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UNIFORM GLOBAL ASYMPTOTIC STABILITY OF DIFFERENTIAL INCLUSIONS

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Abstract. Stability of differential inclusions defined by locally Lipschitz compact valued mappings is considered. It is shown that if such a differential inclusion is globally asymptotically stable, then, in fact, it is uniformly globally asymptotically stable (with respect to initial states in compacts). This statement is trivial for differential *equations*, but here we provide the extension to compact- (not necessarily convex-) valued differential inclusions. The main result is presented in a context which is useful for control-theoretic applications: a differential inclusion with two outputs is considered, and the result applies to the property of global error detectability.

1. Introduction

A fundamental notion in the stability analysis of dynamical systems is that of global asymptotic stability (GAS) which characterizes systems for which all trajectories converge to some equilibrium in a reasonable manner. When considering differential equations, there are two equivalent definitions of the GAS property. The more common definition is that a system is GAS if it is both (locally) stable and satisfies an attractivity property. Alternatively, one can define GAS in terms of a single bound involving a $\mathcal{KL}$ function (definition below). The latter definition makes explicit an important property of globally asymptotically stable differential equations – that initial conditions in a compact set give rise to trajectories which approach the equilibrium *uniformly*.

When generalizing to differential inclusions (or control systems), it becomes clear that the notions of global asymptotic stability and uniform global asymptotic stability (UGAS) do not coincide. In this paper we consider a special case of differential inclusions for which GAS and UGAS are

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equivalent. This is the case of inclusions defined by locally Lipschitz set-valued mappings (see definition below) whose values are nonempty compact sets. The motivation for considering locally Lipschitz set-valued mappings comes from applications to control systems with compact input-value sets, which give rise to just such differential inclusions through the definition of a set-valued mapping by

$$\begin{gathered} F(x) := \{f(x,u) : u \in U\}, \\ U \text{ is compact}, \quad f \text{ is locally Lipschitz}. \end{gathered} \tag{1.1}$$

A control-theoretic motivation also provides the ground for the form of the main result of the paper. We equip the differential equation with two outputs, one of which is considered as providing a description of the error associated with some system performance and the other is considered as a measurement of the system's state. Generalizing asymptotic stability to this case, we obtain a partial detectability property referred to as *error detectability*. Similar notions have appeared in both the control literature (output regulation, error feedback [13]) and literature on differential equations (partial stability [17], stability in two measures [14]).

Special cases of our main result (treating control systems of the form (1.1)) have appeared in [2, 11, 15]. The contribution of the current paper is to extend these relaxation results to more general differential inclusions, as well as to unify the results of [2, 11].

The main technical result used in this paper is a theorem on approximation of solutions of relaxed differential inclusions which appeared in [12], and which complements the standard relaxation theorem of Filippov and Ważewski (cf. [4, 5, 8, 9]). In order to keep this work self-contained, a statement of the main result in [12] is included in the appendix.

2. Basic definitions and notation

We consider the stability of differential inclusions of the type

$$\dot{x}(t) \in F(x(t)). \tag{2.1}$$

We suppose that the state $x(\cdot)$ evolves in $\mathbb{R}^n$, and the set-valued function F is locally Lipschitz (precise definition is given below) and takes values which are nonempty compact subsets of $\mathbb{R}^n$. In addition to the dynamics F, we suppose that two continuous functions $h : \mathbb{R}^n \to \mathbb{R}^{p_y}$ and $k : \mathbb{R}^n \to \mathbb{R}^{p_w}$ are given, which will be referred to as the outputs of the system. More precisely, we will call $y(t) := h(x(t))$ the *error* signal for the system, while $w(t) := k(x(t))$ will be called the *measurement* signal.

For any positive integer m, the Euclidean norm in the space $\mathbb{R}^m$ is denoted simply by $|\cdot|$. For each $\xi \in \mathbb{R}^m$ and $\mathcal{A} \subseteq \mathbb{R}^m$, the point to set distance will be denoted by

$$d(\xi, \mathcal{A}) := \inf\{|\xi - \eta| : \eta \in \mathcal{A}\}.$$

We will study stability notions with respect to nonempty subsets $\mathcal{A}$ in the state space $\mathbb{R}^n$. For each such $\mathcal{A}$, we will use the notation

$$|\xi|_{\mathcal{A}} := d(\xi, \mathcal{A}).$$

(Therefore, for the special case $\mathcal{A} = \{0\}$, $|\xi|_{\mathcal{A}} = |\xi|$.) For each $\varepsilon > 0$ and each nonempty $\mathcal{A}$, we set

$$B(\mathcal{A}, \varepsilon) := \{\xi : |\xi|_{\mathcal{A}} < \varepsilon\}, \quad \overline{B}(\mathcal{A}, \varepsilon) := \{\xi : |\xi|_{\mathcal{A}} \leq \varepsilon\}.$$

In the case of a singleton $\mathcal{A} = \{\eta\}$, we write

$$B(\eta, \varepsilon) := B(\{\eta\}, \varepsilon), \quad \overline{B}(\eta, \varepsilon) := \overline{B}(\{\eta\}, \varepsilon).$$

The complement of a set Ω will be denoted by Ω^C. Given a set $A \subseteq \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$, we will use the abbreviation $cA := \{c\xi : \xi \in A\}$.

For any positive integer m, the supremum norm of a continuous function $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ over an interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$ will be denoted by $\|z\|_{\mathcal{I}}$, i.e.,

$$\|z\|_{\mathcal{I}} = \sup_{t \in \mathcal{I}} |z(t)|.$$

Since the output mappings h and k are assumed to be continuous on $\mathbb{R}^n$, they are uniformly continuous on each bounded subset of $\mathbb{R}^n$. On each bounded C, we may choose a modulus of continuity $\omega_C^h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ of h; this is a function which satisfies the following property: for all $\delta > 0$, if x_1, $x_2 \in C$ and

$$|x_1 - x_2| \leq \omega_C^h(\delta),$$

then

$$|h(x_1) - h(x_2)| \leq \delta.$$

The corresponding modulus of continuity for k will be denoted by ω_C^k.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *of class* $\mathcal{K}$ (or a "$\mathcal{K}$-function") if it is continuous, positive definite, and strictly increasing. A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{L}$ if it is continuous, decreasing, and tends to zero as its argument tends to $+\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is of class $\mathcal{L}$.

In addition to system (2.1), we will also consider solutions of its relaxation

$$\dot{x}(t) \in \operatorname{co} F(x(t)), \tag{2.2}$$

where co denotes the convex hull.

2.1. **Differential inclusions.** We recall some standard definitions and results.

Definition 2.1. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. A set-valued mapping F is said to be *locally Lipschitz* on $\mathcal{O}$ if for each $\xi \in \mathcal{O}$, there exists a neighborhood $U \subset \mathcal{O}$ of ξ and $L > 0$ such that for any η and ζ in U,

$$F(\eta) \subseteq F(\zeta) + L|\eta - \zeta|\overline{B}(0, 1).$$

An immediate consequence of the definition of a locally Lipschitz set-valued mapping is as follows.

Lemma 2.2. *Suppose that the set-valued mapping F is locally Lipschitz on an open subset $\mathcal{O}$ of $\mathbb{R}^n$. Then for any compact set $K \subset \mathcal{O}$, there exists some L_K such that for any η, ζ in K,*

$$F(\eta) \subseteq F(\zeta) + L_K|\eta - \zeta|\overline{B}(0,1).$$

The following elementary remark will be needed.

Lemma 2.3. *Suppose that the set-valued function F is locally Lipschitz with nonempty compact values. For any compact set $C \in \mathbb{R}^n$, the set $F(C) := \bigcup_{x \in C} F(x)$ is bounded.*

Proof. Let a compact set C be given. Chhose, for each $x \in C$, a bounded open neighborhood U_x of x and a Lipschitz constant L_x for F on U_x. Since C is compact, we may choose a finite sequence $x_1, x_2, \dots, x_n$ so that the sets $U_{x_1}, U_{x_2}, \dots U_{x_n}$ cover C. Let

$$L := \max\{L_{x_1}, L_{x_2}, \dots, L_{x_n}\}, \quad r := \max_{i=1,\dots,n} \sup\{|x_i - y| : y \in U_i\}.$$

Then

$$F(C) \subseteq \bigcup_{i=1}^{n} F(x_i) + Lr\overline{B}(0,1).$$

Hence $F(C)$ is bounded. □

Definition 2.4. Let $T > 0$. A function $x : [0,T) \to \mathbb{R}^n$ is said to be a *solution of the differential inclusion* (2.1) if it is absolutely continuous and satisfies

$$\dot{x}(t) \in F(x(t))$$

for almost every $t \in [0,T)$. A solution $x : [0,T) \to \mathbb{R}^n$ is called a *maximal solution of the differential inclusion* (2.1) if it does not have an extension which is a solution in $\mathbb{R}^n$. That is, either $T = \infty$ or there does not exist a solution $y : [0,T_+) \to \mathbb{R}^n$ with $T_+ > T$ such that $y(t) = x(t)$ for all $t \in [0,T)$.

For each $C \subseteq \mathbb{R}^n$, let $\mathbf{S}(C)$ denote the set of maximal solutions of (2.1) satisfying $x(0) \in C$. If C is a singleton $\{\xi\}$, we will use the abbreviation $\mathbf{S}(\xi)$. We set $\mathbf{S} := \mathbf{S}(\mathbb{R}^n)$, the set of all maximal solutions. The domain of a maximal solution $x(\cdot)$ will be denoted by $[0, T^{\max}_{x(\cdot)})$. In the sequel, for each $x(\cdot) \in \mathbf{S}$ we denote the output signals as

$$y(t) = h(x(t)), \quad w(t) = k(x(t)),$$

defined for all $t \in [0, T^{\max}_{x(\cdot)})$.

Definition 2.5. The differential inclusion (2.1) is said to be *forward complete* on $\mathbb{R}^n$ if all solutions $x(\cdot) \in \mathbf{S}$ are defined for all $t \geq 0$.

Definition 2.6. The differential inclusion (2.1) is said to satisfy the *unbounded observability property* (*through* w) (denoted UO) if each $x(\cdot) \in \mathbf{S}$ such that $T^{\max}_{x(\cdot)} < \infty$ satisfies

$$\limsup_{t \to T^{\max}_{x(\cdot)}} |w(t)| = \infty.$$

Of course, a forward complete system is necessarily unbounded observable. A system satisfies the UO property if its measurement output w provides a "sufficient" information about any finite-time explosions of the state.

For the differential inclusion (2.1), we will use the notation $\mathcal{R}^T_{\mathcal{W}}(C)$ for the reachable set in time up to T starting in a compact set $C \subseteq \mathbb{R}^n$ and with measurements in some set $\mathcal{W} \subseteq \mathbb{R}^{p_w}$. That is, for each $T > 0$, $C \subseteq \mathbb{R}^n$ compact, and $\mathcal{W} \subseteq \mathbb{R}^{p_w}$,

$$\mathcal{R}^T_{\mathcal{W}}(C) := \Big\{ \eta \in \mathbb{R}^n : \eta = x(t) \text{ for some } x(\cdot) \in \mathbf{S}(C),$$
$$t \in [0, \min\{T^{\max}_{x(\cdot)}, T\}], \text{ so that } k(x(s)) \in \mathcal{W}\ \forall s \in [0,t] \Big\}.$$

We write $\mathcal{R}^T_{\mathcal{W}}(p)$ for a singleton $C = \{p\}$. We will use the notation $\widehat{\mathcal{R}}^T_{\mathcal{W}}(C)$ for the corresponding reachable set for the convexified system (2.2).

2.2. **Basic results on differential inclusions.** Before presenting our main result, we state a few technical lemmas which will be used in the proof. Most of them are not stated in their full generality (the regularity assumptions on F can be relaxed); we give the results in the form which will be needed later.

Lemma 2.7. *Suppose that a system as in* (2.1) *is given, where F is locally Lipschitz, and suppose that $F(\xi) = \{0\}$ for some $\xi \in \mathbb{R}^n$. Then $\mathbf{S}(\xi)$ consist of the single trajectory which is constantly equal to ξ.*

Proof. Suppose that $F(\xi) = \{0\}$. Since F is locally Lipschitz, there exist some neighborhood U_ξ of ξ and some $L > 0$ such that

$$F(\eta) \subseteq F(\xi) + L|\eta - \xi|\overline{B}(0,1) = L|\eta - \xi|\overline{B}(0,1) \quad \forall\, \eta \in U_\xi\,.$$

Let $x(\cdot) \in \mathbf{S}(\xi)$ and $x(t) \not\equiv \xi$. Then there exist some $r > 0$ and some t such that $|x(t) - \xi| \geq r$. Without loss of generality, we assume that $B(\xi, r) \subseteq U_\xi$. Let

$$t_1 = \min\{t \geq 0 : \ |x(t) - \xi| \geq r\}.$$

For $t \in [0, t_1]$, since $\dot{x}(t) \in F(x(t))$ almost everywhere and $x(t) \in U_\xi$, it follows that

$$\dot{x}(t) \in F(x(t)) \subseteq L|x(t) - \xi|\overline{B}(0,1),$$

i.e., $|\dot{x}(t)| \leq L|x(t) - \xi|$ for almost all t. Hence, for $t \in [0, t_1)$,

$$|x(t) - \xi| \leq \int_0^t |\dot{x}(s)|\, ds \leq \int_0^t L|x(s) - \xi|\, ds.$$

By the Gronwall inequality, $|x(t) - \xi| = 0$ for all $t \in [0, t_1)$. This contradicts the assumption that $|x(t_1) - \xi| \geq r$ and, therefore, $x(t) \equiv \xi$. □

The next result follows from [9, Sec. 7, Theorem 3].

Lemma 2.8. *Suppose that the set-valued mapping F has nonempty compact convex values and is locally Lipschitz on $\mathbb{R}^n$. Suppose further that a compact set $C \subset \mathbb{R}^n$ and $T > 0$ are such that all solutions $x(\cdot) \in \mathbf{S}(C)$ are defined on $[0, T]$. Then the reachable set up to time T starting in C (i.e., $\mathcal{R}^T_{\mathbb{R}^{p_w}}(C)$) is bounded.*

The following lemma generalizes the previous result. The proof (which is postponed to the appendix) is a minor extension of the proof of [3, Lemma 2.2]. This statement also provides a minor generalization of Lemma 2.1 from [3].

Lemma 2.9. *The following statements are equivalent.*

1. *System* (2.1) *satisfies the unboundedn observability property.*
2. *There exist $\mathcal{K}$-functions χ_1, χ_2, and χ_3 and a constant c such that*

$$|x(t)| \leq \chi_1(t) + \chi_2(|x(0)|) + \chi_3\left(\sup_{s\in[0,t]} |w(s)|\right) + c \tag{2.3}$$

   for all solutions $x(\cdot)$ of system (2.1) *and each $t \in [0, T^{\max}_{x(\cdot)})$.*
3. *The convexified system* (2.2) *satisfies the unbounded observability property.*
4. *There exist $\mathcal{K}$-functions χ_1, χ_2, and χ_3 and a constant c such that the bound* (2.3) *holds for all solutions $x(\cdot)$ of the convexified system* (2.2) *and each $t \in [0, T^{\max}_{x(\cdot)})$.*

The following lemma is an immediate consequence.

Lemma 2.10. *Suppose that system* (2.1) *satisfies the unbounded observability property with respect to the measurement output $w = k(x)$. Then for each $T > 0$, each bounded set $C \subset \mathbb{R}^n$, and each bounded set $\mathcal{W} \subset \mathbb{R}^{p_w}$, the reachable set $\mathcal{R}^T_{\mathcal{W}}(C)$ is bounded.*

The following generalization of Gronwall's lemma will be used below. This is a special case of [8, Lemma 8.3].

Lemma 2.11. *Suppose that the set-valued mapping G defined on $\mathbb{R}^n$ has closed nonempty values and is* globally *Lipschitz with constant L. Let $T > 0$ be given. Then for any solution $x(\cdot)$ of*

$$\dot{x} \in G(x)$$

defined for $t \in [0, T]$ *and any* $p \in \mathbb{R}^n$*, there exists a solution* $z_p(\cdot)$ *of* (2.1) *on* $[0, T]$ *which has* $z_p(0) = p$ *and satisfies*

$$|x(t) - z_p(t)| \leq |x(0) - p|e^{Lt} \qquad \forall t \in [0, T].$$

We next make use of this statement to derive a Gronwall-type result for UO systems which satisfy an output constraint.

Lemma 2.12. *Suppose that system* (2.1) *satisfies the unbounded observability property. Let bounded sets* $C \subset \mathbb{R}^n$ *and* $\mathcal{W} \subset \mathbb{R}^{p_w}$ *and a time* $T > 0$ *be given. Then there exists* $L > 0$ *such that for any solution* $x(\cdot)$ *of* (2.1) *defined on* $[0, T]$ *which satisfies* $x(0) \in C$ *and has* $k(x(t)) \in \mathcal{W}$ *for all* $t \in [0, T]$ *and any* $p \in C$*, there exists an absolutely continuous function* $z_p(\cdot) : [0, T] \to \mathbb{R}^n$ *which satisfies the following properties*:

1. $z_p(0) = p$;
2. *the restriction of* $z_p(\cdot)$ *to the interval* $[0, \min\{\bar{t}, T\}]$ *is a solution of* (2.1)*, where* $\bar{t} := \inf\{t \geq 0 : k(z_p(t)) \notin \mathcal{W}\}$;
3. $|x(t) - z_p(t)| \leq |x(0) - p|e^{Lt}$ *for all* $t \in [0, T]$.

Proof. Choose $m \geq 0$ so that the ball $B(0, m) \subset \mathbb{R}^{p_w}$ contains the set $\mathcal{W}$. Take a smooth function $\varphi_1 : \mathbb{R} \to [0, 1]$ so that

$$\varphi_1(s) = \begin{cases} 1 & \text{if } s \leq m, \\ 0 & \text{if } s \geq m + 1. \end{cases}$$

Introduce the auxiliary system

$$\dot{x} \in \varphi_1(|k(x)|)F(x), \tag{2.4}$$

which is forward complete, by Lemma 7.1. Note that the set-valued mapping $x \mapsto \varphi_1(|k(x)|)F(x)$ is locally Lipschitz with nonempty compact values. By [12, Corollary 3.4], the convexification of (2.4) is also forward complete and, therefore, by Lemma 2.8, the reachable set $\widetilde{\mathcal{R}}^T_{\mathbb{R}^{p_w}}(C)$ for the convexified system is bounded. Let $\mathcal{R} := \widetilde{\mathcal{R}}^T_{\mathbb{R}^{p_w}}(C)$. Take a smooth function $\varphi_2 : \mathbb{R}^n \to [0, 1]$ such that

$$\varphi_2(x) = \begin{cases} 1 & \text{if } x \in \mathcal{R}, \\ 0 & \text{if } x \notin B(\mathcal{R}, 1). \end{cases}$$

Introduce the additional auxiliary system

$$\dot{x} \in \varphi_2(x)\varphi_1(|k(x)|)F(x). \tag{2.5}$$

Since $\mathcal{R}$ is bounded, the set-valued mapping $x \mapsto \varphi_2(x)\varphi_1(|k(x)|)F(x)$ is globally Lipschitz, say, with constant L. Then by Lemma 2.11, there exists a solution $z_p(\cdot)$ of (2.5) defined on $[0, T]$ which has $z_p(0) = p$ and satisfies

$$|x(t) - z_p(t)| \leq |x(0) - p|e^{Lt} \qquad \forall t \in [0, T]. \tag{2.6}$$

All that remains to show is that $z_p(\cdot)$ is a solution of (2.1) on the interval $[0, \min\{\bar{t}, T\}]$. We first note that $z_p(\cdot)$ is a solution of (2.4), since $\varphi \equiv 1$ on the reachable set $\mathcal{R}$. Moreover, for all $t \leq \bar{t}$,

$$k(z_p(t)) \in \mathcal{W} \subseteq B(0, m)$$

and, therefore, $\varphi_1(|k(z_p(t))|) = 1$. We conclude that the dynamics of systems (2.4) and (2.1) agree along the trajectory $z_p(\cdot)$ on the interval $[0, \min\{\bar{t}, T\}]$ and, therefore, the restriction of $z_p(\cdot)$ to that interval is a solution of (2.1). □

The next statement follows from [9, Sec. 7, Corollary 1]. The result of [9] is stated for solutions defined on finite intervals, but the extension to infinite intervals is immediate.

Lemma 2.13. *Suppose that the set-valued mapping G has nonempty compact convex values and is locally Lipschitz on $\mathbb{R}^n$. Then the limit $z(\cdot)$ of any sequence of solutions of $\dot{z} \in G(z)$ which converges uniformly on compact time intervals is itself a solution.*

3. Stability notions and main results

We define notions of attractivity, stability, and asymptotic stability for the differential inclusion (2.1). The definitions are all made with respect to a given *compact* set $\mathcal{A}$. The notions of attractivity described below are weak attractivity notions (i.e., conditions on inf rather than lim sup). The definitions are given for stability of the error signal y with respect to the magnitude of the measurement w, and, therefore, are described as notions of *detectability* rather than stability. To emphasize that the results to be given are of interest even for $y = x$, $w = 0$, and $\mathcal{A} = \{0\}$, we include explicit definitions in this case.

Definition 3.1. We say that system (2.1) is *globally error-detectable* if there exists some $\gamma \in \mathcal{K}$ such that for the mapping $z(\cdot)$ defined by

$$z(t) = \max\{|y(t)| - \gamma(|w(t)|),\ 0\}, \tag{3.1}$$

the following assertions hold:

- (Local uniform error-stability modulo measurements): for some $\sigma_1, \sigma_2 \in \mathcal{K}$ and some $\delta > 0$, the relation

$$|z(0)| < \delta \Longrightarrow |z(t)| \leq \sigma_1(|z(0)|) + \sigma_2\left(\|w\|_{[0,t]}\right) \qquad \forall\, 0 \leq t < T^{\max}_{x(\cdot)}$$

  holds for all $x(\cdot) \in \mathbf{S}$.
- (Global weak error-attractivity modulo measurements):

$$\inf_{0 \leq t < T^{\max}_{x(\cdot)}} |z(t)| = 0 \tag{3.2}$$

  for all $x(\cdot) \in \mathbf{S}$. □

Definition 3.2. We say that system (2.1) is *uniformly globally error-detectable* if there exists some $\gamma \in \mathcal{K}$ such that for the mapping $z(\cdot)$ defined by (3.1), the following assertions hold:

- the local uniform error-stability modulo measurements property as in Definition 3.1;
- (Uniform global weak error-attractivity modulo measurements): for any $\varepsilon > 0$ and any $\kappa > 0$, there exists $T_{\varepsilon,\kappa}$ such that for any $\xi \in \mathbb{R}^n$ with $|\xi|_{\mathcal{A}} \leq \kappa$ and any $x(\cdot) \in \mathbf{S}(\xi)$, if $T_{\varepsilon,\kappa} < T^{\max}_{x(\cdot)}$, then there exists some $\tau < T_{\varepsilon,\kappa}$ such that

$$|z(\tau)| \leq \varepsilon.$$

Remark 3.3. The property of uniform global weak error-attractivity modulo measurements implies the following: there exists some $\gamma_1 \in \mathcal{K}$ such that for all $\varepsilon > 0$ and all $\kappa > 0$, there exists $T_{\varepsilon,\kappa}$ such that for any $\xi \in \mathbb{R}^n$ with $|\xi|_{\mathcal{A}} \leq \kappa$ and any $x(\cdot) \in \mathbf{S}(\xi)$, if $T_{\varepsilon,\kappa} < T^{\max}_{x(\cdot)}$, then

$$|y(t)| \leq \varepsilon + \gamma_1 \left(\|w\|_{[0,t)} \right)$$

for all $t \in [T_{\varepsilon,\kappa}\, ,\ T^{\max}_{x(\cdot)})$.

These definitions can be specialized to systems which lack one or both output channels, resulting in properties which describe output-stability, detectability, and stability. Since the results of this paper are new even in the simplest case (stability of the state), we will make this special case an explicit corollary of the main result. To facilitate this statement, we give the following specialized definitions.

Definition 3.4. We say that system (2.1) is *globally asymptotically stable* if the following assertions hold:

- (Local uniform stability): for some $\sigma_1 \in \mathcal{K}$ and some $\delta > 0$, the condition

$$|x(0)|_{\mathcal{A}} < \delta \Longrightarrow |x(t)| \leq \sigma_1(|x(0)|_{\mathcal{A}}) \qquad \forall\, 0 \leq t < T^{\max}_{x(\cdot)}$$

  holds for all $x(\cdot) \in \mathbf{S}$.
- (Global weak attractivity):

$$\inf_{0 \leq t < T^{\max}_{x(\cdot)}} |x(t)|_{\mathcal{A}} = 0$$

  for all $x(\cdot) \in \mathbf{S}$. □

Definition 3.5. We say that system (2.1) is *uniformly globally asymptotically stable* if the following assertions hold:

- the local uniform stability property as in Definition 3.4;

- (Uniform global weak-attractivity): for any $\varepsilon > 0$ and any $\kappa > 0$, there exists $T_{\varepsilon,\kappa}$ such that for any $\xi \in \mathbb{R}^n$ with $|\xi|_{\mathcal{A}} \leq \kappa$ and any $x(\cdot) \in \mathbf{S}(\xi)$, if $T_{\varepsilon,\kappa} < T^{\max}_{x(\cdot)}$, then there exists some $\tau < T_{\varepsilon,\kappa}$ such that

$$|x(\tau)|_{\mathcal{A}} \leq \varepsilon.$$

Remark 3.6. By [1, Lemma 4.1], the uniform global asymptotic stability property is equivalent to the following: there exists some $\beta \in \mathcal{KL}$ such that for each $\xi \in \mathbb{R}^n$ and any $x(\cdot) \in \mathbf{S}(\xi)$,

$$|x(t)|_{\mathcal{A}} \leq \beta(|\xi|_{\mathcal{A}}, t)$$

for all $t \in [0, T^{\max}_{x(\cdot)})$.

Our main result is as follows.

Theorem 1. *Let system* (2.1) *which is unbounded observable through the measurement w be given. Then this system satisfies the global weak error-attractivity modulo measurements property if and only if it satisfies the uniform global weak error-attractivity modulo measurements property.*

The following corollaries are immediate from the definitions.

Corollary 3.7. *Let system* (2.1) *which is unbounded observable through the measurement w be given. Then this system is globally error-detectable if and only if it is uniformly globally error-detectable.*

Applying Corollary 3.7 to the case where $y = x$, $w = 0$, and $\mathcal{A} = \{0\}$, we obtain the following corollary.

Corollary 3.8. *System* (2.1) *is globally asymptotically stable if and only if it is uniformly globally asymptotically stable.*

Remark 3.9. Note that the uniform detectability and stability properties extend directly from the original to the convexified system (by a standard limiting argument). Thus these statements also yield *relaxation results*, e.g., system (2.1) is globally asymptotically stable if and only if its convexification (2.2) is globally asymptotically stable. Such statements can also be applied to control systems employing *relaxed controls* (see [18]), where the relaxation amounts to the convexification of a differential inclusion.

4. Uniform reachability times

Given system (2.1), a solution $x(\cdot) \in \mathbf{S}$, and a subset $S \subset \mathbb{R}^n$ of the state space, we denote the "first crossing time" of $x(\cdot)$ into S as

$$\tau(x(\cdot), S) := \inf\{t \geq 0 : x(t) \in S\}$$

with the convention that $\tau(x(\cdot), S) = \infty$ if $x(t) \notin S$ for all $t \in [0, T^{\max}_{x(\cdot)})$.

Also, for a solution $x(\cdot) \in \mathbf{S}$ and a subset $S^o \subset \mathbb{R}^{p_y}$ of the error-output space, we denote the "first crossing time" for the output as

$$\tau^o(x(\cdot), S^o) := \inf\{t \geq 0 : y(t) \in S^o\}$$

with the convention that $\tau^o(x(\cdot), S^o) = \infty$ if $y(t) \notin S^o$ for all $t \in [0, T^{\max}_{x(\cdot)})$.

The following assertion is a generalization of the main lemma of [15].

Lemma 4.1. *Let an unbounded observable system of the form* (2.1) *be given. Assume that the following objects are given*:

- *an open subset* Ω_0 *of the state space* $\mathbb{R}^n$;
- *a continuous nonincreasing function* $r_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$;
- *a point* $p \in \mathbb{R}^n$ *of the state space*;
- *a neighborhood* $\mathcal{N}$ *of* p;
- *a compact subset* $\mathcal{W} \subset \mathbb{R}^{p_w}$ *of the measurement-output space*

such that

$$\sup\Big\{\tau(x(\cdot), \Omega_0) : x(\cdot) \in \mathbf{S}(p) \text{ so that } k(x(s)) \in \mathcal{W} \ \forall s \in [0, \tau(x(\cdot), \Omega_0)]\Big\} = +\infty.$$

Then there exist some point $q \in \mathcal{N}$ *and some* $x(\cdot) \in \mathbf{S}(q)$ *such that* $T^{\max}_{x(\cdot)} = \infty$ *and*

$$d(x(t), \Omega_0^C) \leq r_0(t) \quad \forall t \geq 0, \tag{4.1}$$

$$d(k(x(t)), \mathcal{W}) \leq r_0(t) \quad \forall t \geq 0. \tag{4.2}$$

Proof. From Lemma 2.10, we have that for each $t > 0$, the reachable set of constrained trajectories $\mathcal{R}^t_{\mathcal{W}}(p)$ is bounded. Combining this lemma with Corollary 2.9, we can make the same statement about the reachable sets $\widehat{\mathcal{R}}^t_{\mathcal{W}}(p)$ for the convexified system (2.2). Choose a continuous function $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ which satisifes $B(p, r(0)) \subseteq \mathcal{N}$ and

$$r(t) \leq \min\Big\{r_0(t), \omega^k_{B(\widehat{\mathcal{R}}^t_{\mathcal{W}}(p), r_0(0))}(r_0(t))\Big\} \qquad \forall t \geq 0.$$

(Recall that, for each bounded $C \subseteq \mathbb{R}^n$, $\omega^k_C(\cdot)$ denotes the modulus of continuity of k over the set C.)

Let $\Omega := \Omega_0 \cup k^{-1}(\mathcal{W}^C)$. Then, since k is continuous and $\mathcal{W}$ is closed, the set Ω is open. By the assumption, for each integer $j \geq 1$ we can choose some $x_j(\cdot) \in \mathbf{S}(p)$ so that $x_j(t)$ is defined for all $t \in [0, j]$, $k(x_j(t)) \in \mathcal{W}$ for all $t \in [0, j]$, and $x_j(t) \notin \Omega_0$ for all $t \in [0, j]$. That is, $x_j(t) \notin \Omega$ for all $0 \leq t \leq j$.

Choose a compact set D_1 containing $\mathcal{R}^1_{\mathcal{W}}(p)$. Then $x_j(t) \in D_1$ for all $t \in [0, 1]$, for all $j \geq 1$. Consider the restrictions of the functions $x_j(\cdot)$ to the interval $[0, 1]$. By Lemma 2.3, the set

$$F(D_1) = \bigcup_{\xi \in D_1} F(\xi)$$

is bounded. Let $M = \sup\{|\zeta| : \zeta \in F(D_1)\}$. Then

$$\left|\frac{d}{dt}x_j(t)\right| \leq M$$

for each j and almost all $0 \leq t \leq 1$. Thus the sequence $\{x_j(t)\}_{j\geq 1}$ is uniformly bounded and equicontinuous on $[0, 1]$ and, therefore, by the Arzela–Ascoli theorem, we can choose a subsequence $\{\sigma_1(j)\}_{j\geq 1}$ of $\{j\}_{j\geq 1}$ with the property that $\{x_{\sigma_1(j)}(\cdot)\}_{j\geq 1}$ converges to a continuous function $z_1(\cdot)$ uniformly on $[0, 1]$.

Now we consider $\{x_{\sigma_1(j)}(\cdot)\}_{j\geq 2}$ as a sequence of functions defined on $[1, 2]$. Using the same argument as above, one proves that there exists a subsequence $\{\sigma_2(j)\}_{j\geq 2}$ of $\{\sigma_1(j)\}_{j\geq 2}$ such that $\{x_{\sigma_2(j)}(\cdot)\}_{j\geq 2}$ converges uniformly to a function $z_2(\cdot)$ for $t \in [1, 2]$. Since $\{\sigma_2(j)\}$ is a subsequence of $\{\sigma_1(j)\}$, it follows that $z_2(1) = z_1(1)$.

Repeating the above procedure, one obtains by induction a subsequence $\{\sigma_{k+1}(j)\}_{j\geq k+1}$ of $\{\sigma_k(j)\}_{j\geq k}$ such that the sequence $\{x_{\sigma_{k+1}(j)}(\cdot)\}_{j\geq k+1}$ converges uniformly to a continuous function $z_{k+1}(\cdot)$ on $[k, k+1]$. Clearly, $z_k(k) = z_{k+1}(k)$ for all $k \geq 1$. Let $z(\cdot)$ be the continuous function defined by

$$z(t) = z_k(t) \qquad \forall t \in [k-1, k).$$

Then, on each interval $[0, k]$, $z(\cdot)$ is the uniform limit of the sequence $\{x_{\sigma_k(j)}(\cdot)\}_{j\geq k}$.

Consider the new subsequence corresponding to the first term in each of these subsequences, which we denote by

$$\{x_{\sigma(k)}(\cdot)\}_{k\geq 1} := \{x_{\sigma_k(k)}(\cdot)\}_{k\geq 1}.$$

This sequence converges uniformly to $z(\cdot)$ on each finite interval and satisfies $x_k(t) \notin \Omega$ for all $t \in [0, k]$ if each $\sigma_k(k) \geq k$.

Next, we note that $z(t)$ lies outside of Ω for all $t \geq 0$. Indeed, let $\bar{t} \geq 0$. Since the trajectories satisfy $x_k(t) \notin \Omega$ for $t \leq k$, it follows that $x_{\sigma(k)}(\bar{t}) \notin \Omega$ for all $k \geq \bar{t}$. Since the complement of Ω is closed, we have

$$z(\bar{t}) = \lim_{k\to\infty} x_{\sigma(k)}(\bar{t}) \notin \Omega.$$

We note that since for each k, the trajectory $x_k(\cdot)$ is a solution of (2.1) on $[0, k]$, it is also a solution of the convexified system (2.2) on $[0, k]$. Then, by Lemma 2.13, the function $z(\cdot)$ is also a solution of (2.2) on $[0, \infty)$, since it is a uniform limit of solutions on each finite interval. If $z(\cdot)$ were a trajectory of system (2.1), the result would be proved (with $q = p_0$). Of course, there is no reason for $z(\cdot)$ to be a trajectory of (2.1), but it can be well approximated by an appropriate trajectory.

Recalling that $B(p, r(0)) \subseteq \mathcal{N}$, we apply Theorem 1 of [12] (which appears as Proposition 7.2 in the appendix) to provide a solution $x(\cdot)$ of (2.1) which satisfies

$$|x(t) - z(t)| \le r(t) \le r_0(t) \qquad \forall t \ge 0$$

and $x(0) \in \mathcal{N}$. Since $z(t) \notin \Omega$ for all $t \ge 0$, we have $z(t) \notin \Omega_0$ for all $t \ge 0$, which implies (4.1).

To verify (4.2), we first note that since $z(t)$ lies outside of Ω for all $t \ge 0$, we have

$$k(z(t)) \in \mathcal{W} \qquad \forall t \ge 0. \tag{4.3}$$

Also, for each $t \ge 0$,

$$z(t) \in \widehat{\mathcal{R}}^t_{\mathcal{W}}(p),$$

and, therefore, (4.1) gives

$$x(t) \in B(\widehat{\mathcal{R}}^t_{\mathcal{W}}(p), r_0(t)) \subseteq B(\widehat{\mathcal{R}}^t_{\mathcal{W}}(p), r_0(0)) \qquad \forall t \ge 0.$$

Then, since

$$|x(t) - z(t)| \le r(t) \le \omega^k_{B(\widehat{\mathcal{R}}^t_{\mathcal{W}}(p), r_0(0))}(r_0(t)) \qquad \forall t \ge 0,$$

it follows that

$$|k(x(t)) - k(z(t))| \le r_0(t) \qquad \forall t \ge 0.$$

Together with (4.3), we conclude that (4.2) holds. □

We will make use of a corollary of this result, which is a slight variation of the contraposition.

Corollary 4.2. *Suppose that an unbounded observable system of the form* (2.1) *is given. Assume that the following objects are given*:

- *a compact subset C of the state space $\mathbb{R}^n$ and a bounded open neighborhood $\widetilde{C}$ of C*;
- *an open subset Φ of the error output space $\mathbb{R}^{p_y}$*;
- *a compact subset $J \subset \Phi \subseteq \mathbb{R}^{p_y}$*;
- *an open subset $\mathcal{W}_0$ of the measurement output space $\mathbb{R}^{p_w}$*;
- *a compact subset $\mathcal{W} \subset \mathcal{W}_0 \subseteq \mathbb{R}^{p_w}$*

such that for all $x(\cdot) \in \mathbf{S}(\widetilde{C})$, there exists $t \in [0, T^{\max}_{x(\cdot)})$ such that

$$y(t) \in J \text{ or } w(t) \notin \mathcal{W}_0. \tag{4.4}$$

Then

$$\sup\{\tau^o(x(\cdot), \Phi) : x(\cdot) \in \mathbf{S}(C) \text{ with } k(x(s)) \in \mathcal{W}\ \forall s \in [0, \tau^o(x(\cdot), \Phi)]\} < \infty$$

(*with the convention that* $\sup \emptyset = -\infty$).

Proof. Let an open set Φ_0 in $\mathbb{R}^p$, a bounded open set $\mathcal{W}_1$ in $\mathbb{R}^{p_w}$, and a number $r > 0$ be such that

$$B(J, r) \subseteq \Phi_0 \subset B(\Phi_0, r) \subseteq \Phi, \quad B(\mathcal{W}, 2r) \subset \mathcal{W}_1 \subset B(\mathcal{W}_1, r) \subset \mathcal{W}_0.$$

For any $t \ge 0$, let $R(t) := \overline{B}(\mathcal{R}^t_{B(\mathcal{W}_1, r)}(\widetilde{C}), 1)$ be a bounded set.

Claim. *For each $p \in C$, there exists $T_p \geq 0$ such that for each $x(\cdot) \in \mathbf{S}(p)$, there exists $t \in [0, T_p]$ such that either $y(t) \in \Phi_0$ or $w(t) \notin \mathcal{W}_1$.*

Proof. Assume that the claim fails for some $p \in C$. Define the open set $\Omega_0 := h^{-1}(\Phi_0) \subseteq \mathbb{R}^n$. If the claim were false, we would have

$$\sup \Big\{ \tau(x(\cdot), \Omega_0) : x(\cdot) \in \mathbf{S}(p) \quad \text{with } k(x(s)) \in \mathcal{W}_1 \ \forall s \in [0, \tau(x(\cdot), \Omega_0)] \Big\} = +\infty.$$

Applying Lemma 4.1 with Ω_0, $\mathcal{N} = \widetilde{C}$, $\mathcal{W}_1$, and any continuous, nonincreasing function $r_0(\cdot)$ such that

$$0 < r_0(t) < \frac{1}{2} \min \Big\{ \omega^h_{R(t)}(r/2), r, 1 \Big\} \qquad \forall t \geq 0,$$

we find that there exist some $q \in \widetilde{C}$ and $x(\cdot) \in \mathbf{S}(q)$ defined for all $t \geq 0$ such that

$$d\Big(x(t), [h^{-1}(\Phi_0)]^C\Big) \leq r_0(t) \qquad \forall t \geq 0$$

and

$$d(w(t), \mathcal{W}_1) \leq r_0(t) \qquad \forall t \geq 0.$$

From the latter inequality, $d(w(t), \mathcal{W}_1) \leq r$ for all $t \geq 0$. It follows that $x(t) \in \mathcal{R}^t_{B(\mathcal{W}_1,1)}(\widetilde{C})$ and $w(t) \in \mathcal{W}_0$ for all $t \geq 0$. Also, for each $t \geq 0$, there exists a point $\eta \in h^{-1}(\Phi_0)^C$ such that $|x(t) - \eta| \leq 2r_0(t)$. This gives $|x(t) - \eta| \leq \min\{1, \omega^h_{R(t)}(r/2)\}$, from which we conclude that $\eta \in R(t)$ and $|y(t) - h(\eta)| \leq r/2$. Also, since $h(\eta) \notin \Phi_0$, we have $y(t) \notin J$, which contradicts assumption (4.4). This proves the claim. □

For each $p \in C$, let L_p be the Lipschitz constant given by Lemma 2.12 for the sets C and $\mathcal{W}_1$ and the interval $[0, T_p]$. For each $p \in C$, let and

$$\mathcal{R}_p := \mathcal{R}^{T_p}_{\mathcal{W}_1}(C), \quad \delta_p := \frac{\min\{\omega^h_{\mathcal{R}_p}(r), \omega^k_{\mathcal{R}_p}(r)\}}{e^{L_p T_p}}.$$

Choose a finite cover of C by sets of the form $B(p, \delta_p)$. Let T be the largest of T_p in this cover. Now, given any $\xi \in C$ and any $x(\cdot) \in \mathbf{S}(\xi)$, choose p so that $\xi \in B(p, \delta_p)$ for one of the balls in this finite cover. Then by Lemma 2.12, there exists an absolutely continuous function $z_p : [0, T_p] \to \mathbb{R}^n$ such that $z_p(0) = p$ and

$$|x(t) - z_p(t)| \leq |x(0) - p| e^{L_p T_p} \leq \min \big\{ \omega^h_{S_p}(r), \omega^k_{S_p}(r) \big\} \quad \forall t \in [0, T_p]. \quad (4.5)$$

Moreover, $z_p(\cdot)$ is a solution of system (2.1) on the interval $[0, \min\{\bar{t}, T_p\}]$, where $\bar{t} = \min\{t \geq 0 : k(z_p(t)) \notin \mathcal{W}_1\}$.

Consider the following two cases.

Case 1. If $\bar{t} > T_p$, then $z_p(\cdot)$ is a solution of (2.1) on the interval $[0, T_p]$ and, therefore, by the claim, there exists some $t_0 \in [0, T_p]$ such that either

$h(z_p(t_0)) \in \Phi_0$ or $k(z_p(t_0)) \notin \mathcal{W}_1$. The latter contradicts the assumption the inequality $\bar{t} > T_p$ and, therefore, we conclude that $h(z_p(t_0)) \in \Phi_0$. Moreover, since $k(z_p(t)) \in \mathcal{W}_1$ for all $t \in [0, T_p]$, we have that both $x(t_0)$ and $z_p(t_0)$ lie in $\mathcal{R}_p$. From (4.5) we have

$$\left|h(x(t_0)) - h(z_p(t_0))\right| \leq r.$$

Since $B(\Phi_0, r) \subseteq \Phi$, we conclude that $h(x(t_0)) \in \Phi$.

Case 2. If $\bar{t} \leq T_p$, let $t_1 := \min\{t \geq 0 : k(z_p(t)) \notin B(\mathcal{W}, r)\}$. Note that $t_1 < \bar{t}$. Since both $x(t_1)$ and $z_p(t_1)$ lie in $\mathcal{R}_p$, we conclude from (4.5) that

$$\left|k(x(t_1)) - k(z_p(t_1))\right| \leq r,$$

and, therefore, $x(t_1) \notin \mathcal{W}$. Since this holds for each $\xi \in C$ and $x(\cdot) \in \mathbf{S}(C)$, we conclude that

$$\sup\Big\{\tau^o(x(\cdot), \Phi) : x(\cdot) \in \mathbf{S}(C) \quad \text{with } k(x(s)) \in \mathcal{W}\ \forall s \in [0, \tau^o(x(\cdot), \Phi)]\Big\} \leq T.$$

The proof is complete. □

We mention the following assertion on the boundedness of reachable sets.

Corollary 4.3. *Let the assumptions of Corollary* 4.2 *hold. If, in addition, there exists a closed set* Ψ *in the error-output space* $\mathbb{R}^{p_y}$ *such that* $h^{-1}(\Phi) \subset h^{-1}(\Psi) \subseteq C$, *then there exists some* $T \geq 0$ *such that*

$$\bigcup_{t\geq 0} \mathcal{R}^t_{\mathcal{W}}(C) = \mathcal{R}^T_{\mathcal{W}}(C),$$

and, in particular, $\bigcup_{t\geq 0} \mathcal{R}^t_{\mathcal{W}}(C)$ *is bounded.*

Proof. Let

$$T = \sup\Big\{\tau^o(x(\cdot), \Phi) \mid x(\cdot) \in \mathbf{S}(C) \quad \text{with } \ k(x(s)) \in \mathcal{W}\ \forall s \in [0, \tau^o(x(\cdot), \Phi)]\Big\}.$$

Choose any $\eta \in \bigcup_{t\geq 0} \mathcal{R}^t_{\mathcal{W}}(C)$. Then $\eta = x(t_0)$ for some $\xi \in C$, $x(\cdot) \in \mathbf{S}(\xi)$, and $t_0 \geq 0$ such that $k(x(t)) \in \mathcal{W}$ for all $t \in [0, t_0]$. Let

$$t_1 = \max\{t \leq t_0 : h(x(t)) \in \Psi\}$$

and $p := x(t_1)$. Note that $h(p) \in \Psi$. For each $t \geq 0$, we define $\widehat{x}(t) := x(t + t_1)$ and note that $\widehat{x}(\cdot) \in \mathbf{S}(p)$. By the definition of t_1, $h(\widehat{x}(t)) \notin \Psi$ for all $t \in [0, t_0 - t_1]$ and, therefore, $h(\widehat{x}(t)) \notin \Phi$ for all $t \in [0, t_0 - t_1]$. Moreover, since $\widehat{x}(t) = x(t + t_1)$ for all $t \in [0, t_0 - t_1]$, we have $k(\widehat{x}(t)) \in \mathcal{W}$ for all $t \in [0, t_0 - t_1]$. Since $p \in h^{-1}(\Psi) \subseteq C$, we conclude by the definition of T that $t_0 - t_1 < T$. Thus, $\eta = \widehat{x}(t_0 - t_1) \in \mathcal{R}^T_{\mathcal{W}}(C)$. □

5. Proof of Theorem 1

Here we make use of Corollary 4.2. We need not make use of the output constraint conditions allowed in the corollary (i.e., it will be applied with $k \equiv 0$, $\mathcal{W} = \{0\}$).

Proof of Theorem 1. One implication of Theorem 1 is immediate. Suppose that the system satisfies the global weak error-attractivity modulo measurements property with $\mathcal{K}$-function γ as in Definition 3.1. To prove Theorem 1, we need to verify the following: for $z(\cdot)$ defined as in (3.1),

$$\begin{aligned} &\forall \varepsilon > 0,\ \forall \kappa > 0\ \exists T_{\varepsilon,\kappa} > 0 \text{ such that } \forall x(\cdot) \in \mathbf{S}(\xi) \\ &\text{with } |\xi|_{\mathcal{A}} \le \kappa,\ T_{\varepsilon,\kappa} < T^{\max}_{x(\cdot)} \ \Rightarrow\ \exists \tau \le T_{\varepsilon,\kappa} \text{ such that } |z(\tau)| < \varepsilon. \end{aligned} \tag{5.1}$$

For each $r \ge 0$, let $D_r = \{\xi :\ |h(\xi)| - \gamma(|k(\xi)|) \ge r\}$.

Let $\varepsilon > 0$ be given. If $D_\varepsilon = \emptyset$, then $T_{\varepsilon,\kappa}$ in (5.1) can be chosen to be 0. Suppose that $D_\varepsilon \ne \emptyset$. Consider two sets

$$\begin{aligned} W_0 &:= \{\xi :\ |h(\xi)| - \gamma(|k(\xi)|) > \varepsilon/4\}, \\ W_1 &:= D_{\varepsilon/2} = \{\xi : |h(\xi)| - \gamma(|k(\xi)|) \ge \varepsilon/2\}. \end{aligned}$$

Clearly, W_0 is open, W_1 is closed, and $W_1 \subseteq W_0$. Therefore, there exists a smooth function $\varphi : \mathbb{R}^n \to [0,1]$ such that $\varphi(\xi) = 1$ for all $\xi \in W_1$ and $\varphi(\xi) = 0$ for all $\xi \notin W_0$. Consider the differential inclusion

$$\dot{x} \in G(x) := \varphi(x)F(x). \tag{5.2}$$

Claim. *Differential inclusion* (5.2) *is forward complete.*

Proof. Let V_1 be the zero set of φ, i.e., $V_1 = \{\xi : \varphi(\xi) = 0\}$. By Lemma 2.7, for any $\xi \in V_1$, $\widetilde{\mathbf{S}}(\xi)$ consists of only the equilibrium solution, and, therefore, each $\widetilde{x}(\cdot) \in \widetilde{\mathbf{S}}(V_1)$ has a maximal interval of existence $[0,\infty)$. □

Let $\xi \in V_1^C$. Choose any $\widetilde{x}(\cdot) \in \widetilde{\mathbf{S}}(\xi)$ defined on some maximal interval $[0,T)$. Below we show that $T = \infty$.

Suppose that $T < \infty$. Then $\widetilde{x}(t) \notin V_1$ for all $t \in [0,T)$ and $|\widetilde{x}(t)| \to \infty$ as $t \to T$ (applying [16, Lemma 2] to the convexification of (5.2)). Consider the initial value problem

$$\dot{\tau} = \frac{1}{\varphi(\widetilde{x}(\tau))}, \quad \tau(0) = 0, \tag{5.3}$$

defined on some maximal interval $[0,t_1)$. Then $\widetilde{x}(\tau(t))$ is a solution of (2.1) with initial state ξ, i.e., $\widetilde{x}(\tau(t)) = x(t)$ on $[0,t_1)$ for some $x(\cdot) \in \mathbf{S}(\xi)$. Consider two cases.

Case 1. $t_1 = \infty$. Then $\widetilde{x}(\tau(t)) \in \mathbf{S}(\xi)$. Hence, by (3.2) there exists some t_2 such that $\widetilde{x}(\tau(t_2) \notin W_0$ and, therefore, $\widetilde{x}(\tau(t_2)) \in V_1$, a contradiction. Thus, it is impossible to have $t_1 = \infty$.

Case 2. $t_1 < \infty$. In this case, $\tau(t) \to T$ as $t \to t_1$ (note that $\tau(\cdot)$ is increasing). Since $x(t) = x(\tau(t)) \in V_1^C \subseteq D_{\varepsilon/4}$ for all $t \in [0,t_1)$, it follows that $T^{\max}_{x(\cdot)} > t_1$. Let $p = x(t_1)$. Then $\widetilde{x}(\tau) \to p$ as $\tau \to T$. This contradicts the fact that $|\widetilde{x}(t)| \to \infty$ as $t \to T$.

Hence, we have proved that $T = \infty$.

Now, for each ξ, let $\widetilde{\mathbf{S}}(\xi)$ denote the set of maximal solutions of (5.2) with initial state ξ, and let $\widetilde{x}(\cdot)$ denote a maximal solution of (5.2). Let $\kappa > 0$ be given, and let $C = B(\mathcal{A}, \kappa)$. Note that for any $\xi \in C$, if $\widetilde{x}(s) \in D_{\varepsilon/2}$ for all $0 \le s \le t$, then $\widetilde{x}(s) = x(s)$ for all $s \in [0, t]$ for some $x(\cdot) \in \mathbf{S}(\xi)$. Combining this with (3.2), we see that for any $\xi \in C$ and any maximal solution $\widetilde{x}(\cdot) \in \widetilde{\mathbf{S}}(\xi)$, there exists some $\tau > 0$ such that $\widetilde{x}(\tau)$ exists and $\widetilde{x}(\tau) \notin D_{\varepsilon/2}$, i.e.,

$$0 \le \widetilde{z}(\tau) < \varepsilon/2,$$

where

$$\widetilde{z}(t) = \max\{|h(\widetilde{x}(t))| - \gamma(|k(\widetilde{x}(t))|), 0\}$$

for $t > 0$.

Applying Corollary 4.2 to inclusion (5.2) with $C = B(\mathcal{A}, \kappa)$, $J = [0, \varepsilon/2]$, $\Phi = (-\infty, \varepsilon)$, and with y and w from Corollary 4.2 replaced by $\widetilde{z}$ and 0, respectively, we see that there exists some $\widetilde{T}(\varepsilon, \kappa) > 0$ such that for any $\xi \in C$ and any $\widetilde{x}(\cdot) \in \widetilde{\mathbf{S}}$, there exists some $s \le \widetilde{T}(\varepsilon, \kappa)$ such that $\widetilde{z}(s) < \varepsilon$.

We now prove (5.1) with $T_{\varepsilon,\kappa} := \widetilde{T}(\varepsilon, \kappa)$ for any given ε and κ. Consider $\xi \in B(\mathcal{A}, \kappa)$ and choose any $x(\cdot) \in \mathbf{S}(\xi)$ for which $T_{\varepsilon,\kappa} < T^{\max}_{x(\cdot)}$.

Suppose, on the contrary, that (5.1) fails, i.e., $x(t) \in D_\varepsilon \subseteq W_1$ for all $t \in \big[0, T_{\varepsilon,\kappa}\big]$. Then $x(t) = \widetilde{x}(t)$ for some $\widetilde{x}(\cdot) \in \widetilde{\mathbf{S}}(\xi)$ on $\big[0, T_{\varepsilon,\kappa}\big]$. But then we know that there exists some $s \le T_{\varepsilon,\kappa}$ such that $\widetilde{z}(s) < \varepsilon$. Consequently, $z(s) = \widetilde{z}(s) < \varepsilon$. This contradicts the assumption that $x(t) \in D_\varepsilon$ for all $t \in \big[0, T_{\varepsilon,\kappa}\big]$. Thus, statement (5.1) is proved. □

6. Final remarks

One of our main motivations for this work is to provide a unified tool that can be applied to obtain asymptotic characterizations of the input-output-to-state stability property (see [2]) and input-to-output stability property (see [11]). In some contexts, the results of this work can be applied to systems of the following type:

$$\dot{x}(t) = f(x(t), d(t)), \tag{6.1}$$

where $d(\cdot)$ is a measurable function which takes values in a compact subset U of some Euclidean space and $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a locally Lipschitz mapping.

We associate with system (6.1) the differential inclusion

$$\dot{x}(t) \in F(x(t)) := \{f(x(t), v) : \ v \in U\}. \tag{6.2}$$

Clearly, the mapping F is compact-valued and locally Lipschitz. As above, we use $\mathbf{S}(\xi)$ to denote the set of maximal solutions of (6.2) starting from ξ.

For system (6.1), let $\widehat{\mathbf{S}}(\xi)$ denote the set of all maximal solutions starting from ξ, i.e., $z(\cdot) \in \widehat{\mathbf{S}}(\xi)$ if and only if $z(\cdot) = x(\cdot, d)$ for some input function d. To make this work self-contained, we make the following observation.

Proposition 6.1. *Suppose that U is compact. Then, for each $\xi \in \mathbb{R}^n$, $\mathbf{S}(\xi) = \widehat{\mathbf{S}}(\xi)$.*

This result follows immediately from the measurable selection principle (cf. [6, Theorem 5.3, Chap. 3]). We note that, by virtue of this result, the statements obtained in this work remain valid for systems of the type (6.1).

7. Appendix

If we are only interested in trajectories which satisfy an appropriate measurement constraint, then the UO condition provides that we can adjust the dynamics to yield a forward complete system, as we show next.

Lemma 7.1. *Suppose that system (2.1) satisfies the unbounded observability property with respect to the measurement output $w = k(x)$. For any $m > 0$, define a smooth function $\theta_m : \mathbb{R} \to [0, 1]$ which satisfies*

$$\theta_m(s) = \begin{cases} 1 & \text{if } s \le m, \\ 0 & \text{if } s \ge m+1. \end{cases}$$

Introduce the auxiliary system

$$\dot{x} \in \theta_m(|k(x)|)F(x). \tag{7.1}$$

Then system (7.1) is forward complete.

Proof. We note that the set-valued mapping $x \mapsto \theta_m(|k(x)|)F(x)$ is locally Lipschitz with nonempty compact values. Assume that system (7.1) is not complete. Then we can choose a maximal solution $z(\cdot)$ of (7.1) defined on a maximal interval $[0, S)$, where $S < \infty$. By [16, Lemma 2] (applied to the convexification),

$$|z(t)| \to \infty \quad \text{as} \quad s \nearrow S. \tag{7.2}$$

We note that $|k(z(s))| \le m + 1$ for all $s \in [0, S)$. If this were not the case, there would be some $s_0 \in [0, S)$ for which $|k(z(s_0))| > m + 1$. Then Lemma 2.7 says that this solution would satisfy $z(s_0 + \tau) = z(s_0)$ for all $\tau \ge 0$, which contradicts (7.2). Thus $|k(z(s))| \le m + 1$ for all $s \in [0, S)$, which gives $\theta_m(|k(z(s))|) > 0$ for all $s \in [0, S)$. Hence the function

$$\varphi(\tau) := \int_0^\tau \theta_m(|k(z(s))|)\, ds$$

is strictly increasing and maps $[0, S)$ onto an interval $[0, T)$ with $T \le S \le \infty$, since $\theta_m \le 1$ everywhere. We define $x(t) := z(\varphi^{-1}(t))$ for all $t \in [0, T)$.

Then x is absolutely continuous and satisfies

$$\dot{x}(t) = \frac{d}{dt} z(\varphi^{-1}(t)) = \dot{z}(\varphi^{-1}(t)) \frac{1}{\varphi'(\varphi^{-1}(t))}$$
$$= \dot{z}(\varphi^{-1}(t)) \frac{1}{\theta_m\big(\big|k(z(\varphi^{-1}(t)))\big|\big)} \in F(z(\varphi^{-1}(t))) = F(x(t))$$

for almost all $t \in [0, T)$, i.e., $x(\cdot)$ is a solution of (2.1). From (7.2) we have $x(t) \to \infty$ as $t \nearrow T$, and, therefore, $T = T^{\max}_{x(\cdot)}$. The unbounded observability condition then says that $w(t) = k(x(t))$ is unbounded on $[0, T)$. But $w(t) = k(z(s))$, where $s = \varphi^{-1}(t)$, and we have shown that $|k(z(s))| \leq m+1$ for all $s \in [0, S)$, which is a contradiction. We conclude that system (7.1) is forward complete. ☐

Next, we give the proof of Lemma 2.9.

Proof. Clearly, 3 implies 1 and 4 implies 2.

To show that 2 implies 3, suppose that there exist $\mathcal{K}$-functions χ_1, χ_2, and χ_3 and a constant c such that (2.3) holds for all $x(\cdot) \in \mathbf{S}$ and each $t \in [0, T^{\max}_{x(\cdot)})$. A standard limiting argument using the Filippov–Ważewski relaxation theorem shows that the same bound holds for all solutions of the convexified system (since $k(\cdot)$ is continuous, it is uniformly continuous in a neighborhood of each compact trajectory). Suppose that a maximal solution $x(\cdot)$ of the convexified system has $T^{\max}_{x(\cdot)} < \infty$. By [16, Lemma 2], $|x(t)| \to \infty$ as $t \nearrow T^{\max}_{x(\cdot)}$. Then bound (2.3) gives $|w(t)| \to \infty$ as $t \nearrow T^{\max}_{x(\cdot)}$.

Finally, we show that 1 implies 4. Suppose that system (2.1) is unboundedly observable through w. We verify (2.3) for all solutions of the convexification. For all nonnegative numbers m, r, and T, taking reachable sets for the convexified system (2.2), we define

$$R(T, r, m) := \widehat{\mathcal{R}}^T_{\overline{B}(0,m)}(\overline{B}(0, r)).$$

We observe that the function

$$\gamma(T, r, m) := \sup\{|\eta| : \eta \in R(T, r, m)\}$$

(which *a priori* may have infinite values) is nondecreasing separately on each of the variables T, r, and m.

Claim. *$\gamma(T, r, m) < \infty$ for all nonnegative T, r, and m.*

Proof. Let T, r, and m be given. Choose any smooth function $\theta_m : \mathbb{R} \to [0, 1]$ such that

$$\theta_m(s) = \begin{cases} 1 & \text{if } s \leq m, \\ 0 & \text{if } s \geq m + 1. \end{cases}$$

and introduce the auxiliary system

$$\dot{x} \in \theta_m(|k(x)|)F(x). \tag{7.3}$$

We note that the set-valued mapping $x \mapsto \theta_m(|k(x)|)F(x)$ is locally Lipschitz with nonempty compact values.

By Lemma 7.1, system (7.3) is forward complete. It follows from [12, Corollary 3.4] that the convexification of (7.3) is forward complete as well. Since the reachable set $R(T, r, m)$ for the convexified auxiliary system is the same as that for the convexification of the original system (since the dynamics agree whenever the measurement $w(\cdot)$ satisfies $|w| \leq m$), the claim follows from as application of Lemma 2.8. □

Finally, for each $\xi \in \mathbb{R}^n$, each solution $x(\cdot)$ of (2.2), and each $t \in [0, T^{\max}_{x(\cdot)})$, we set

$$r = |\xi|, \quad m = \sup_{s \in [0,t]} |w(s)|.$$

It follows that $x(t) \in R(t, r, m)$ and, therefore,

$$|x(t)| \leq \gamma(t, r, m) \leq \chi(t) + \chi(r) + \chi(m),$$

where $\chi(r) := \gamma(r, r, r)$. Then $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is nondecreasing and, therefore, there exist a function $\widetilde{\chi} \in \mathcal{K}$ and a constant c_0 such that $\chi(r) \leq \widetilde{\chi}(r) + c_0$ for all $r \geq 0$. The result follows with each $\chi_i = \widetilde{\chi}$ and $c = 4c_0$. □

Finally, we state the result from [12], specialized to the case treated in this paper.

Proposition 7.2. *Let system* (2.1) *be given. Fix* $\xi \in \mathbb{R}^n$ *and let* $z(\cdot)$ *be a solution of the convexified system* (2.2) *defined on some maximal interval* $T^{\max}_{z(\cdot)}$. *Let* $r : [0, T^{\max}_{z(\cdot)}) \to \mathbb{R}$ *be a continuous function satisfying* $r(t) > 0$ *for all* $t \in [0, T^{\max}_{z(\cdot)})$. *Then there exist* $\eta \in B(\xi, r(0))$ *and a solution* $x(\cdot) \in \mathbf{S}(\eta)$ *which satisfies*

$$|z(t) - x(t)| \leq r(t)$$

for all $t \in [0, T^{\max}_{z(\cdot)})$.

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