so that

$$L_A V(x) \leq -\alpha_3(|x - \xi|) + \sigma_1(|u|) + \sigma_2(|y|)$$
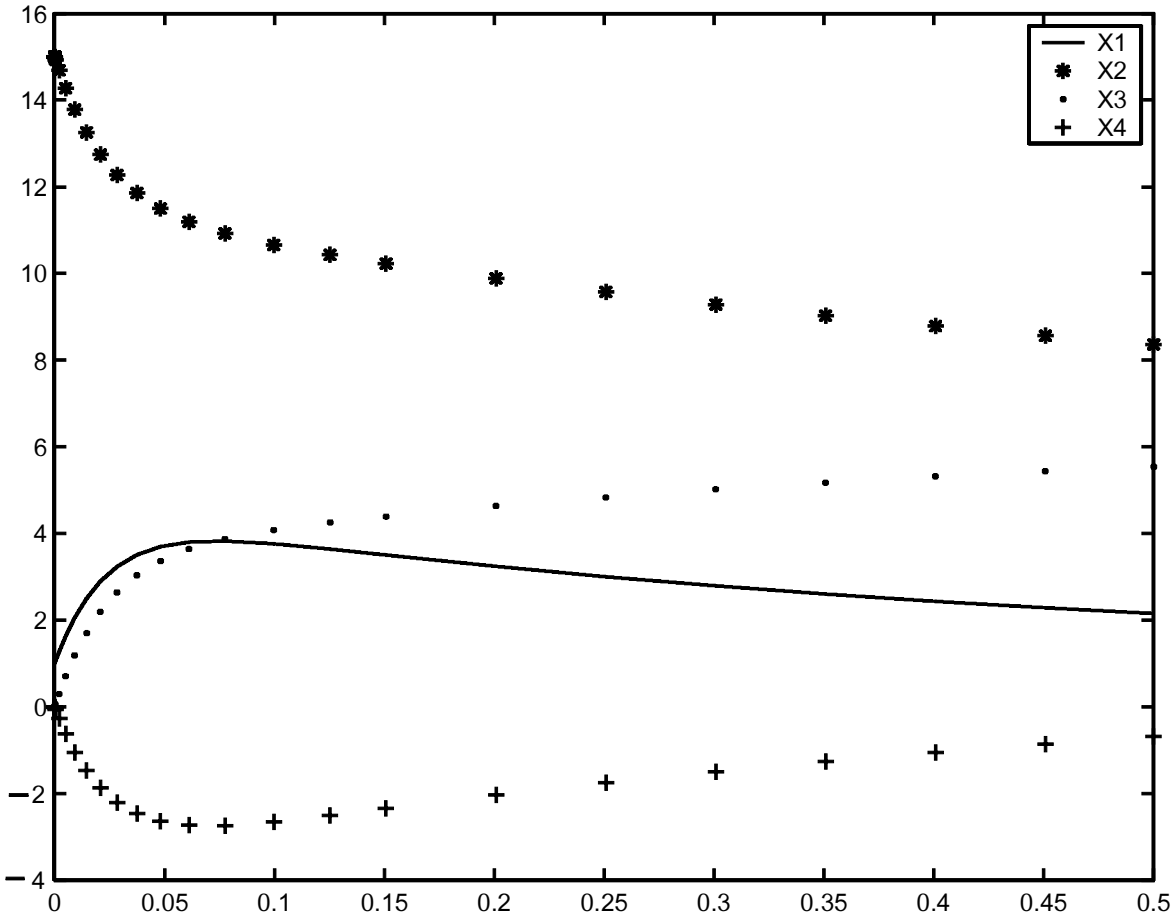
with σ_1 as above

Figure 4. $x_1(0) = 1$, $x_2(0) = 15$, $x_3(0) = x_4(0) = 0$, $0 \le t \le 0.5$. Large unobserved component decays until bounded by the output gain.

$$\sigma_2(s) := (\tilde{\gamma}(s))^2/2$$

$$+ 2\left(\sum_{k=1}^{2n} d_k |g_{k,2n-1}| + \sum_{k=1}^{2n} d_k |g_{k,2n}| + 1\right)s$$

and α_3 as in (65).

6.1. *Numerical example*

Consider a network with two transistors with the following coefficients:

- $p_1 = q_1 = p_2 = q_2 = 1$,
- $r_1 = r_2 = 1/2$,
- $b = (1/2, 1/2, 1/2, 1/2)$,
- $C_i(s) \equiv 1, \quad i = 1, 2, 3, 4$,
- $G = \begin{bmatrix} 11 & -10 & 10 & -11 \\ -10 & 11 & -11 & 10 \\ 10 & -11 & 11 & -10 \\ -11 & 10 & -10 & 11 \end{bmatrix}$,
- $f_1(s) = f_2(s) = f_3(s) = f_4(s) = e^{s/10}$

Thus, the equations of the system are

$$\dot{x}_1 = -e^{x_1/10} + e^{x_2/10}/2$$
$$- (11x_1 - 10x_2 + 10x_3 - 11x_4) + 1/2$$

$$\dot{x}_2 = e^{x_1/10}/2 - e^{x_2/10}$$
$$- (-10x_1 + 11x_2 - 11x_3 + 10x_4) + 1/2$$

$$\dot{x}_3 = -e^{x_3/10} + e^{x_4/10}/2$$
$$- (10x_1 - 11x_2 + 11x_3 - 10x_4) + 1/2$$

$$\dot{x}_4 = e^{x_3/10}/2 - e^{x_4/10}$$
$$- (-11x_1 + 10x_2 - 10x_3 + 11x_4) + 1/2$$

Note that, while the matrix T is column-sum dominant, the matrix G is singular, and thus Δ^G cannot be made column-sum dominant with any choice of D.

The system has a equilibrium at 0. Since Δ^G cannot me made column-sum dominant with any choice of D, we are unable to apply the 'classical' Sandberg's theorem to establish the global asymptotic stability of this system. However, if we are able to measure the voltages
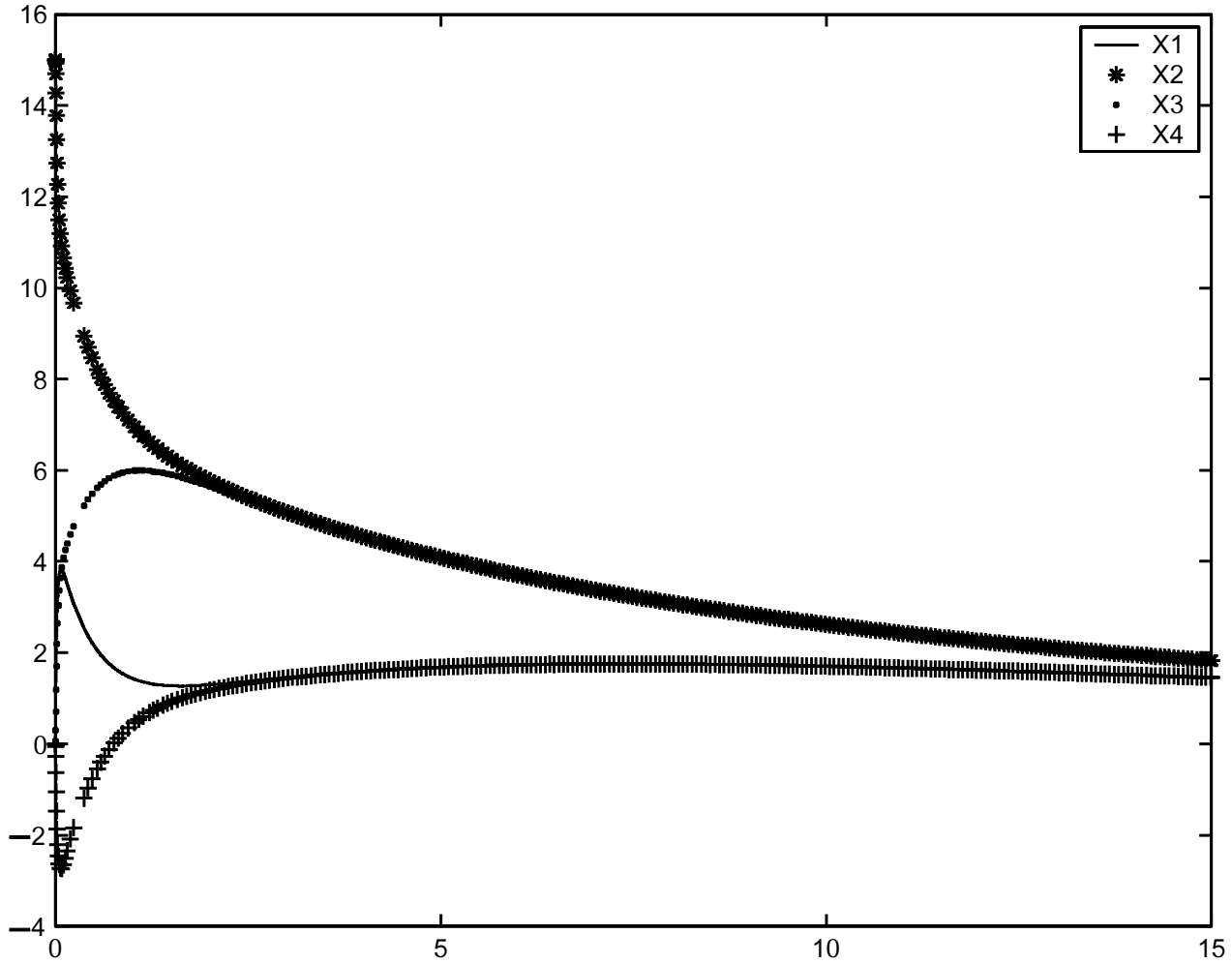
Figure 5. $x_1(0) = 1$, $x_2(0) = 15$, $x_3(0) = x_4(0) = 0$, $0 \le t \le 15$. Long term behaviour.

x_3 and x_4 on the second transistor, we can use Theorem 3 to show that the system is OSS with respect to the outputs $y_1 := x_3$ and $y_2 := x_4$. Indeed, the upper left minor

$$G_2 = \begin{bmatrix} 11 & -10 \\ -10 & 11 \end{bmatrix}$$

of G, is column-sum dominant, and so is

$$T_2 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

Let $D = \text{diag}\, [1, 1, 1/100, 1/100]$ (notice that (59) holds for G under this definition of D). Define a Lyapunov function V as in (56), i.e.

$$V(x) = |x_1| + |x_2| + (|x_3| + |x_4|)/100$$

To show that V is an OSS-Lyapunov function for our network, we next calculate the gains for its dissipation inequality.

According to (65), the dissipation rate $\alpha_3(s)$ must minorize

$$\left(11 - 10 - \frac{10 + 11}{100}\right)s \ge \frac{s}{2}$$

so, we can take

$$\alpha_3(s) := s/2$$

The output gain $\sigma_2(s)$ can be computed according to (66), majorizing

$$\left(\frac{1}{100} + \frac{1}{200} + \frac{1}{200} + \frac{1}{100}\right)(e^{s/10} - 1)$$

$$+ 2(10 + 11 + 21/100 + 1)s$$

thus we can take

$$\sigma_2(s) := (e^{s/10} - 1) + 100s$$

According to our predictions, if $|x_1(0)|$ and $|x_2(0)|$ (unobserved components) are large enough, and $|x_3(0)|$ and $|x_4(0)|$ (the outputs) are small enough, the Lyapunov function must decrease along a trajectory as long as

$$\alpha_3(|x|) \geq \sigma_2(|y|) \qquad (67)$$

The following simulation shows the behaviour of the system when the initial voltages on the first transistor are $x_1(0) = 1$ and $x_2(0) = 15$, and the second, 'observed' transistor starts with zero voltage ($x_3(0) = x_4(0) = 0$). Since (67) holds in the beginning, $V(x(t))$ must decrease. As expected, the largest voltage x_2 on the first, unobserved, transistor is steadily decreasing, and x_1 decreases after a small overshoot. The observed states x_3 and x_4 increase until they dominate x_1 and x_2 in magnitude. The plots in figures 4 and 5 show the overshoot and long-term behaviour of the system.

Independent disclaimer

The views expressed are those of the author(s) and do not reflect the official policy or position of The MITRE Corporation.

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