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STABILIZATION WITH SATURATED ACTUATORS, A WORKED EXAMPLE:F-8 LONGITUDINAL FLIGHT CONTROL*

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Abstract

The authors and coworkers recently proved general theorems on the global stabilization of linear systems subject to control saturation. This paper develops in detail an explicit design for the linearized equations of longitudinal flight control for an F-8 aircraft, and tests the obtained controller on the original nonlinear model. This paper represents the first detailed derivation of a controller using the techniques in question, and the results are very encouraging.

Keywords: Aircraft control, saturated controllers

1. Introduction

In [5], [6], [12], and [8], the authors and Sussmann considered linear time-invariant continuous time systems of the type

$$\dot{x} = Ax + B\sigma(u)\,,$$

where as usual $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, for some integers n, the dimension of the system, and m, the number of inputs. The function σ is supposed to be a saturation nonlinearity such as $\sigma(u) = u$ for $|u| < \varepsilon$ and $\sigma(u) = \varepsilon\,\mathrm{sign}\,u$ otherwise; if u is a vector, then σ is understood as being applied coordinatewise. (The positive constant ε indicates the level at which the actuator saturates. Alternatively, one may think of linear systems $\dot{x} = Ax + Bu$ for which the control values $u(t)$ are restricted to a bounded set containing the origin in its interior; mathematically there is no difference.) The imposition of control constraints is a realistic addition to the standard linear systems model, and it reflects practical limitations on actuators. The above-mentioned papers presented results pertaining to the global stabilization of such systems.

The main result in the above references was that, subject only to the more or less obvious necessary conditions, namely: all eigenvalues of A have nonpositive real part, and all eigenvalues of the uncontrollable part of the system have strictly negative real parts —that is, the pair (A, B) is stabilizable in the ordinary sense, e.g. [4],— there exists an infinitely differentiable global feedback stabilizer $u = k(x)$. Notice that this result is nontrivial because there may exist non-simple Jordan blocks corresponding to critical eigenvalues, so that the system $\dot{x} = Ax$ is not even marginally stable. Moreover, Fuller had already shown long ago in [2] (see [7] for a related result) that the "naive" technique consisting of just using a linear controller will in general lead to oscillations and even instability.

In the first paper, [5], we presented a fairly abstract existence theorem. Motivated by this, Teel introduced a constructive approach to the problem, in the particular case of single-input multiple integrators. He proposed a feedback law $u = k(x)$ which consisted of certain linear combinations of the basic linear saturation σ used above. In the technical report [6], we generalized Teel's construction to the full case considered in the original work [5], allowing us to deal with multivariable inputs, non-zero eigenvalues, and output stabilization. Also in [8], we extended this in such a manner as to allow much more general saturation nonlinearities σ in the feedback loop, and we also showed that an alternative design based on a "feedforward neural network" architecture also provides global stabilization. The paper [8] will present complete details of the most general possible theorem along these lines.

This paper develops in detail an explicit design, following the general philosophy outlined in [12], for the linearized equations of longitudinal flight control for an F-8 aircraft, and tests —via simulations— the obtained controller on the original nonlinear model. We feel that it is appropriate to provide the example, in order to illustrate the power of our techniques, as they have not been applied before in a realistic problem. Reasons of space precluded including such a detailed example in the main theoretical papers [12, 8]. In the process of working out this example, we were able to obtain tighter bounds, in certain particular cases, than in [12, 8], for some critical estimates. With these improved bounds, we can achieve better performance.

We picked the model in [3], as expanded and corrected in [10], since this has been often considered as a paradigm for many aircraft control problems. The paper uses the exact constants and trim conditions —that is, the desired operating points— given in references [1, 10, 11]; we thank Jianliang Wang for providing us with the preprints of his work. (Note that the problem considered here is different from the question treated in those references, and only the data is employed. Also, we add a saturation to the model. We pick relative small values of these saturations, to analyze our control design under demanding

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conditions.)

In the next section we provide the detailed model and all constants. After that, we linearize the system about an operating point and construct a globally stabilizing controller for the resulting linearization, following the steps of the proof in [12] and [8]. Then we proceed to compare the performance of our controller, applied to the original nonlinear airplane, and starting reasonably far from the desired operating point, with the "naive" controller that would result from applying a linear feedback law which would stabilize in the absence of saturations.

2. The Model

We rely on [10, 11] (see also [3]) for the following nonlinear model for the F8 aircraft longitudinal flight dynamics. For the reader's convenience, we repeat here all the relevant equations, using the same notations as in the above references:

$$\begin{array}{rcl} \dot{\mu} & = & -\mu q \tan\alpha - g\sin\theta + \frac{L_\omega}{m}\sin\alpha + \frac{L_t}{m}\sin\alpha_t \\ \dot{\alpha} & = & q + \frac{g}{\mu}\cos\alpha\cos(\alpha-\theta) - \frac{L_\omega}{\mu m}\cos\alpha \\ & & -\frac{L_t}{\mu m}\cos\alpha\cos(\alpha-\alpha_t) \\ \dot{\theta} & = & q \\ \dot{q} & = & (M_\omega + lL_\omega\cos\alpha - l_t L_t \cos\alpha_t - cq)/I_y\ . \end{array} \tag{1}$$

We are using the following symbols:

$$\begin{array}{rcl} \alpha_t & = & (1-a_\epsilon)\alpha + \delta_\epsilon \\ L_\omega & = & C_L(\alpha)\bar{q}S \\ L_t & = & C_{L_t}(\alpha_t,\delta_e)\bar{q}S_t \\ \bar{q} & = & \frac{\rho\mu^2}{2\cos^2\alpha} \\ \gamma & = & \theta-\alpha\,. \end{array} \tag{2}$$

Still quoting from [10, 11], the meaning of the above coefficients and variables, and their units, are:

μ: forward speed, in ft/sec,

α: wing angle of attack, in rad,

θ, q: pitch angle, in rad, and pitch rate, in rad/sec,

γ: flight path angle, in rad,

α_t: tail angle of attack, in rad,

δ_e: elevator angle, in rad,

m: mass of aircraft, in slugs,

I_y: moment of inertia of aircraft about Y axis, in slugs ft^2,

L_ω, L_t: wing and tail lifts, in lb,

M_ω: wing moment,

l: distance between wing a.c and aircraft c.g., in ft,

l_t: distance between tail a.c and aircraft c.g., in ft,

c: damping coefficient, in lb ft sec,

C_L, C_{L_t}: wing and tail lift coefficients,

$\bar{q}$: dynamic pressure, in lb/ft^2,

S, S_t: wing and tail area, in ft^2, and

ρ: atmospheric density, in slugs ft^3.

The lift coefficients are complicated nonlinear functions of the angles of attack and elevator angle; for simplicity, and again following [10, 11], we use a cubic approximation:

$$\begin{array}{rcl} C_L(\alpha) & = & (C_L^1\alpha - C_L^2\alpha^3) \\ C_{L_t}(\alpha_t,\delta_e) & = & (C_L^1\alpha_t - C_L^2\alpha_t^3 + a_e\delta_e)\ . \end{array} \tag{3}$$

Define

$$\sigma(s) = \operatorname{sign}(s)\min\{|s|, 0.01\}\,. \tag{4}$$

The control u is applied to δ_e, by means of an actuator with the following dynamics:

$$\dot{\delta}_e = \sigma(-\delta_e + \delta_{e0} + u)\,, \tag{5}$$

where δ_{e0} is a desired equilibrium (see below). We have included a saturation nonlinearity, which saturates the right hand side at 0.01.

2.1. Trim Conditions

The desired operating point corresponds to an altitude of 30,000 ft, again as in the references [10, 11], and we also choose, as there, the following values for all parameters:

$$\begin{array}{rlrl} \rho & = 0.00089\,\text{slugs/ft}^3 & C_L^1 & = 4.0 \\ C_L^2 & = 12 & a_e & = 0.1 \\ a_\epsilon & = 0.75 & S & = 375\ \text{ft}^2 \\ S_t & = 93.4\ \text{ft}^2 & m & = 667.7\ \text{slugs} \\ I_y & = 96800\ \text{slugs ft}^2 & l & = 0.189\ \text{ft} \\ l_t & = 16.7\ \text{ft} & M_\omega & = 0\ \text{lb ft} \\ c & = 38332.8\ \text{lb ft sec} & g & = 32.2\ \text{ft/sec}^2\ . \end{array} \tag{6}$$

The equations (1) with (2) and (3), and (5) give rise to a control system. We consider the system with constants given in (6) and the desired equilibrium at:

$$\begin{array}{llllll} \mu_0 & = 389.1, & \alpha_0 & = .24007, & \theta_0 & = .23759, \\ q_0 & = 0, & \delta_{e0} & = -0.05\,. \end{array} \tag{7}$$

The above values are essentially those given in [1], but calculated more precisely using the Maple symbolic manipulation system (the numbers in that reference, namely: $\mu_0 = 388.7, \alpha_0 = 0.240, \theta_0 = 0.238, q_0 = 0, \delta_{e0} = -0.05$, give numerical values significantly far from zero when substituted in the equations).

3. Controller Design

The linearization of the model at the equilibrium (7) (but not linearizing the saturation, of course), is as follows:

$$\begin{array}{rcl} \dot{\mu} & = & a(\alpha-\alpha_0) + b(\theta-\theta_0) + cq + d(\delta_e-\delta_{e0}) \\ \dot{\alpha} & = & e(\alpha-\alpha_0) + q + f(\delta_e-\delta_{e0}) \\ \dot{\theta} & = & q \\ \dot{q} & = & g(\alpha-\alpha_0) + hq + i(\delta_e-\delta_{e0}) \\ \dot{\delta}_e & = & \sigma(-(\delta_e-\delta_{e0})+u)\,, \end{array} \tag{8}$$

where $a = 53.20473$, $b = -31.29545$, $c = -95.25528$, $d = .76017$, $e = -.25642$, $f = -.09966$, $g = -1.06173$, $h = -.39600$, $i = -4.71364$, and some small coefficients are eliminated. Disregarding the saturation, the eigenvalues of (8) are $0, 0, 0$ and:

$$1/2(e+h) \pm 1/2(-e^2 + 2eh - h^2 - 4g)^{1/2}\sqrt{-1}\,.$$

Following the procedure in [12, 8], we wish to transform (8) into a system of the following form:

$$\begin{array}{rcl} \dot{x}_1 &=& 1/2(e+h)x_1 + \lambda x_2 \\ \dot{x}_2 &=& -\lambda x_1 + 1/2(e+h)x_2 + x_4 \\ \dot{x}_3 &=& x_4 \\ \dot{x}_4 &=& x_5 \\ \dot{x}_5 &=& \sigma(-x_5 + u)\,, \end{array} \tag{9}$$

where $\lambda = 1/2(-e^2 + 2eh - h^2 - 4g)^{1/2}$. This can be achieved by changing into the variables:

$$\begin{array}{rcl} x_1 &=& \frac{i^2\lambda}{\Delta}(\alpha-\alpha_0) - \frac{\lambda}{ei-gf}(\theta-\theta_0) - \frac{if\lambda}{\Delta}q\,, \\ x_2 &=& \frac{i(ei-hi-2gf)}{2\Delta}(\alpha-\alpha_0) \\ && +\frac{e+h}{2(ei-gf)}(\theta-\theta_0) - \frac{f(ei-hi-2gf)}{2\Delta}q\,, \\ x_3 &=& -\frac{eh-g}{(ei-gf)b}(\mu-\mu_0) \\ && +\frac{ieha-dehg-agi-ibg+dg^2}{(e^2i^2-2fgei+g^2f^2)b}(\alpha-\alpha_0) \\ && +\frac{N}{(e^2i^2-2fgei+g^2f^2)b}(\theta-\theta_0) \\ && +\frac{dhe^2-efah-egd+bgf+agf}{(e^2i^2-2fgei+g^2f^2)b}q \\ x_4 &=& -\frac{g}{ei-gf}(\alpha-\alpha_0) - \frac{eh-g}{ei-gf}(\theta-\theta_0) + \frac{e}{ei-gf}q \\ x_5 &=& (\delta_e - \delta_{e0}) \end{array} \tag{10}$$

where $N = e^2bi + e^2chi - e^2h^2d + efah^2 - ebgf - echgf - igec - ieha + 2dehg - fbgh - fahg + fg^2c + agi + ibg - dg^2$, and $\Delta = e^2fi^2 + i^3e - efi^2h - 2ef^2gi - gfi^2 + f^2ihg + f^3g^2$.

Finally, we carry out the design following the general outline of the construction in [12, 8], as applied to a system in the form (9) and provide a very simple feedback, which uses one saturation. (Note that in the closed-loop system there is one other saturation, which is part of the original system: the one that already exists in the equation for $\dot{x}_5$.)

The feedback we obtain is as follows:

$$u = -0.9\,\sigma(0.2x_3 + 2.2x_4 + 2x_5)\,. \tag{11}$$

These constants 0.9, 0.2, and so forth, are not the ones that would follow from using the very conservative bounds in the proof of the theorems in [12, 8]. We picked much better constants, taking advantage of the knowledge of the equations. (Of course, one may expect that a fair amount of such fine tuning will be necessary in any realistic application.) Using the transformation (10), we get a control law for the original system (1).

3.1. Stability of the Linear System

We sketch next a proof of the fact that the above design globally stabilizes the linearized system. This is of course not enough in order to guarantee stability when the nonlinear model is used, except locally, but we think it is nonetheless of interest to show the computations. Later, we investigate experimentally the domain of stability for the nonlinear system.

Consider the system

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& x_3 \\ \dot{x}_3 &=& \sigma(-x_3 + u)\,. \end{array} \tag{12}$$

Let $y_1 = x_1 + x_2$, $y_2 = x_2 + x_3$, $y_3 = x_3$. Then we have

$$\begin{array}{rcl} \dot{y}_1 &=& y_2 \\ \dot{y}_2 &=& y_3 + \sigma(-y_3 + u) \\ \dot{y}_3 &=& \sigma(-y_3 + u)\,. \end{array} \tag{13}$$

Let

$$u = -\rho\sigma(ay_1 + by_2)\,. \tag{14}$$

Then we have the following result.

Theorem 1 *For any $\varepsilon > 0$, let $\sigma(s) = \mathrm{sign}(s)\min\{|s|, \varepsilon\}$. Let a, b, ρ be positive numbers such that $b \geq 2a$, $2b^2 > 5a$, and $1/2 < \rho < 1$. Then, the closed-loop system consisting of (13) together with the control (14) is globally asymptotically stable.*

Choosing $\varepsilon = 0.01$, $\rho = 0.9, a = 0.2, b = 2$ in the theorem, we obtain the feedback (11). For reasons of space, we cannot provide the proof here. Interested people can ask a copy from either of the authors.

Remark 3.1 In [8], [6], [12], and [9], only the case $\rho \leq \frac{1}{2}$ was discussed. Therefore, the conclusion of the theorem can be shown to be true for any $a, b > 0$. In the above theorem, we allow ρ to be bigger than $\frac{1}{2}$, and for the particular model in studied in this paper, the value $\rho > \frac{1}{2}$ indeed makes performance far better. □

4. Simulation Results

It is guaranteed by our theorems that the control law that we obtained globally stabilizes the *linearized* model, and hence locally the original model. But local stabilization of the nonlinear airplane could be achieved in principle also by using linear feedback. Thus we intend to discuss next, via simulations, the advantages of our control law with regards to such linear feedback, when used for the nonlinear model. Essentially, we obtain a relatively large domain of attraction with respect to the desired equilibrium. Providing explicit bounds on this region of attraction is in principle a difficult problem, and we do not in any way attempt to do so.

We provide now several plots regarding the closed-loop system consisting of (1-5) when using the control

(11). We also provide comparisons between our control and the "naive" design, which does not use the saturation in (11).

The initial values of $\alpha, \theta, q, \delta_e$ in all plots are:

$$\begin{aligned} \alpha(0) &= \alpha_0 + 0.25 = 0.4900685620 \\ \theta(0) &= \theta_0 + 0.2 = 0.4375883269 \\ q(0) &= 0 \\ \delta_e(0) &= \delta_{e0} - 0.04 = 0.01 \end{aligned}$$

These represent fairly large displacements from equilibrium. The different plots will differ on the initial value taken for μ.

Figures 1 and 2 are as follows. In each case, there are two vertical sets of three plots each. The three left plots are for the design that results when using (11), while the three right plots are for the design obtained when using instead the "naive" linear control law

$$u = -0.9(0.2x_3 + 2.2x_4 + 2x_5)\,, \tag{15}$$

which would stabilize in the absence of the saturations in the elevator rate. Observe that (11) and (15) coincide for all small deviations x from the desired equilibrium. The top two plots are for the forward speed μ, the bottom plots are for $\dot{\delta}_e$ (so that the effective control being applied is illustrated; notice the saturation at 0.01), and the middle plot displays the variables $(\alpha, \theta, q, \delta_e)$, plotted according to the linestyles explained in the captions.

In Figure 1, the initial value for μ is

$$\mu_0 - 18 = 371.1315833$$

while in Figure 2 it is

$$\mu_0 - 63 = 326.1315833$$

These values were chosen for the following reasons. Simulations show that our design stabilizes the nonlinear system for a wide range of values. But the "naive" design is stable roughly only in the range $[-62, -18]$ (the domain of attraction is a connected subset of $\mathbb{R}^5$, but it is not convex, of course), thus we considered the extreme cases -18, -63. In the first of these cases, the "naive" design would stabilize as well, as shown, though the difference in performance is quite striking. In the second case, the "naive" design results in instability.

5. Conclusions

We carried out an explicit design of a control law for a realistic model of a system with actuator saturation. The objective was to show that the calculations in the abstract proofs can indeed be carried out explicitly (though this an extremely simple case compared to the generality of the results in [12, 8]), and moreover, to study the performance of the resulting controller when used for the original, nonlinear, model. We consider the results to be encouraging and indicating the usefulness of further work along this direction. Among topics being considered now are questions of tracking and disturbance rejection for linear systems with saturation.

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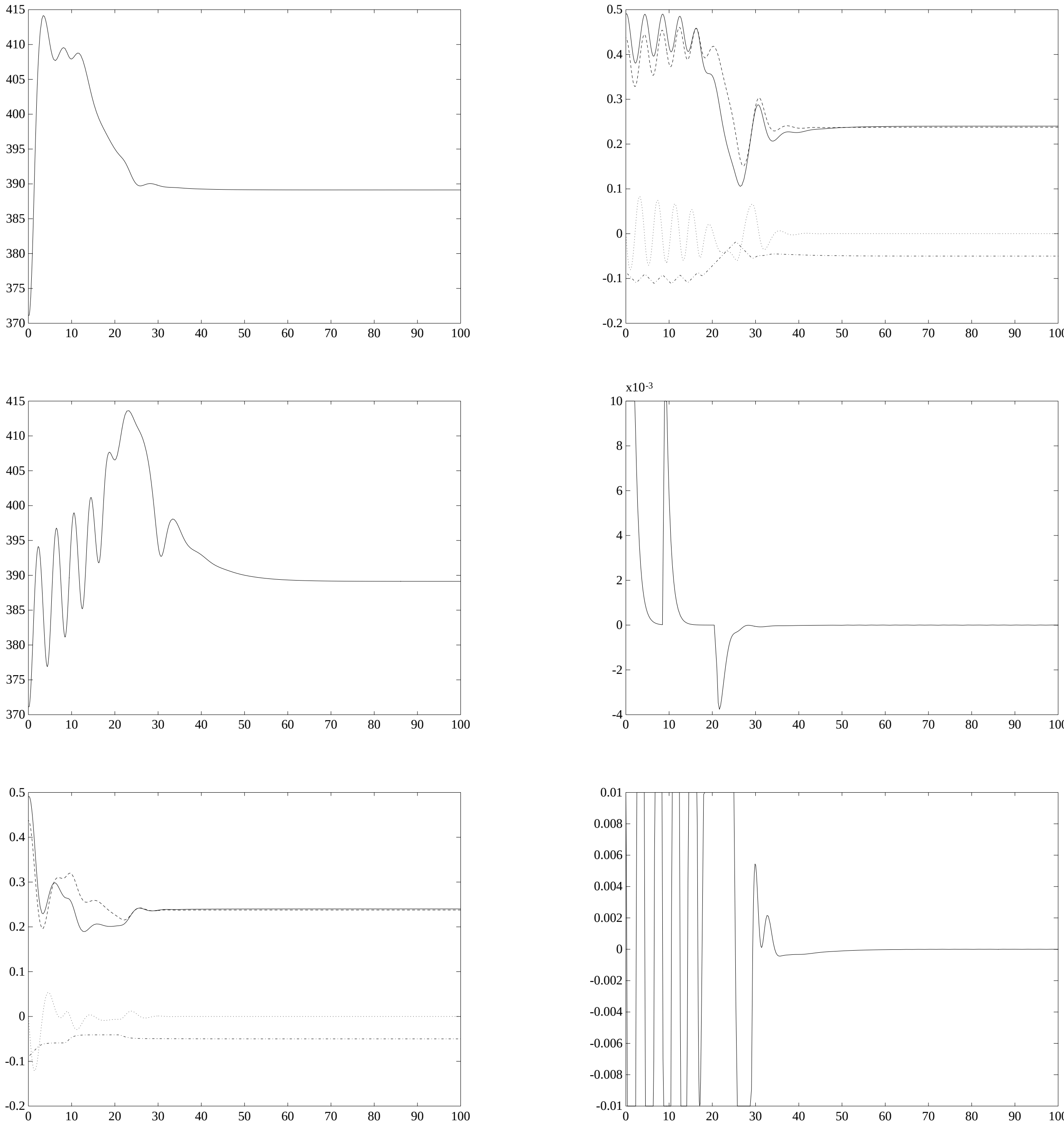

Fig 1: $\mu_0 = 371$, saturated and naive design

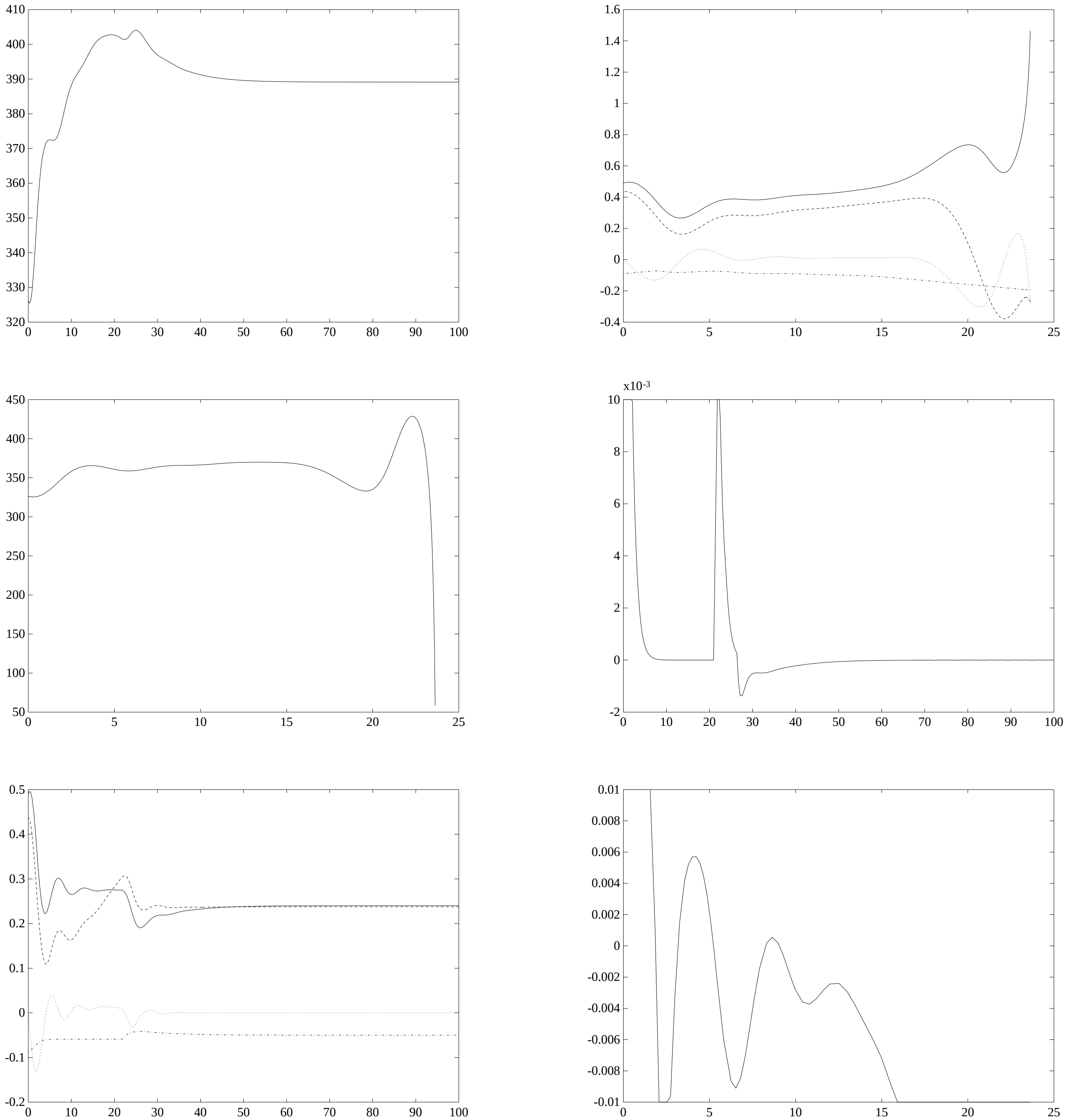

Figure 2: $\mu_0 = 326$, saturated and naive design