Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation

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Abstract

We study nonlinear systems with both control and disturbance inputs. The main problem addressed in the paper is design of state feedback control laws that render the closed-loop system integral-input-to-state stable (iISS) with respect to the disturbances. We introduce an appropriate concept of control Lyapunov function (iISS-CLF), whose existence leads to an explicit construction of such a control law. The same method applies to the problem of input-to-state stabilization. Converse results and techniques for generating iISS-CLFs are also discussed. © 2002 Published by Elsevier Science B.V.

1. Introduction

Since the concept of input-to-state stability (ISS) was first introduced in [21], there has been a great deal of research on the problem of designing input-to-state stabilizing control laws [7,13,12,25,30,31]. In its usual setting, this problem consists in finding a state feedback control law that makes the closed-loop system input-to-state stable with respect to external disturbances. Most of this activity has centered around the concept of ISS-control Lyapunov function (ISS-CLF). It has been shown that the knowledge of an ISS-CLF leads to explicit formulas for input-to-state stabilizing control laws (see [7] and [25,31] for two different constructions). For certain classes of systems, ISS-CLFs can be systematically generated via backstepping [13,12]. In addition, input-to-state stabilizing control laws possess desirable properties associated with inverse optimality [7,12].

In parallel with these developments, an integral variant of input-to-state stability (iISS) has been introduced and studied in [2,27]. Intuitively, while the state of an input-to-state stable system is small if inputs are small (a type of "L^∞ to L^∞ stability"), the state of an integral-input-to-state stable system is small if inputs have finite energy as defined by an appropriate integral (analogous to "L^2 to L^∞ stability"). The concept of iISS is weaker than that of ISS, in the sense that every input-to-state stable system is necessarily integral-input-to-state stable, but the converse is not true. From the viewpoint of control design for systems with disturbances, this

leads to the existence of systems that are integral-input-to-state stabilizable but not input-to-state stabilizable (an example is given below). The notion of iISS has proved to be useful in a variety of nonlinear control contexts, including control of robotic manipulators [2] and switching control of uncertain systems [9]. (One may also introduce an ISS-like notion analogous to H_∞, i.e., L^2 to L^2 stability. Interestingly, this turns out to be equivalent to plain ISS; see [27].)

This paper is concerned primarily with the problem of designing integral-input-to-state stabilizing control laws. We introduce the concept of iISS-CLF, whose existence leads to an explicit construction of an integral-input-to-state stabilizing state feedback control law. We present a unified approach for both input-to-state and integral-input-to-state stabilization, which applies to nonlinear systems that are not affine in disturbances. The developments reported here are based on characterizations of iISS obtained in [2] by David Angeli and two of the authors. The present paper is an updated version of our earlier conference paper [15], expanded and improved using ideas from the recent work of Teel and Praly [29].

In the next two sections we recall necessary definitions and review background results. In Section 4 we present explicit formulas for input-to-state and integral-input-to-state stabilizing control laws. In Section 5 we discuss our findings in the context of previous work. An illustrative example is given in Section 6. In Section 7 we show how integral-input-to-state stabilizing control laws for certain cascade systems can be recursively constructed via backstepping. Section 8 is devoted to converse results and the issue of regularity at the origin. Finally, Section 9 contains some remarks on other types of CLFs.

2. ISS and integral-ISS

A function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If in addition α is unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geqslant 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geqslant 0$.

Consider a general system of the form

$$\dot{x} = f(x, d), \tag{1}$$

where f is a locally Lipschitz function and d is a locally essentially bounded disturbance input. We recall from [21] that system (1) is called *input-to-state stable* (ISS) with respect to d if for some functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every d the corresponding solution of (1) satisfies the following estimate:

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|d_t\|) \quad \forall t \geqslant 0,$$

where $\|d_t\| := \operatorname{ess\,sup}\{|d(s)| : s \in [0, t]\}$. As shown in [26], a necessary and sufficient condition for ISS is the existence of an *ISS-Lyapunov function*, i.e., a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions α and χ we have

$$\nabla V(x) f(x, d) \leqslant -\alpha(|x|) + \chi(|d|) \quad \forall x, d. \tag{2}$$

In this paper, we are particularly interested in the integral variant of the ISS property, introduced in [27]. The system (1) is called *integral-input-to-state stable* (iISS) with respect to d if for some functions $\alpha_0, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for every initial state $x(0)$, and every d the corresponding solution of (1) satisfies the following estimate:

$$\alpha_0(|x(t)|) \leqslant \beta(|x(0)|, t) + \int_0^t \gamma(|d(s)|)\, \mathrm{d}s \quad \forall t \geqslant 0. \tag{3}$$

The result stated below summarizes equivalent characterizations of iISS obtained in [2,15].

Theorem 1. *The following statements are equivalent*:

1. *System* (1) *is iISS.*
2. *There exists an* iISS-Lyapunov *function, i.e., a positive definite radially unbounded smooth function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that* (2) *holds for some continuous positive definite function* α *and some class* $\mathcal{K}_\infty$ *function* χ.
3. *System* (1) *is* 0-GAS (*i.e., the system* $\dot{x} = f(x, 0)$ *is globally asymptotically stable*) *and* zero-output *dissipative, i.e., there exist a positive definite radially unbounded smooth function* $V : \mathbb{R}^n \to \mathbb{R}$ *and a class* $\mathcal{K}_\infty$ *function* v *such that*

$$\nabla V(x) f(x, d) \leqslant v(|d|) \quad \forall x, d.$$

4. *There exist a positive definite radially unbounded smooth function* $W : \mathbb{R}^n \to \mathbb{R}$, *two class* $\mathcal{K}_\infty$ *functions* ρ *and* γ, *and a positive definite function* b *with* $\int_0^{+\infty} 1/(1 + b(r)) \, dr = +\infty$ *such that for all* $x \neq 0$ *and all* d *we have*

$$|x| \geqslant \rho(|d|) \Rightarrow \nabla W(x) f(x, d) < \gamma(|d|) b(W(x)).$$

5. *There exist functions* W, ρ, γ, b *as in* 4 *and a positive definite function* α *such that for all* x *and* d *we have*

$$|x| \geqslant \rho(|d|) \Rightarrow \nabla W(x) f(x, d) \leqslant -\alpha(|x|) + \gamma(|d|) b(W(x)).$$

Comparing item 2 of Theorem 1 with the characterization of ISS given by (2), where α is required to be of class $\mathcal{K}_\infty$, we see that iISS is a weaker property than ISS. This characterization of iISS in terms of iISS-Lyapunov functions will be our main tool for introducing a proper notion of a control Lyapunov function to study the integral-input-to-state stabilization problem. Item 3 will be needed in some of the proofs. Items 4 and 5 will not be used and are given here for completeness. However, there exist alternative constructions which utilize these characterizations; see [15] for details.

We remark that the control laws considered in this paper will lead to closed-loop systems that are in general just continuous at the origin (and smooth or at least locally Lipschitz everywhere else). The above necessary and sufficient conditions for iISS and ISS in terms of Lyapunov functions remain valid for such systems. The sufficiency part is not difficult to check, while necessity requires more attention; see Section 8 for a detailed discussion of this issue.

We conclude this section with a simple result on how ISS and iISS systems behave under series connections. Consider the cascade system

$$\dot{x} = f_1(x, u),$$
$$\dot{z} = f_2(z, x). \tag{4}$$

Assume that the x-subsystem is iISS with respect to u, so that for some functions $\alpha_0, \gamma_1 \in \mathcal{K}_\infty$ and $\beta_1 \in \mathcal{K}\mathcal{L}$ we have

$$\alpha_0(|x(t)|) \leqslant \beta_1(|x(0)|, t) + \int_0^t \gamma_1(|u(s)|) \, ds$$

and assume also that the z-subsystem is ISS with respect to x, so that for some functions $\gamma_2 \in \mathcal{K}_\infty$ and $\beta_2 \in \mathcal{K}\mathcal{L}$ we have

$$|z(t)| \leqslant \beta_2(|z(0)|, t) + \gamma_2(\|x_t\|).$$

Proposition 2. *Under the above assumptions, the cascade system* (4) *is iISS with respect to the input u.*

Proof (*Sketch*). We employ a standard argument used for analysis of cascade systems (cf. [21,23,9]). In view of time-invariance, the ISS property of the z-subsystem can be written as

$$|z(t)| \leqslant \beta_2(|z(t/2)|, t/2) + \gamma_2(\|x\|_{[t/2, t]}),$$

where $\|x\|_{[t/2, t]} := \operatorname{ess\,sup}\{|x(s)|: s \in [t/2, t]\}$. Using straightforward manipulations, we obtain

$$|z(t)| \leqslant \bar{\beta}_1(|x(0)|, t/2) + \bar{\beta}_2(|z(0)|, t/2) + \bar{\gamma}\left(\int_0^t \gamma_1(|u(s)|)\,\mathrm{d}s \right),$$

where

$$\bar{\beta}_1(r, s) := \beta_2(3\gamma_2(2\alpha_0^{-1}(2\beta_1(r, 0))), s) + \gamma_2(2\alpha_0^{-1}(2\beta_1(r, s))),$$

$$\bar{\beta}_2(r, s) := \beta_2(3\beta_2(r, s), s), \quad \bar{\gamma}(r) := \beta_2(3\gamma_2(2\alpha_0^{-1}(2r)), 0) + \gamma_2(2\alpha_0^{-1}(2r)).$$

Applying $\bar{\gamma}^{-1}$ to both sides of the last inequality, we arrive at the desired result. □

Proposition 2 can be added to the series of (well-known) results saying that a cascade of an ISS system driving another ISS system is ISS and a cascade of a GAS system driving an ISS system is GAS (see, e.g., [23,24]). Cascades in which the driven system is iISS are studied in [3].

3. Control Lyapunov functions

A positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *control Lyapunov function* (CLF) for the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \ u \in \mathscr{U} \subset \mathbb{R}^m$$

if we have

$$\inf_{u \in \mathscr{U}} \{\nabla V(x) f(x, u)\} < 0 \quad \forall x \neq 0.$$

This notion goes back to Artstein [4] (see also [20] for parallel work which studied nonsmooth CLFs). If the system is affine in controls, as given by $\dot{x} = f(x) + G(x)u$, then the knowledge of a CLF V often leads to an explicit formula for a state feedback control law that makes the closed-loop system globally asymptotically stable (with Lyapunov function V). For example, in the case when $\mathscr{U} = \mathbb{R}^m$, the "universal" formula derived in [22] yields the stabilizing feedback law $u = K(a(x), b^{\mathrm{T}}(x))$, where $a(x) := \nabla V(x) f(x)$, $b(x) := \nabla V(x) G(x)$, and the function $K : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is defined by the formula

$$K(a, b) := \begin{cases} -\dfrac{a + \sqrt{a^2 + |b|^4}}{|b|^2} b, & b \neq 0, \\ 0, & b = 0. \end{cases} \tag{5}$$

The above control law is smooth on $\mathbb{R}^n \setminus \{0\}$ if the functions f and G are smooth. It is in addition continuous at 0 if V satisfies the *small control property*: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$ there exists some u with $|u| < \varepsilon$ for which $a(x) + b(x)u < 0$. We will sometimes also say that the pair $(a(x), b(x))$ satisfies the small control property. Functions that are smooth away from the origin and continuous at the origin are in this context called *almost smooth*. Similar universal formulas have later been obtained for controls bounded in magnitude [16], positive controls [17], and controls restricted to Minkowski balls [18].

In this paper we will be concerned with systems of the form

$$\dot{x} = f(x,d) + G(x)u, \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^k$ is a disturbance, $u \in \mathscr{U}$ is a control input, $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are smooth functions, and $\mathscr{U}$ is a nonempty closed subset of $\mathbb{R}^m$ containing the origin. A natural counterpart of the feedback stabilization problem in the presence of disturbances is the problem of achieving disturbance attenuation, in the ISS or iISS sense, by choice of feedback. The first step is to introduce a suitable notion of control Lyapunov function.

Definition 1. We will say that a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-CLF for system (6) if there exist class $\mathscr{K}_\infty$ functions α and χ such that we have

$$\inf_{u \in \mathscr{U}} \{a(x,d) + b(x)u\} \leqslant -\alpha(|x|) + \chi(|d|) \quad \forall x, d, \tag{7}$$

where

$$a(x,d) := \nabla V(x)f(x,d), \quad b(x) := \nabla V(x)G(x).$$

We will say that a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is an iISS-CLF for system (6) if inequality (7) holds for some continuous positive definite function α and some class $\mathscr{K}_\infty$ function χ. We will refer to the triple (V, α, χ) as an ISS-CLF (respectively, iISS-CLF) triple.

The above definition of ISS-CLF is equivalent to the ones previously proposed in [12,25,31]. The concept of iISS-CLF was first introduced in [15].

4. Universal formulas

Consider system (6). Assume that an iISS-CLF triple (V, α, χ) for system (6) is given (see Definition 1 in Section 3). Increase the function χ if necessary, so that for every fixed x the expression $a(x,d) - \chi(|d|)$ is negative for sufficiently large $|d|$. Then the function ω given by the formula

$$\omega(x) := \max_d \{a(x,d) - \chi(|d|)\} \tag{8}$$

is well defined. It is always possible to modify the function χ so that the above property holds; for example, replace $\chi(r)$ by $\chi(r) + \bar{\chi}(r)$ where $\bar{\chi}(r) := \max_{|\xi|, |\eta| \leqslant r} a(\xi, \eta)$ for each $r \geqslant 0$. This modification serves to guarantee that the right-hand side of inequality (7) is an "assignable upper bound" for the derivative of V, according to the terminology of [29]. To be more precise, one can always choose χ so that (8) can be rewritten as

$$\omega(x) = \max_{|d| \leqslant \rho(|x|)} \{a(x,d) - \chi(|d|)\}$$

for some $\rho \in \mathscr{K}_\infty$.

We also assume that V satisfies the following variant of the small control property: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$ there exists some u with $|u| < \varepsilon$ for which

$$\omega(x) + b(x)u \leqslant -\alpha(|x|). \tag{9}$$

Note that taking the function α in (9) to be the same as in Definition 1 introduces no loss of generality, because we can always decrease this function in a neighborhood of 0 if necessary.

Let us take the disturbance-affine system

$$\dot{x} = \hat{f}(x) + \hat{G}(x)d + G(x)u \tag{10}$$

as an example. Then $a(x,d) = \hat{a}(x) + \hat{b}(x)d$, where

$$\hat{a}(x) := \nabla V(x)\hat{f}(x), \quad \hat{b}(x) := \nabla V(x)\hat{G}(x).$$

To ensure that ω is well defined, it is sufficient to demand that χ grow faster than any linear function at infinity, i.e., $\chi(r)/r \to \infty$ as $r \to \infty$. Increasing χ if necessary so that it becomes greater than some linear function for all d, we see that (9) is equivalent to the condition $\hat{a}(x) + b(x)u \leqslant -\alpha(|x|)$. This corresponds to the standard small control property for the disturbance-free case.

The function ω defined by (8) is locally Lipschitz [4] but not necessarily smooth. Take another function, $\bar{\omega}$, which is smooth away from 0 and continuous at 0 (i.e., almost smooth) and satisfies

$$\omega(x) + \alpha(|x|)/3 \leqslant \bar{\omega}(x) \leqslant \omega(x) + 2\alpha(|x|)/3 \quad \forall x. \tag{11}$$

Such a function can be constructed using standard smooth approximation techniques (cf. [6, Lemma 4.9]).

From (7), (8) and (11) we have

$$\inf_{u \in \mathcal{U}} \{\bar{\omega}(x) + b(x)u\} \leqslant -\alpha(|x|)/3. \tag{12}$$

Moreover, using (11) and the small control property, we see that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$ there exists some u with $|u| < \varepsilon$ for which we have $\bar{\omega}(x) + b(x)u \leqslant -\alpha(|x|)/3$.

In what follows, we take the control set $\mathcal{U}$ to be $\mathbb{R}^m$ and use the "universal" formula for asymptotic stabilization from [22]. As mentioned in Section 3, similar formulas are available for systems with controls taking values in various restricted control spaces. The result given below can be carried over to these systems by simply substituting an appropriate universal formula (cf. [14]).

Consider the feedback control law

$$k(x) := K(\bar{\omega}(x), b^{\mathrm{T}}(x)), \tag{13}$$

where the function K is defined by formula (5). In view of the above developments, it is now not difficult to prove the following.

Theorem 3. *Consider the system* (6) *with* $\mathcal{U} = \mathbb{R}^m$. *If V is an iISS-CLF satisfying the small control property* (9), *then the feedback law* (13) *is almost smooth and integral-input-to-state stabilizing. If V is an ISS-CLF satisfying the small control property* (9), *then the feedback law* (13) *is almost smooth and input-to-state stabilizing.*

Proof. It follows from the above analysis and from the results of [22] that the control law (13) is almost smooth and that we have

$$\bar{\omega}(x) + b(x)k(x) < 0 \quad \forall x \neq 0. \tag{14}$$

The derivative of V along solutions of the closed-loop system

$$\dot{x} = f(x,d) + G(x)k(x) \tag{15}$$

satisfies

$$\dot{V} = a(x,d) + b(x)k(x) \leqslant \omega(x) + b(x)k(x) + \chi(|d|),$$

$$\leqslant \bar{\omega}(x) - \alpha(|x|)/3 + b(x)k(x) + \chi(|d|) \leqslant -\alpha(|x|)/3 + \chi(|d|)$$

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[4] This is because on every compact subset of $\mathbb{R}^n$ it is obtained by taking the maximum of a parameterized family of smooth functions of x, with the parameter d taking values in a compact set.

by virtue of (11) and (14). This shows that V is an iISS-Lyapunov function for the system (15), which implies that (15) is iISS as needed. If V is an ISS-CLF for (6), then the function α is of class $\mathcal{K}_\infty$. In this case V is an ISS-Lyapunov function for (15), which implies ISS. $\square$

Remark 1. In place of $\bar\omega$ we could take a function $\tilde\omega$ which is almost smooth (i.e., smooth away from 0 and continuous at 0) and satisfies

$$\omega(x) \leqslant \tilde\omega(x) \leqslant \omega(x) + \alpha(|x|)/2 \quad \forall x.$$

Applying the feedback law $k(x) := K(\tilde\omega(x), b^{\mathrm{T}}(x))$, we obtain for the closed-loop system

$$\dot V \leqslant \tilde\omega(x) + b(x)k(x) + \chi(|d|) < \chi(|d|) \quad \forall x \neq 0.$$

We conclude that the system is 0-GAS and zero-output dissipative, hence iISS. Such an alternative construction allows one to let $\tilde\omega = \omega$ if this function already happens to be smooth, or if a locally Lipschitz control law is sufficient. Note, however, that this method does not yield an input-to-state stabilizing control law in the case when V is an ISS-CLF.

As mentioned in the Introduction, various methods for designing input-to-state and, more recently, integral-input-to-state stabilizing control laws have already appeared in the literature. The relation of Theorem 3 to these earlier works is clarified in the next section.

5. Discussion

The feedback design procedure described in Section 4 relies on the universal formula from [22] which provides a specific expression for an almost smooth feedback law. This distinguishes the present approach from alternative constructions based on pointwise min-norm control laws [7,19,29], which are in general just continuous, and from non-constructive arguments involving partitions of unity [4]. If one applies the universal formula directly to the appropriate definition of control Lyapunov function, the resulting control law is of the form $u = k(x, d)$ rather than $u = k(x)$. This is why we needed to perform additional manipulations to eliminate the disturbance d and arrive at the expression (12). The case $\mathcal{U} = R^m$ was used merely as an example; other universal formulas can be used to treat different control spaces. Also note that once (12) is known to hold, the *existence* of a desired feedback law, for a general closed convex $\mathcal{U}$, follows as a special case from the main result of Artstein [4]. [5]

In the context of the *input-to-state* stabilization problem for the affine system (10), a different approach is possible. Namely, by using a worst-case disturbance argument, one can obtain an equivalent characterization of ISS-CLF which does not involve d. Indeed, it is not hard to show that the inequality

$$\inf_{u \in \mathcal{U}} \{\hat a(x) + \hat b(x)d + b(x)u\} \leqslant -\alpha(|x|) + \chi(|d|) \quad \forall x, d$$

holds for some $\alpha, \chi \in \mathcal{K}_\infty$ if and only if there exist class $\mathcal{K}_\infty$ functions ρ and $\tilde\alpha$ such that for all x and d we have

$$|x| \geqslant \rho(|d|) \Rightarrow \inf_{u \in \mathcal{U}} \{\hat a(x) + \hat b(x)d + b(x)u\} \leqslant -\tilde\alpha(|x|).$$

This in turn is equivalent to

$$\inf_{u \in \mathcal{U}} \{\hat a(x) + |\hat b(x)|\rho^{-1}(|x|) + b(x)u\} \leqslant -\tilde\alpha(|x|)$$

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[5] Although Artstein's original construction does not give a smooth feedback law, smoothness can be easily achieved by a simple modification using a smooth partition of unity (see [4, Remark 4.5]).

and now a universal formula can be invoked directly by virtue of the last inequality. This was the basic idea behind the constructions of input-to-state stabilizing controllers given in [12,25,31]. A small control property must be imposed to guarantee continuity of the control law at the origin, so that one can apply to the closed-loop system the sufficient condition for ISS in terms of an ISS-Lyapunov function. Since this condition also requires that the system be smooth or at least locally Lipschitz away from the origin, one needs to replace the function $\hat{a}(x) + |\hat{b}(x)|\rho^{-1}(|x|)$ by a suitable smooth approximation.

If the function α is only positive definite and not of class $\mathscr{K}_\infty$, the above equivalences break down. Therefore, the problem of *integral-input-to-state* stabilization requires a different approach. The solution proposed in [15] for affine systems of form (10) involves combining several control laws defined on appropriate regions of the state space. This can be done either by smooth "patching" or by hysteresis switching (the latter method is described in [14] for the case of bounded controls). The construction presented here is much more straightforward, and can be used to solve both input-to-state and integral-input-to-state stabilization problems. It is based on defining the function ω as in (8), an idea that appears in the recent work of Teel and Praly [29] devoted to the general problem of assigning the derivative of a disturbance attenuation CLF. Within the framework of ISS and iISS disturbance attenuation, we described an explicit procedure for modifying the CLF triple to ensure that the function ω is well defined (whereas in the more general setting of [29] this is not always possible to achieve and needs to be assumed a priori). A similar function played a role in the earlier work on robust stabilization reported in [17, Section 5].

Sometimes a control law of the form $u=k(x,d)$ is acceptable, in other words, the disturbance can be directly measured and used in control design. This situation arises, for example, in supervisory control of uncertain nonlinear systems, where the disturbance corresponds to the output estimation error which is available for control [8,9]. In such cases, the universal formula can be applied directly, starting from the definition of a CLF, to yield a control law $u = k(x,d)$, as described in [14,15].

Finding an (integral-)input-to-state stabilizing control law $u=k(x,d)$ can also be a first step towards finding a suitable feedback law $u = k(x)$. Indeed, the closed-loop system

$$\dot{x} = f(x,d) + G(x)k(x,d)$$

will then possess an (i)ISS-Lyapunov function, which is automatically an (i)ISS-CLF for the original system. The only condition that one needs to check before invoking Theorem 3 is the small control property. For example, given an affine system of the form (10), one might first want to look for a smooth[6] integral-input-to-state stabilizing control law $u = k(x,d)$ satisfying $k(0,0) = 0$. If such a control law exists, the closed-loop system

$$\dot{x} = \hat{f}(x) + \hat{G}(x)d + G(x)k(x,d)$$

admits an iISS-Lyapunov function V, which is an iISS-CLF for the original system. This means that we have

$$\hat{a}(x) + \hat{b}(x)d + b(x)k(x,d) \leqslant -\alpha(|x|) + \chi(|d|) \quad \forall x, d, \tag{16}$$

where α is continuous positive definite and $\chi \in \mathscr{K}_\infty$. Moreover, letting $d = 0$ in (16), we obtain $\hat{a}(x) + b(x)k(x,0) \leqslant -\alpha(|x|)$. Using the continuity of k at $(0,0)$ and the fact that $k(0,0) = 0$, we conclude that the small control property is satisfied. Now Theorem 3 can be applied to generate an integral-input-to-state stabilizing state feedback law $u = k(x)$. A specific example along these lines is given in the next section.

Inequality (7) from Definition 1 can be rewritten as

$$\sup_{d\in\mathbb{R}^k} \inf_{u\in\mathscr{U}} \{a(x,d) + b(x)u - \chi(|d|)\} \leqslant -\alpha(|x|) \quad \forall x. \tag{17}$$

Since d and u on the left-hand side are decoupled, the sup and the inf commute and (17) is equivalent to

$$\inf_{u\in\mathscr{U}} \sup_{d\in\mathbb{R}^k} \{a(x,d) + b(x)u - \chi(|d|)\} \leqslant -\alpha(|x|) \quad \forall x. \tag{18}$$

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[6] The smoothness requirement can be relaxed (see Section 8).

We used this fact implicitly in the control construction. For the more general system

$$\dot{x} = f(x, d) + G(x, d)u \tag{19}$$

the sup and the inf may not commute, and the above construction cannot be used. Indeed, an iISS-CLF for system (19) guarantees the existence of an integral-input-to-state stabilizing control law $u = k(x, d)$ but does not guarantee the existence of a feedback law $u = k(x)$ with the same property. In [7], this difficulty is avoided by requiring a stronger condition of kind (18) in the definition of a CLF. See [5] for an interesting discussion of related issues in the context of robust stabilization.

6. Example

It is pointed out in [27] that the scalar system

$$\dot{x} = -x + xd \tag{20}$$

is iISS but not ISS. As an iISS-Lyapunov function one can take $V(x) := \log(1 + x^2)$. Indeed, we have $\dot{V} = (-2x^2 + 2x^2 d)/(1 + x^2) \leqslant -2x^2/(1 + x^2) + 2|d|$. On the other hand, the bounded disturbance $d \equiv 2$ leads to unbounded trajectories, so the system is not ISS.

We will now use the above observation to construct an example of a system that is integral-input-to-state stabilizable but not input-to-state stabilizable[7]. Consider the system

$$\dot{x} = -x + (x - x^2)d + u,$$
$$\dot{y} = -y + (y + x^2)d - u \tag{21}$$

No matter what control law u is applied, $d \equiv 2$ gives $(d/dt)(x + y) = x + y$. This means that the system (21) is not input-to-state stabilizable (and thus does not admit an ISS-CLF). On the other hand, it *is* integral-input-to-state stabilizable: setting $u = x^2 d$, we obtain the system

$$\dot{x} = -x + xd,$$
$$\dot{y} = -y + yd$$

which is iISS in view of the above remarks. It is not at all clear how to integral-input-to-state stabilize system (21) without cancelling some of the terms that contain the disturbance, i.e., how to achieve iISS by applying a state feedback law $u = k(x)$. However, we can use the construction described in Section 4 to demonstrate that this is possible. Indeed, the function $V(x, y) := \log(1 + x^2) + \log(1 + y^2)$ is an iISS-CLF for (21). In fact, all we need is the inequality

$$\inf_{u \in \mathcal{U}} \{\hat{a}(x) + \hat{b}(x)d + b(x)u\} \leqslant -\frac{2x^2}{1 + x^2} - \frac{2y^2}{1 + y^2} + 4|d| \quad \forall x, d.$$

Taking $\chi(|d|) := 4|d| + |d|^2$, it is straightforward to check that function (8) is well defined and equals

$$\omega(x) = \begin{cases} -\dfrac{2x^2}{1 + x^2} - \dfrac{2y^2}{1 + y^2}, & \left| \dfrac{x^2 - x^3}{1 + x^2} + \dfrac{y^2 + x^2 y}{1 + y^2} \right| < 2, \\[4mm] -\dfrac{2x^2}{1 + x^2} - \dfrac{2y^2}{1 + y^2} + \left(\dfrac{x^2 - x^3}{1 + x^2} + \dfrac{y^2 + x^2 y}{1 + y^2} - 2 \right)^2, & \left| \dfrac{x^2 - x^3}{1 + x^2} + \dfrac{y^2 + x^2 y}{1 + y^2} \right| \geqslant 2, \end{cases}$$

---

[7] Incidentally, (20) already provides a trivial example of such a system; here we have $G \equiv 0$.

Since system (21) is affine, and since χ has a linear lower bound, it follows from the discussion given at the beginning of Section 4 that the small control property for the pair $(\omega(x), b(x))$ holds if and only if it holds for the pair $(\hat{a}(x), b(x))$. We have

$$(\hat{a}(x), b(x)) = \left(-\frac{2x^2}{1+x^2} - \frac{2y^2}{1+y^2}, \frac{2x}{1+x^2} - \frac{2y}{1+y^2} \right)$$

and the small control property is obviously satisfied because $\hat{a}(x) < 0$ for all $(x, y) \neq (0, 0)$. Now an integral-input-to-state stabilizing feedback law, locally Lipschitz (actually, continuously differentiable) away from 0 and continuous at 0, is given by the formula $u = K(\omega(x), b^{\mathrm{T}}(x))$. One could also take a function $\tilde{\omega}$ which is almost smooth and satisfies

$$\omega(x) \leqslant \tilde{\omega}(x) \leqslant \omega(x) + \frac{x^2}{1+x^2} + \frac{y^2}{1+y^2}$$

and define an almost smooth desired feedback law by $u = K(\tilde{\omega}(x), b^{\mathrm{T}}(x))$.

7. Backstepping

The following lemma shows that for certain classes of systems, affine in both disturbance and control inputs, integral-input-to-state stabilizing control laws can be systematically constructed by using backstepping. It exactly parallels the corresponding result for the ISS case [13,12].

Lemma 4. *If a system of the form*

$$\dot{x} = \hat{f}(x) + \hat{G}(x)d + G(x)u, \quad x \in R^n, \ d \in \mathbb{R}^k, \ u \in \mathbb{R}^m \tag{22}$$

is integral-input-to-state stabilizable with a smooth control law $u = k(x)$ satisfying $k(0) = 0$, then an augmented system of the form

$$\dot{x} = \hat{f}(x) + \hat{G}(x)d + G(x)\xi,$$

$$\dot{\xi} = u + F(x, \xi)d \tag{23}$$

is integral-input-to-state stabilizable with a smooth control law $u = \hat{k}(x, \xi)$.

Proof. Since the system

$$\dot{x} = \hat{f}(x) + \hat{G}(x)d + G(x)k(x)$$

is iISS, it admits a smooth iISS-Lyapunov function V so that we have

$$\hat{a}(x) + \hat{b}(x)d + b(x)k(x) \leqslant -\alpha(|x|) + \chi(|d|) \quad \forall x, d \tag{24}$$

(in the notation of Section 4), where α is continuous positive definite and $\chi \in \mathcal{K}_\infty$. Regarding (23), we claim that an integral-input-to-state stabilizing feedback control law $u = \hat{k}(x, \xi)$ can be defined by

$$\hat{k}(x, \xi) := -(\xi - k(x))\left(1 + |\nabla k(x)\hat{G}(x)|^2 + |F(x, \xi)|^2\right) - b^{\mathrm{T}}(x) + \nabla k(x)\hat{f}(x) + \nabla k(x)G(x)\xi. \tag{25}$$

To verify this claim, define the function

$$V_a(x, \xi) := V(x) + \tfrac{1}{2}|\xi - k(x)|^2.$$

Calculating the derivative of V_a along solutions of the closed-loop system

$$\dot{x} = \hat{f}(x) + \hat{G}(x)d + G(x)\xi,$$
$$\dot{\xi} = \hat{k}(x, \xi) + F(x, \xi)d \qquad (26)$$

with the help of (24), (25), and square completion, it is not hard to verify that

$$\dot{V}_a = \hat{a}(x) + \hat{b}(x)d + b(x)\xi$$
$$\qquad + (\xi - k(x))^{\mathrm{T}}[\hat{k}(x, \xi) + F(x, \xi)d - \nabla k(x)\hat{f}(x) - \nabla k(x)\hat{G}(x)d - \nabla k(x)G(x)\xi]$$
$$= \hat{a}(x) + \hat{b}(x)d + b(x)k(x)$$
$$\qquad + (\xi - k(x))^{\mathrm{T}}[b^{\mathrm{T}}(x) + \hat{k}(x, \xi) + F(x, \xi)d - \nabla k(x)\hat{f}(x) - \nabla k(x)\hat{G}(x)d - \nabla k(x)G(x)\xi]$$
$$= \hat{a}(x) + \hat{b}(x)d + b(x)k(x) - |\xi - k(x)|^2 - |\xi - k(x)|^2|\nabla k(x)\hat{G}(x)|^2$$
$$\qquad - (\xi - k(x))^{\mathrm{T}}\nabla k(x)\hat{G}(x)d - |\xi - k(x)|^2|F(x, \xi)|^2 + (\xi - k(x))^{\mathrm{T}}F(x, \xi)d$$
$$\leqslant -\alpha(|x|) + \chi(|d|) - |\xi - k(x)|^2 + \tfrac{1}{2}|d|^2 < \chi(|d|) + \tfrac{1}{2}|d|^2$$

for all $(x, \xi) \neq (0, 0)$ and all d. This implies that system (26) is 0-GAS and zero-output dissipative, hence iISS. $\square$

Remark 2. We can actually go further and show that V_a is an iISS-Lyapunov function for the closed-loop system (26), and consequently an iISS-CLF for (23). Let α_a be a continuous positive definite function such that for all $r \geqslant 0$, all x and ξ with $|\binom{x}{\xi}| = r$, and all $|d| \leqslant r$ we have

$$\alpha_a(r) \leqslant -\dot{V}_a + \chi(|d|) + \tfrac{1}{2}|d|^2.$$

Then for all x and d such that $|\binom{x}{\xi}| \geqslant |d|$ we have

$$\dot{V}_a \leqslant -\alpha_a\left(\left|\binom{x}{\xi}\right|\right) + \chi(|d|) + \frac{1}{2}|d|^2. \qquad (27)$$

Next, let $\hat{\chi}_a$ be a class $\mathscr{K}_\infty$ function with the property that

$$\hat{\chi}_a(r) \geqslant \dot{V}_a + \alpha_a\left(\left|\binom{x}{\xi}\right|\right) - \chi(|d|) - \frac{1}{2}|d|^2$$

for all $r \geqslant 0$, all x and ξ with $|\binom{x}{\xi}| \leqslant r$, and all $|d| = r$. Then for all x and d such that $|\binom{x}{\xi}| < |d|$ we have

$$\dot{V}_a \leqslant -\alpha_a\left(\left|\binom{x}{\xi}\right|\right) + \chi(|d|) + \frac{1}{2}|d|^2 + \hat{\chi}_a(|d|).$$

Together with (27) this implies that

$$\dot{V}_a \leqslant -\alpha_a\left(\left|\binom{x}{\xi}\right|\right) + \chi_a(|d|) \quad \forall x, \xi, d$$

where $\chi_a(r) := \hat{\chi}_a(r) + \chi(r) + \tfrac{1}{2}|d|^2$. Thus V_a is an iISS-Lyapunov function for (26), as claimed.

Suppose now that we start with a system of form (22) that satisfies the assumptions of Lemma 4. Repeated application of this lemma leads to an explicit recursive procedure for designing integral-input-to-state stabilizing control laws for cascade systems of the form

$$
\begin{aligned}
\dot{x} &= \hat{f}(x) + \hat{G}(x)d + G(x)\xi_1, \\
\dot{\xi}_1 &= \xi_2 + F_1(x, \xi_1)d, \\
&\vdots \\
\dot{\xi}_{n-1} &= \xi_n + F_{n-1}(x, \xi_1, \dots, \xi_{n-1})d, \\
\dot{\xi}_n &= u + F_n(x, \xi_1, \dots, \xi_n)d.
\end{aligned}
$$

As a byproduct, an iISS-CLF is also generated at each step (see Remark 2). We refer the reader to [29] for a more detailed analysis of these issues and for additional insights.

8. Regularity at the origin and converse results

We have shown that if system (6) admits an iISS-CLF, then there exists an almost smooth integral-input-to-state stabilizing feedback law $u = k(x)$. To prove this result (Theorem 3), we exploited the fact that if a system admits an iISS-Lyapunov function, then it is iISS. This fact was established in [2] under the assumption that the system is locally Lipschitz. Since the control law that we used is not necessarily Lipschitz at the origin, the resulting closed-loop system may not satisfy this assumption. However, it is not difficult to check that the relevant argument given in [2] is still valid if the system is merely continuous at the origin. The same remark applies to input-to-state stabilizing control laws.

We would like to show that the converse also holds, namely, that the existence of an almost smooth integral-input-to-state stabilizing feedback law for system (6) implies the existence of an iISS-CLF. To this end, take such a feedback law $u = k(x)$. If it is smooth or at least locally Lipschitz at the origin, then the closed-loop system (15) satisfies the assumptions of [2]. The converse part of the main theorem in that paper implies that system (15) admits an iISS-Lyapunov function, which is automatically an iISS-CLF for system (6). However, if the feedback law does not have sufficient regularity at the origin, one cannot simply appeal to the converse result of [2], and the above argument requires additional justification.

A similar issue in fact arises in the context of ISS, and it has been frequently overlooked. One way to fix this problem in the case of an almost smooth input-to-state stabilizing feedback law is to consider a positive definite function φ such that the right-hand side of the closed-loop system becomes smooth everywhere when multiplied by $\varphi(x)$. More precisely, let $u = k(x)$ be an almost smooth input-to-state stabilizing feedback law for system (6). Let φ be a positive definite function such that the system

$$
\dot{x} = \varphi(x)[f(x, d) + G(x)k(x)] \tag{28}
$$

is smooth. The results of [26] allow one to show that the modified system (28) is still ISS (see Lemmas 2.12–2.14 in [26]). Therefore, a characterization of ISS proved in [26] guarantees the existence of a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions of (28) satisfies

$$
|x| \geqslant \rho(|d|) \Rightarrow \dot{V} \leqslant -\alpha(|x|),
$$

where $\rho \in \mathcal{K}_\infty$ and α is continuous positive definite. Since φ is positive definite, the derivative of V along solutions of the original closed-loop system (15) also satisfies the same inequality for a different α, hence V is an ISS-CLF for the open-loop system (6). This "time reparameterization" trick was used in [4], although for a different purpose, namely, assuring continuity at the origin of a possibly discontinuous vector field.

Alternatively, one can appeal to the work reported in [31], where a generalization of a converse Lyapunov theorem was proved for systems that do not have regularity on the invariant sets. The desired conclusion also follows from the recent results of Teel and Praly [28].

To make this work self-contained, we now present a construction for the iISS case which is similar to the one described in the previous paragraph, although far less trivial to justify (because the results of [26] do not apply to iISS). Namely, we will prove that if an almost smooth feedback law $u = k(x)$ integral-input-to-state stabilizes system (6) and if φ is a suitably chosen positive definite function which makes the system (28) smooth, then (28) is iISS. We will then show how an iISS-Lyapunov function for the modified closed-loop system (28) can be used to obtain an iISS-Lyapunov function for the original closed-loop system (15), which automatically yields an iISS-CLF for the open-loop system (6). This construction is based on the following two results, which are of independent interest.

Lemma 5. *Suppose that system* (1) *is iISS. Let* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *be a positive definite smooth function. Then the system*

$$\dot{x} = \varphi(x)f(x,d) \tag{29}$$

is iISS.

Proof. Let $\alpha_0, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ be the functions appearing in the iISS estimate (3) for system (1). Pick an arbitrary initial state $z_0 \neq 0$ and a disturbance d such that

$$\int_0^\infty \gamma(|d(s)|)\,\mathrm{d}s \leqslant b < \infty.$$

Denote by $z(\cdot)$ the corresponding solution of (29), defined on some maximal interval $[0, T)$. Let

$$\tau(t) := \int_0^t \varphi(z(s))\,\mathrm{d}s, \quad t \in [0, T).$$

Since $\varphi(z(t)) > 0$ for all $t \in [0, T)$, the function $\tau(\cdot)$ is strictly increasing. Note that τ satisfies the following initial value problem:

$$\dot{\tau}(t) = \varphi(z(t)), \quad \tau(0) = 0.$$

Hence, by the uniqueness property of system (29), it holds that

$$z(t) = x(\tau(t)) \quad \forall t \in [0, T),$$

where $x(\cdot)$ is the solution of (1) with the initial state z_0 and the disturbance function $d_x(t) := d \circ \tau^{-1}(t)$ defined on $[0, \lim_{t \to T^-} \tau(t))$. It then follows that for all $t \in [0, T)$ we have

$$\alpha_0(|z(t)|) \leqslant \beta(|z_0|, \tau(t)) + \int_0^{\tau(t)} \gamma(|d_x(s)|)\,\mathrm{d}s$$

$$\leqslant \beta(|z_0|, \tau(t)) + \int_0^t \gamma(|d(s)|)\,\mathrm{d}s \leqslant \beta(|z_0|, \tau(t)) + b.$$

Hence, $z(t)$ stays bounded on $[0, T)$, and consequently $T = \infty$. It is not hard to see now that system (29) is 0-GAS and satisfies the so-called *bounded energy frequently bounded state* property (see [1]):

$$\int_0^\infty \gamma(|d(s)|)\,\mathrm{d}s < \infty \Rightarrow \liminf_{t \to \infty} |z(t)| < \infty.$$

By Theorem 3 of [1], the system (29) is iISS. $\quad\square$

Lemma 6. *Consider system* (29), *where* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is a positive definite smooth function satisfying* $\varphi(x) \leqslant 1$ *for all* x *and* $\varphi(x) = 1$ *for all* $|x| \geqslant 1$ *such that the right-hand side of* (29) *is smooth everywhere. If* (29) *is iISS, then system* (1) *admits an iISS-Lyapunov function.*

Proof. System (29) is iISS and smooth, thus it admits an iISS-Lyapunov function V which satisfies

$$\nabla V(x)\varphi(x)f(x,d) \leqslant -\alpha(|x|) + \chi(|d|)$$

for some continuous positive definite function α and some class $\mathcal{K}_\infty$ function χ. Let ρ_0 be some smooth class $\mathcal{K}$ function such that $\rho_0(V(x)) \leqslant \varphi(x)$ for all x, and define

$$W(x) := \int_0^{V(x)} \rho_0(\sigma)\, d\sigma.$$

It then follows that W is positive definite, smooth, radially unbounded, and its derivative along solutions of (1) satisfies

$$\nabla W(x)f(x,d) = \rho_0(V(x))\nabla V(x)f(x,d) \leqslant -\rho_0(V(x))\alpha(|x|) + \chi(|d|).$$

Therefore, W is an iISS-Lyapunov function for system (1). $\quad\square$

Choosing a function φ satisfying the hypotheses of Lemma 6 and applying the two lemmas to the system (15), we immediately arrive at the following result.

Corollary 7. *Consider system* (15) *with* f, G *smooth and* k *almost smooth. If this system is iISS, then it admits an iISS-Lyapunov function.*

We can now summarize our main findings as follows.

Theorem 8. *System* (6) *admits an iISS-CLF* (*respectively, an ISS-CLF*) *satisfying the small control property* (9) *if and only if there exists an almost smooth integral-input-to-state stabilizing* (*respectively, input-to-state stabilizing*) *feedback law* $u = k(x)$.

9. Remarks on other types of CLFs

The results discussed in Sections 3 and 4 can be generalized to deal with other types of control Lyapunov functions. Consider a system with output:

$$\begin{aligned} \dot{x} &= f(x,d) + G(x)u, \\ y &= h(x), \end{aligned} \tag{30}$$

where f and G are as before and the output map $h : \mathbb{R}^n \to \mathbb{R}^q$ is continuous. Suppose that for this system there exists a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\inf_u \{\nabla V(x)f(x,d) + \nabla V(x)G(x)u\} \leqslant -\alpha(|x|) + \chi(|d|) + \rho(|y|), \tag{31}$$

where α is continuous positive definite and $\chi, \rho \in \mathcal{K}_\infty$. Similarly to the discussion in Section 3, we let

$$a(x,d) := \nabla V(x)f(x,d) - \rho(|h(x)|), \quad b(x) := \nabla V(x)G(x).$$

Then (31) becomes

$$\inf_u \{a(x,d) + b(x)u\} \leqslant -\alpha(|x|) + \chi(|d|) \quad \forall x, d. \tag{32}$$

As explained in Section 4, by modifying χ if necessary, one may assume that the function ω given by

$$\omega(x) = \sup_d \{a(x,d) - \chi(|d|)\}$$

is well defined and continuous. Let $\bar{\omega}(\cdot)$ be an almost smooth function as in (11). Then (32) together with (11) yield

$$\inf_u \{\bar{\omega}(x) + b(x)u\} \leqslant -\alpha(|x|)/3. \tag{33}$$

With the feedback control law

$$k(x) = K(\bar{\omega}(x), b^{\mathrm{T}}(x)) \tag{34}$$

where K is defined by the universal formula (5), we obtain

$$\nabla V(x) f(x,d) + \nabla V(x) G(x) k(x) \leqslant -\alpha(|x|)/3 + \chi(|d|) + \rho(|y|).$$

Hence, we arrive at the following generalization of Theorem 3.

Theorem 9. *Suppose that system* (30) *admits a positive definite radially unbounded smooth function V that satisfies inequality* (31) *for some continuous positive function α and some class $\mathcal{K}_\infty$ functions χ and ρ. Assume that the small control property* (9) *holds for the pair* $(\omega(x), b(x))$. *Then, with the almost smooth feedback law $k(x)$ given by* (34), *we have the following*:

1. *The closed-loop system is* integral-input integral-output to state stable, *that is, there exist some $\alpha_0, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for each trajectory $x(\cdot)$ with disturbance d we have*

$$\alpha_0(|x(t)|) \leqslant \beta(|x(0)|, t) + \int_0^t \gamma_1(|d(s)|)\,\mathrm{d}s + \int_0^t \gamma_2(|y(s)|)\,\mathrm{d}s$$

   for all $t \geqslant 0$ (see [10]).
2. *If $\alpha \in \mathcal{K}_\infty$, then the closed-loop system is* input-output-to-state stable, *that is, there exist some $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for each trajectory $x(\cdot)$ with disturbance d we have*

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma_1(\|d\|) + \gamma_2(\|y_t\|)$$

   for all $t \geqslant 0$ (see [11]).
3. *If $\alpha \in \mathcal{K}_\infty$ and $\chi \equiv 0$, then the closed-loop system is* robustly output-to-state stable, *that is, there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that*

$$|x(t)| \leqslant \beta(|x(0)|, t) + \gamma(\|y_t\|)$$

   for all $t \geqslant 0$ and all d (see [11]).
4. *If $\rho \equiv 0$, then the closed-loop system is iISS.*
5. *If $\rho \equiv 0$ and $\alpha \in \mathcal{K}_\infty$, then the closed-loop system is ISS.*

Remark 3. By an argument similar to the one used in Section 8, one can show that the converse of statement 2 in the above theorem about input-output-to-state stabilizability is also true. Namely, if there is an almost smooth feedback $u = k(x)$ under which the closed-loop system is input-output-to-state stable, then the system

admits a CLF V that satisfies inequality (31) and the small control property (9), with α of class $\mathscr{K}_\infty$. As for statement 1, it is not clear to us at this stage if the converse statement is true (it was shown in [10] that integral-input integral-output to state stability implies the existence of a *continuous* V). Regarding statement 3, it was shown in [11] that robust output-to-state stability implies the existence of a smooth V when the disturbance d takes values in a compact set, but this is in general not true for unbounded d.

Acknowledgements

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